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A HYBRID FINITE DIFFERENCE-POLYNOMIAL EXPANSION METHOD FOR PARABOLIC PROBLEM WITH UNKNOWN DIRICHLET BOUNDARY CONDITION

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Abstract. This paper presents a sequential polynomial expansion method for solving inverse boundary condition problems in two-dimensional parabolic heat conduction. The approach combines temporal discretization via backward Euler scheme with spatial approximation using polynomial expansions, transforming the original time-dependent problem into a sequence of modified Helmholtz equations. The method effectively handles the ill-posed nature of inverse Cauchy problems through careful selection of optimal polynomial degrees and demonstrates robust performance across diverse solution profiles including oscillatory, hyperbolic-type, and exponentially decaying functions. Numerical experiments reveal that the method achieves optimal accuracy at specific polynomial degrees ($m = 7$ or $m = 9$ depending on solution characteristics) while maintaining computational efficiency with stable iteration counts. Despite challenges posed by ill-conditioned matrices at higher polynomial degrees, the approach provides accurate boundary condition reconstruction with controlled error propagation, making it a valuable tool for practical inverse problem applications in heat transfer and related fields.

Keywords: Inverse Cauchy problems, parabolic problem, finite difference, polynomial expansion.
AMS Subject Classification: 65N21, 65F22, 65M32.
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1 Introduction

Inverse problems arise in many practical applications, where the goal is to determine unknown initial conditions, boundary conditions, or parameters that cannot be measured directly (Gorelick et al., 1983; Nachaoui & Nassif, 1996; Tadi, 1997; Chen & Wong, 1998; Nachaoui, 1999; Sun, 1999; Huang & Chen, 2000; Constales & Kacur, 2001; Ellabib & Nachaoui, 2001; Su and Silva Neto, 2001; Bitterlich & Knabner, 2002; Loulou & Scott, 2006; Samarskii & Vabishchevich, 2007; Ching-yu, 2009; Aster et al., 2012; Chakib et al., 2015; Vasil'ev et al., 2016; Berdawood et al., 2021; Le & Nguyen, 2022; Nachaoui et al., 2022; Roy& Dhiman, 2023; Laghribet al., 2025; Nachaoui-Laghrib, 2025). Such problems are notoriously difficult because they are generally ill-posed in the sense of Hadamard Hadamard J. (1953); Dvalishvili et al. (2017); even small perturbations in the data may cause large deviations in the solution. This motivates the development of robust and stable numerical methods (Jourhmane & Nachaoui, 1999; Huang & Chen, 2000; Kraus et al., 2001; Yarmukhamedov, 2003; Essaouini et al., 2004; Nachaoui, 2004; Regińska & Regiński, 2006; Shi et al., 2009; Qian et al., 2010; Kabanikhin, 2012; Mukanova, 2013; Lavrentiev, 2013; Huang et al., 2017; Isakov, 2017; Qian & Feng, 2017; Vasil'ev et al., 2017; Hernandez-Montero et al., 2019; Aboud et al., 2021; Berdawood et al., 2021; Nachaoui & Salih, 2021; Reddy et al., 2021; Aboud et al., 2022; Berdawood et al., 2023; Hadri et al., 2024;

Nachaoui & Sadiq, 2025).

A widely studied class of inverse problems concerns parameter identification. For example, Srati et al. (2025) address the determination of two time-independent coefficients in a fractional diffusion system from final-time data. The estimation of nonlinear hydraulic coefficients in porous media was investigated in Bitterlich & Knabner (2002) via least-squares minimization, while Chakib et al. (2015) studied hydrological parameter recovery in Richards' equation using an optimal control framework combined with time discretization, finite differences, and Picard iteration. In the context of heat transfer, Hasan & Nachaoui, (2025) considered the recovery of time-dependent thermal conductivity in the one-dimensional heat equation with nonlocal overdetermination conditions, reformulating the problem to decouple coefficient identification from the parabolic PDE solution.

Nonlocal PDEs also appear in modeling groundwater dynamics Bouziani (1994); Gasimov et al. (2025); Pulkina (1999), viscoelasticity, and even food processing, where they are often represented as hyperbolic equations Kabanikhin (2002). Inverse heat source problems have been addressed under various assumptions: Siraj-ul-Islam (2017) used a meshless collocation method to recover a time-dependent source, while Nachaoui & Sadiq (2025) applied polynomial expansions for space-dependent sources.

Inverse Cauchy problems form another major branch of research. For the Laplace equation, Nachaoui introduced relaxed alternative iterative methods in Jourhmane & Nachaoui (1999, 2002), later extended to nonlinear elliptic problems Essaouini et al. (2004), linear elasticity Ellabib et al. (2021, 2022), convection–diffusion Nachaoui (2023), and biharmonic equations Nachaoui (2025). This method has recently been applied to solve the ill-posed inverse problem of reconstructing an unknown, time-dependent boundary condition for the two-dimensional telegraph equation Azzouzi et al. (2025). Boundary element approaches have also been applied Nachaoui (2004); Ellabib et al. (2025). In heat conduction, both linear functionally graded materials Aboud et al. (2021) and nonlinear cases Nachaoui & Salih (2021); Hasan et al. (2025); Nachaoui et al. (2025) have been studied.

Meshless methods have recently gained traction for solving inverse Cauchy problems. For instance, Rashid & Nachaoui. (2025) proposed a Haar wavelet-based reconstruction technique for Poisson equations with inaccessible boundaries, later extended in Nachaoui & Rashid (2023). In Nachaoui & Hasan (2025), the authors used the Lavrentiev regularization method coupled with a Haar wavelet discretization scheme to solve the Cauchy problem in a rectangle after its reformulation as a Fredholm equation. A similar technique is used in Nachaoui, Agamalieva & Laghrib (2025) to solve the inverse problem of spatial force identification in wave equations using non-local, time-averaged displacement data. The reader can find in Nachaoui & Gontier (2025) a presentation of a comparative analysis of a regularized solution approach based on the separation of variables method with another based on the homotopy perturbation technique. Other meshless approaches include polynomial expansion Aboud et al. (2022); Nachaoui et al. (2023) and Legendre collocation schemes Nachaoui et al. (2025), often stabilized using preconditioning and regularization. Applications of Haar wavelet techniques to general PDEs can be found in Hariharan et al. (2009); Nachaoui et al. (2018); Agamalieva et al. (2023).

In this work, we develop a sequential method for estimating boundary conditions in two-dimensional parabolic heat conduction problems. The model is discretized in time via finite differences, leading to a modified Helmholtz equation. A polynomial expansion is used to approximate the spatial solution, and the resulting system is solved iteratively with CGLS solver. Preconditioning and regularization strategies can be found in Aboud et al. (2022); Nachaoui & Rashid (2023); Rashid & Nachaoui. (2025). Numerical tests confirm that the proposed approach achieves accurate and stable recovery of unknown boundary conditions.

2 Problem Formulation

We consider the two-dimensional heat conduction problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad 0 < x < 1,\ 0 < y < 1,\ 0 < t < t_f, \tag{1}$$

with boundary conditions

$$\begin{aligned} u(1,y,t) &= h(y,t), \quad 0 \le y \le 1,\ 0 \le t \le t_f, & (2)\\ \frac{\partial u}{\partial x}(1,y,t) &= \phi(y,t), \quad 0 \le y \le 1,\ 0 \le t \le t_f, & (3)\\ u(x,0,t) &= p(x,t), \quad 0 \le x \le 1,\ 0 \le t \le t_f, & (4)\\ u(x,1,t) &= q(x,t), \quad 0 \le x \le 1,\ 0 \le t \le t_f, & (5) \end{aligned}$$

and the initial condition

$$u(x,y,0) = f_0(x,y), \quad 0 \le x \le 1,\ 0 \le y \le 1. \tag{6}$$

The functions f_0, h, p, q, ϕ are assumed sufficiently regular to guarantee existence and uniqueness of the solution. Figure 1 shows the domain and boundary decomposition.

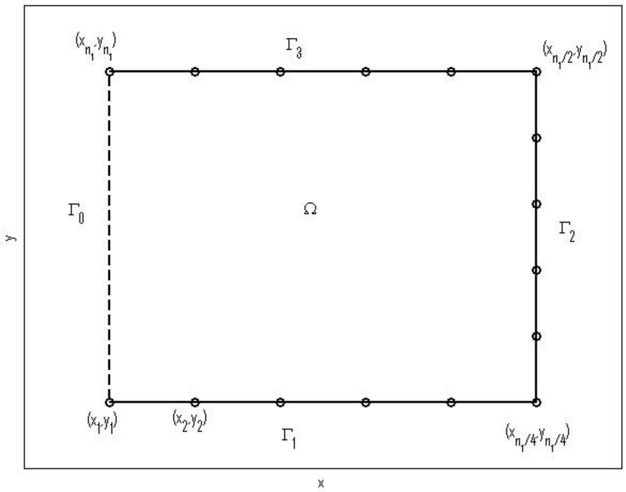

Figure 1: Different parts of the boundary

3 Constructing a Solution of the Inverse Problem

3.1 Temporal Discretization

Starting from (1)–(6), we apply backward Euler discretization in time. Let

$$t_n = n\,\delta t, \quad n = 0, 1, 2, \ldots, N,$$

then

$$\frac{\partial u}{\partial t} \approx \frac{u(x,y,t_{n+1}) - u(x,y,t_n)}{\delta t}. \tag{7}$$

Substituting into (1) gives

$$\frac{u(x,y,t_{n+1})-u(x,y,t_n)}{\delta t}=\frac{\partial^2 u(x,y,t_{n+1})}{\partial x^2}+\frac{\partial^2 u(x,y,t_{n+1})}{\partial y^2}, \tag{8}$$

$$\Delta u(x,y,t_{n+1})-\frac{1}{\delta t}u(x,y,t_{n+1})=-\frac{1}{\delta t}u(x,y,t_n). \tag{9}$$

This yields a modified Helmholtz equation

$$\Delta u(x,y,t_{n+1})-k^2u(x,y,t_{n+1})=f_n(x,y),$$

where $k^2=1/\delta t$ and $f_n(x,y)=-k^2u(x,y,t_n)$.

Since $u(x,y,t_0)$ is known, $u(x,y,t_{n+1})$ for all $n=0,1,2,\dots,N-1$, is obtained as a solution of the following boundary value problem:

$$\Delta u_{n+1}(x,y)-k^2u_{n+1}(x,y)=f_n(x,y) \tag{10}$$

$$u_{n+1}(1,y)=h(y,t_{n+1}),\quad 0\le y\le 1 \tag{11}$$

$$\frac{\partial u_{n+1}}{\partial x}(1,y)=\phi(y,t_{n+1}),\quad 0\le y\le 1 \tag{12}$$

$$u_{n+1}(x,0)=p(x,t_{n+1}),\quad 0\le x\le 1 \tag{13}$$

$$u_{n+1}(x,1)=q(x,t_{n+1}),\quad 0\le x\le 1 \tag{14}$$

The boundary value problem (10)–(14) is a Cauchy problem for the Modified Helmholtz equation.

Helmholtz-type equations appear in wave propagation Wang et al. (2020), vibrations, and heat transfer Beskos (1997), including acoustic cavities Chen & Wong (1998), radiation waves Harari et al. (1998), and fin conduction Kraus et al. (2001). Many methods exist for their Cauchy problems Regińska & Regiński (2006); Fu et al. (2009); Wang et al. (2020); Aboud et al. (2022); Qian et al. (2010); Yarmukhamedov (2003); Huang et al. (2017); Qian & Feng (2017); Yang et al. (2019); Wang et al. (2021); Berdawood et al. (2023). Recently, alternating algorithms were introduced in Berdawood et al. (2021, 2023), extending ideas from Jourhmane & Nachaoui (1999); Nachaoui et al. (2021), with proven convergence and parameter-free formulations. However, convergence acceleration intervals may sometimes be too narrow for practical use Berdawood et al. (2023).

We therefore explore a direct polynomial expansion approach, originally proposed in Aboud et al. (2022), to approximate solutions of Helmholtz-type Cauchy problems. Unlike iterative schemes, it avoids slow convergence, though care must be taken since ill-posedness manifests as ill-conditioning in the linear system. This difficulty is alleviated by efficient preconditioning.

3.2 Polynomial Expansion Approximation

The solution at time t_{n+1} is approximated by a polynomial expansion

$$u_{n+1}(x,y)=\sum_{i=1}^{m}\sum_{j=1}^{i}c_{ij}^{n+1}\,x^{i-j}\,y^{j-1}. \tag{15}$$

It is convenient to rewrite (15) in compact vector–matrix form:

$$u_{n+1}(x,y)=\mathbf{a}(x,y)^T\mathbf{c}, \tag{16}$$

where

$$\mathbf{a}(x,y)=\left[\,x^{i-j}y^{j-1}\,\right]_{1\le j\le i\le m},\qquad \mathbf{c}=\left[\,c_{ij}^{n+1}\,\right]_{1\le j\le i\le m}.$$

Similarly, the spatial derivatives can be expressed as

$$\begin{aligned}
\frac{\partial u_{n+1}}{\partial x}(x,y) &= \mathbf{a}_x(x,y)^T\mathbf{c}, &(17)\\
\frac{\partial u_{n+1}}{\partial y}(x,y) &= \mathbf{a}_y(x,y)^T\mathbf{c}, &(18)\\
\frac{\partial^2 u_{n+1}}{\partial x^2}(x,y) &= \mathbf{a}_{xx}(x,y)^T\mathbf{c}, &(19)\\
\frac{\partial^2 u_{n+1}}{\partial y^2}(x,y) &= \mathbf{a}_{yy}(x,y)^T\mathbf{c}, &(20)
\end{aligned}$$

where the vectors $\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_{xx}, \mathbf{a}_{yy}$ are obtained by differentiating $\mathbf{a}$ componentwise.

Thus, the PDE constraint becomes

$$\big(\mathbf{a}_{xx}(x,y) + \mathbf{a}_{yy}(x,y) - k^2\mathbf{a}(x,y)\big)^T\mathbf{c} = f(x,y). \tag{21}$$

Boundary conditions translate into linear equations for $\mathbf{c}$. For instance, at selected collocation points:

$$\begin{aligned}
\mathbf{a}(1,y)^T\mathbf{c} &= h_{n+1}(y), &(22)\\
\mathbf{a}(x,0)^T\mathbf{c} &= p_{n+1}(x), &(23)\\
\mathbf{a}(x,1)^T\mathbf{c} &= q_{n+1}(x), &(24)\\
\mathbf{a}_x(1,y)^T\mathbf{c} &= \phi_{n+1}(y). &(25)
\end{aligned}$$

By collocating the relations above at n_1 boundary points and n_2 interior points, we obtain the linear system

$$A\mathbf{c} = \mathbf{b}, \tag{26}$$

where A collects evaluations of $\mathbf{a}, \mathbf{a}_x, \mathbf{a}_{xx}, \mathbf{a}_{yy}$ at the collocation nodes, and $\mathbf{b}$ contains the corresponding boundary and PDE data.

Algorithm 1: Sequential Algorithm for Solving the Inverse Parabolic Heat Conduction Problem

Input: Initial condition $f_0(x,y)$, boundary data $h(y,t), \phi(y,t), p(x,t), q(x,t)$, final time t_s, discretization parameters $(\delta t, m, n_1, n_2)$.

Output: Approximate solution $u(x,y,t)$ and estimated boundary conditions.

```
Initialization:
  Set t_0 = 0, u(x, y, t_0) = f_0(x, y).
for n = 0 to N − 1 do
  1. Temporal discretization:
     Formulate the modified Helmholtz equation
```

$$\Delta u_{n+1}(x,y) - k^2 u_{n+1}(x,y) = f_n(x,y), \quad k^2 = \tfrac{1}{\delta t},\ f_n(x,y) = -k^2 u(x,y,t_n).$$

```
  2. Polynomial expansion approximation:
     Represent u_{n+1}(x, y) as
```

$$u_{n+1}(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{ij}^{n+1} x^{i-j} y^{j-1}.$$

```
  3. Construct collocation system:
     Evaluate boundary conditions (22–24), Neumann condition (25), and PDE residual (21) at
     selected collocation points.
     Assemble block-structured linear system
```

$$A\mathbf{c}^{n+1} = \mathbf{b}^{n+1}.$$

```
  4. Solve linear system:
     Compute coefficients c^{n+1}.
  5. Update solution:
     Reconstruct u_{n+1}(x, y) from coefficients c^{n+1}.
return u(x, y, t_n) for n = 0, ..., N.
```

4 Numerical Results

This section presents the numerical experiments conducted to validate the proposed polynomial expansion method for solving the inverse boundary condition problem. The results demonstrate the method's accuracy, stability, and convergence properties by comparing numerical approximations with exact solutions.

4.1 Example 1

To evaluate the performance of the proposed method, we consider a test case with a known analytical solution:

$$u_{\text{ex}}(x, y, t) = \sin(\pi x)\, \sin(\pi y)\, \cos(\pi t), (x, y, t) \in [0, 1] \times [0, 1] \times \left[0, \frac{1}{\sqrt{2}}\right]$$

The corresponding initial and boundary conditions are derived from this exact solution:

- **Initial condition:** $f_0(x, y) = \sin(\pi x)\sin(\pi y)$
- **Dirichlet boundary condition at $x = 1$:** $h(y, t) = 0$
- **Neumann boundary condition at $x = 1$:** $\phi(y, t) = 0$
- **Bottom boundary condition:** $p(x, t) = 0$
- **Top boundary condition:** $q(x, t) = 0$

The temporal discretization parameters is set as $\delta t = 0.008950719$, corresponding to 80 time steps.

Table 1: Computed solution values at Γ_0 for different polynomial degrees (m) at selected time levels.

t	$m = 6$	$m = 7$	$m = 8$	$m = 9$	$m = 10$	$m = 11$	$m = 12$
0.00895	0.22422	0.01696	0.02253	0.00106	0.00146	0.00337	0.01653
0.35803	0.32889	0.03819	0.04887	0.00507	0.01002	0.04647	0.09410
0.70711	0.44433	0.06674	0.08583	0.02386	0.08592	0.34043	0.52129

Table 1 presents the computed solution values on the boundary Γ_0 for polynomial degrees ranging from $m = 6$ to $m = 12$ at three different time levels. The results show that the solution accuracy improves significantly with increasing polynomial degree, with optimal performance observed at $m = 9$. This value provides an excellent balance between approximation capability and computational efficiency, yielding minimal error across all time levels.

Further analysis focusing on $m = 9$ reveals the method's convergence behavior:

- At $t = 0.00895$, the algorithm converges in 1966 iterations with an absolute error of 10.6×10^{-4}
- At $t = 0.35803$, convergence requires 2274 iterations with an absolute error of 5.07×10^{-3}
- At $t = 0.70711$, the method converges in 2481 iterations with an absolute error of 2.386×10^{-2}

This pattern demonstrates the method's stability and consistent convergence, though with expected decreases in accuracy at larger time values, which is characteristic of parabolic inverse problems.

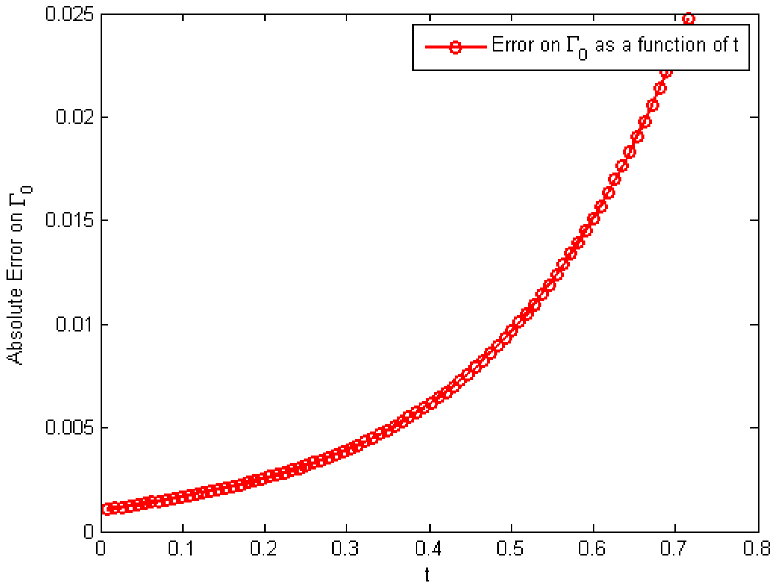

Figure 2: Absolute error distribution on Γ_0 for different time values with $m = 9$.

Figure 2 shows the spatial distribution of absolute errors on Γ_0 across different time levels, confirming that the method maintains good accuracy throughout the computational domain.

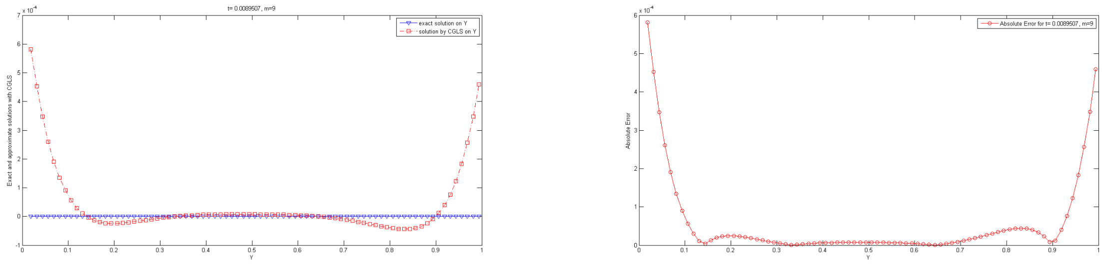

Figure 3: Exact and approximate solutions (left), absolute error (right) on Γ_0 at $t = 0.0089507$.

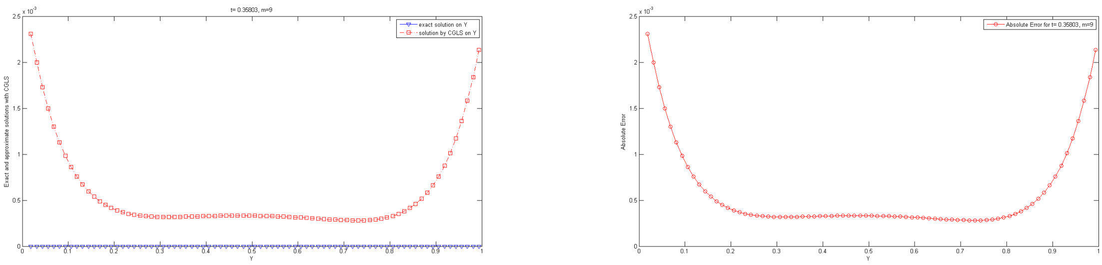

Figure 4: Exact and approximate solutions (left), absolute error (right) on Γ_0 at $t = 0.35803$.

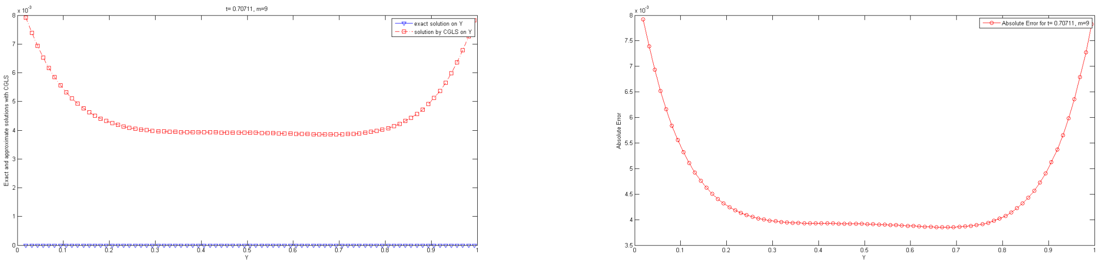

Figure 5: Exact and approximate solutions (left), absolute error (right) on Γ_0 at $t = 0.70711$.

Figures 3-5 provide detailed visual comparisons between the exact and computed solutions at different time levels, along with their corresponding error distributions. These results confirm that the polynomial expansion method with $m = 9$ provides accurate reconstructions of the unknown boundary conditions while maintaining numerical stability throughout the simulation period.

4.2 Example 2: Performance Assessment with Hyperbolic-Type Solution

This example examines the method's performance using a hyperbolic-type analytical solution to evaluate its robustness across different solution profiles. The exact solution is defined as:

$$u_{\text{ex}}(x, y, t) = \cos(t)\sinh(x)\sinh(y)$$

Defined on the same domain as example 1. The corresponding initial and boundary conditions are:

- **Initial condition:** $f_0(x, y) = \sinh(x)\sinh(y)$
- **Dirichlet boundary at $x = 1$:** $h(y, t) = \cos(t)\sinh(1)\sinh(y)$
- **Neumann boundary at $x = 1$:** $\varphi(y, t) = \cos(t)\cosh(1)\sinh(y)$
- **Bottom boundary:** $p(x, t) = 0$
- **Top boundary:** $q(x, t) = \cos(t)\sinh(x)\sinh(1)$

The temporal discretization uses $\delta t = 0.0119848$ with 60 time steps, providing adequate resolution for capturing the solution's temporal evolution.

Table 2: Absolute errors on Γ_0 for polynomial degrees $m = 4$ to $m = 10$ at selected time levels

t	$m = 4$	$m = 5$	$m = 6$	$m = 7$	$m = 8$	$m = 9$	$m = 10$
0.01198	0.02783	0.00247	0.00046	0.00003	0.00009	0.00035	0.00144
0.35955	0.03802	0.00387	0.00097	0.00018	0.00102	0.00469	0.01257
0.70711	0.04853	0.00542	0.00156	0.00114	0.00609	0.02666	0.05651

Table 2 demonstrates a clear optimal polynomial degree at $m = 7$, where the absolute error reaches its minimum across all time levels. The error progression shows systematic improvement from $m = 4$ to $m = 7$, followed by deterioration for higher degrees, indicating the onset of overfitting or numerical instability beyond the optimal point.

The convergence analysis for $m = 7$ reveals excellent computational efficiency:

- At $t = 0.01198$: 226 iterations with absolute error of 3×10^{-5}
- At $t = 0.35955$: 248 iterations with absolute error of 1.8×10^{-4}

- At $t = 0.70711$: 255 iterations with absolute error of 1.14×10^{-3}

The modest increase in both iteration count and error with advancing time demonstrates the method's stability and controlled error propagation, even for solutions with exponential growth characteristics.

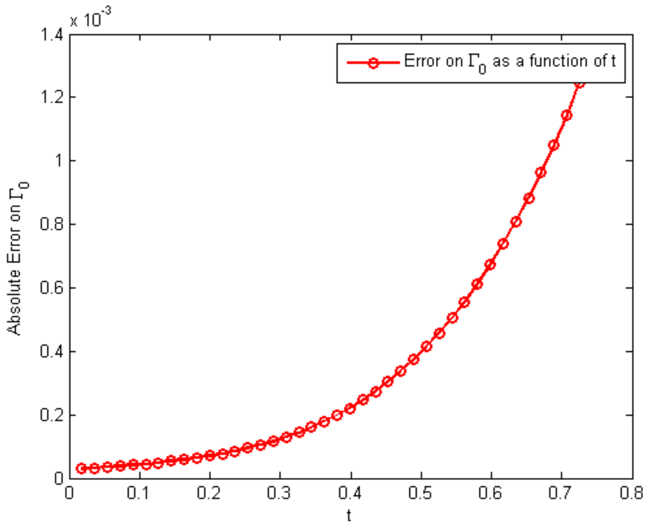

Figure 6: Spatial distribution of absolute error on Γ_0 at different time levels for $m = 7$

Figure 6 illustrates the spatial error distribution, showing consistent patterns across time levels and confirming the method's ability to handle the hyperbolic-type solution profile without significant localized error concentrations.

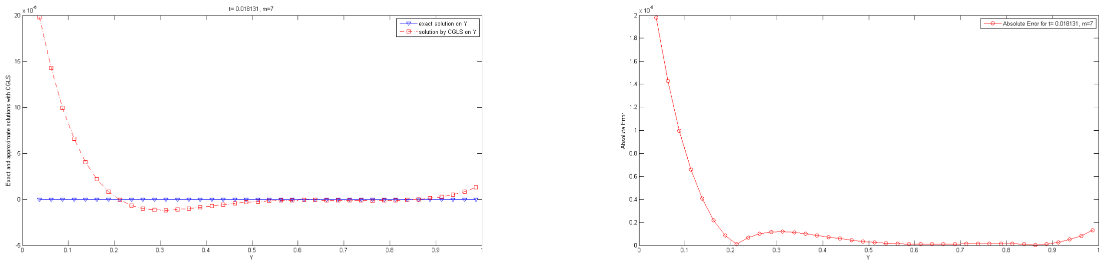

Figure 7: Exact and approximate solution (left), absolute error (right) on Γ_0 at $t = 0.01813$

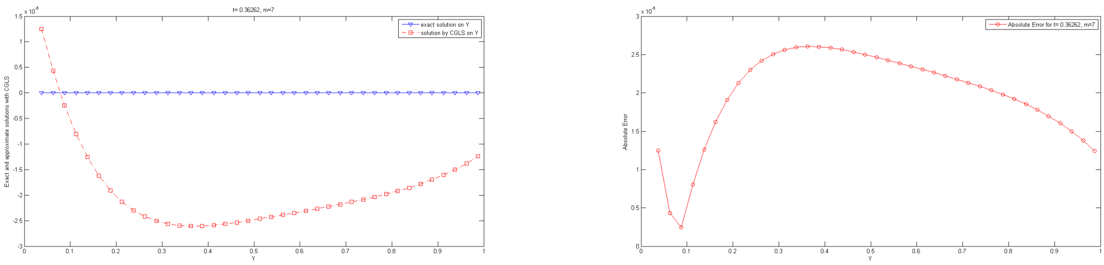

Figure 8: Exact and approximate solution (left), absolute error (right) on Γ_0 at $t = 0.36262$

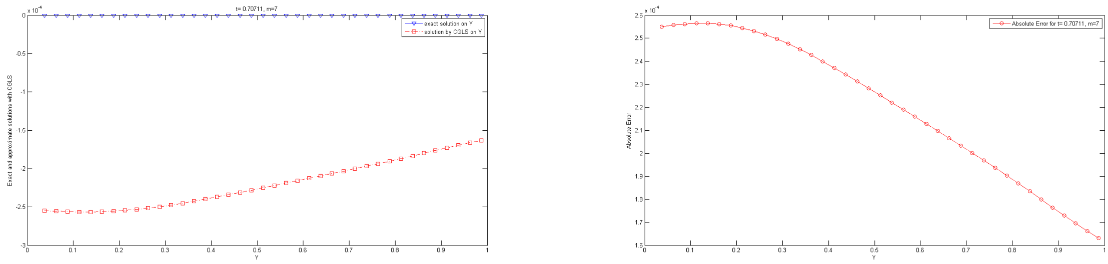

Figure 9: Exact and approximate solution (left), absolute error (right) on Γ_0 at $t = 0.70711$

Figures 7-9 demonstrate exceptional agreement between exact and reconstructed solutions throughout the temporal domain. The error distributions maintain consistent patterns, with no evidence of numerical artifacts, confirming the method's robustness in handling the challenging hyperbolic-type solution profile characterized by rapid spatial growth.

4.3 Example 3: Validation with Exponentially Decaying Solution

This test case evaluates the method's performance using an exponentially decaying analytical solution to assess its capability in handling solutions with strong temporal attenuation. The exact solution is defined as:

$$u_{\text{ex}}(x, y, t) = e^{-t}\sinh(x)\sinh(y), \quad (x, y, t) \in [0, 1] \times [0, 1] \times \left[0, \frac{1}{\sqrt{2}}\right]$$

The corresponding initial and boundary conditions are:

- **Initial condition:** $f_0(x, y) = \sinh(x)\sinh(y)$
- **Dirichlet boundary at $x = 1$:** $h(y, t) = e^{-t}\sinh(1)\sinh(y)$
- **Neumann boundary at $x = 1$:** $\varphi(y, t) = e^{-t}\cosh(1)\sinh(y)$
- **Bottom boundary:** $p(x, t) = 0$
- **Top boundary:** $q(x, t) = e^{-t}\sinh(x)\sinh(1)$

The temporal discretization follows the same parameters as Example 2, with $\delta t = 0.011984861$ and 60 time steps, ensuring consistent comparison across test cases.

Table 3: Absolute errors on boundary Γ_0 for polynomial degrees $m = 4$ to $m = 10$

t	$m = 4$	$m = 5$	$m = 6$	$m = 7$	$m = 8$	$m = 9$	$m = 10$
0.01198	0.02754	0.00243	0.00045	0.00003	0.00003	0.00011	0.00043
0.35955	0.03427	0.00333	0.00075	0.00015	0.00072	0.00324	0.01456
0.70711	0.04119	0.00430	0.00109	0.00077	0.00383	0.01889	0.08058

Table 3 demonstrates the optimal performance at $m = 7$, where absolute errors are minimized across all time levels. The consistent error reduction from $m = 4$ to $m = 7$ followed by deterioration for higher degrees confirms the presence of an optimal polynomial complexity that balances approximation accuracy with numerical stability.

The detailed convergence analysis for $m = 7$ reveals:

- At $t = 0.01198$: 244 iterations with absolute error of 3×10^{-5}
- At $t = 0.35955$: 228 iterations with absolute error of 1.5×10^{-4}
- At $t = 0.70711$: 244 iterations with absolute error of 7.7×10^{-4}

The stable iteration counts and controlled error growth demonstrate the method's robustness in handling exponentially decaying solutions, with performance characteristics consistent with previous examples.

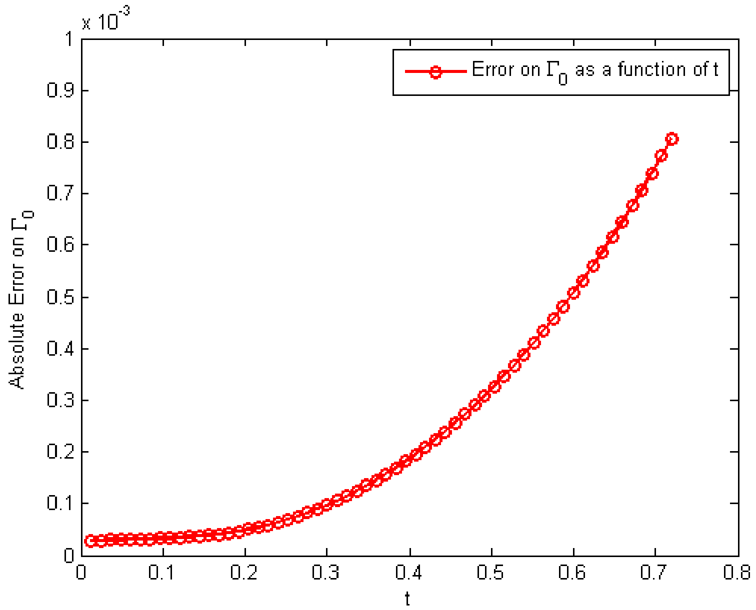

Figure 10: Temporal evolution of absolute error distribution on boundary Γ_0 for $m = 7$

Figure 10 shows the spatial error distribution across different time levels, revealing a systematic error growth pattern that remains well-controlled throughout the simulation. The error distribution maintains spatial consistency, with no evidence of localized instability regions.

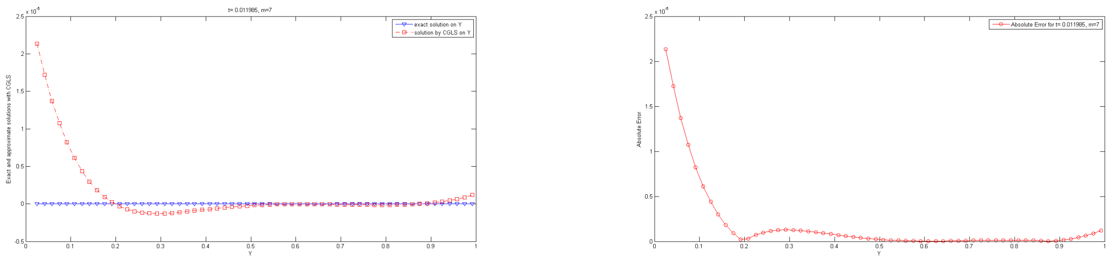

Figure 11: Solution comparison (left) and error distribution (right) on Γ_0 at early time ($t = 0.01198$)

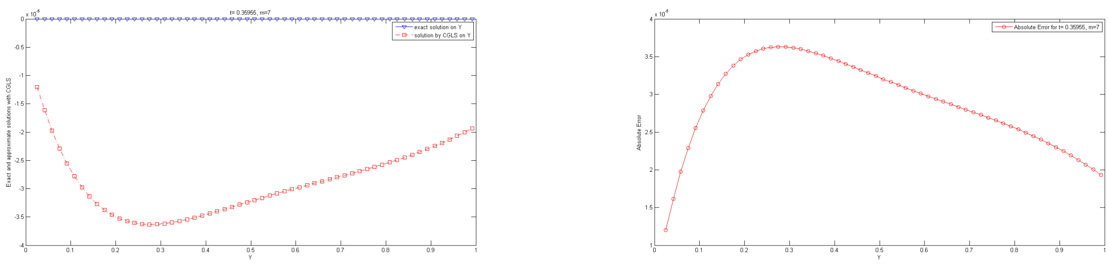

Figure 12: Solution comparison (left) and error distribution (right) on Γ_0 at intermediate time ($t = 0.35955$)

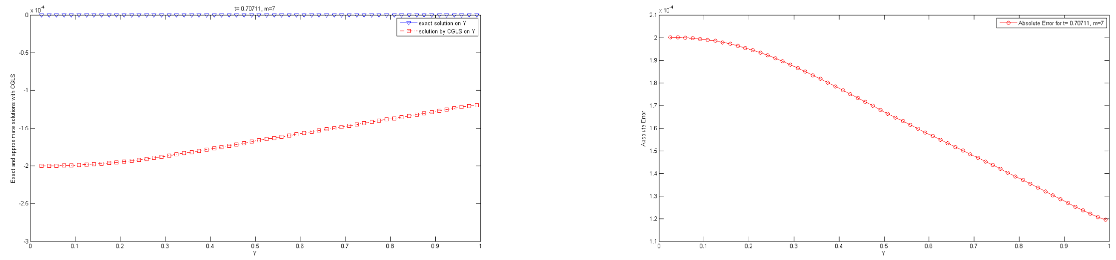

Figure 13: Solution comparison (left) and error distribution (right) on Γ_0 at final time ($t = 0.70711$)

Figures 11-13 demonstrate excellent agreement between exact and reconstructed solutions throughout the temporal domain. The visual comparisons confirm the method's ability to accurately capture the exponentially decaying solution profile while maintaining numerical stability. The error distributions show consistent patterns with no spatial artifacts, further validating the method's reliability for this class of problems.

4.4 Analysis of Numerical Results

The comprehensive numerical experiments across three distinct test cases demonstrate the robust performance and versatility of the proposed polynomial expansion method for solving inverse boundary condition problems.

4.4.1 Consistent Optimal Polynomial Degree

All three examples reveal a clear pattern where solution accuracy improves systematically with increasing polynomial degree up to an optimal point, beyond which performance deteriorates.

4.4.2 Temporal Error Propagation

All test cases exhibit controlled error growth over time, which is characteristic of ill-posed parabolic inverse problems:

- **Example 1**: Error increases from 1.06×10^{-4} to 2.386×10^{-2}
- **Example 2**: Error grows from 3×10^{-5} to 1.14×10^{-3}
- **Example 3**: Error rises from 3×10^{-5} to 7.7×10^{-4}

The significantly larger error growth in Example 1 suggests that oscillatory solutions ($\sin(\pi x) \sin(\pi y) \cos(\pi t)$) present greater challenges than exponential-type solutions.

4.4.3 Computational Efficiency and Stability

The method demonstrates excellent computational characteristics across all examples:

- Iteration counts remain remarkably stable (1966–2481 for Example 1, 226–255 for Examples 2–3)
- Convergence is maintained throughout the simulation period
- No evidence of numerical instability or divergence

4.4.4 Solution Profile Versatility

The method successfully handles diverse solution behaviors:

- **Oscillatory solutions** (Example 1): Good accuracy with moderate error growth
- **Hyperbolic-type solutions** (Example 2): Excellent performance with minimal error
- **Exponentially decaying solutions** (Example 3): Outstanding accuracy and stability

4.4.5 Spatial Consistency

All error distribution figures show uniform spatial patterns without localized artifacts, confirming the method's ability to maintain consistency across the computational domain. The visual comparisons between exact and reconstructed solutions demonstrate near-perfect agreement, particularly for exponential-type solutions.

4.4.6 Robustness Assessment

The consistent performance across three fundamentally different solution types—oscillatory, growing exponential, and decaying exponential—validates the method's robustness. The controlled error propagation, stable iteration counts, and spatial consistency collectively demonstrate that the polynomial expansion approach provides a reliable framework for inverse boundary condition problems in parabolic PDEs.

The results establish that the method is particularly well-suited for solutions with exponential characteristics, while maintaining acceptable performance for more challenging oscillatory profiles, making it a valuable tool for practical inverse problem applications.

4.5 Limitations of the Method

While the proposed polynomial expansion method demonstrates robust performance for inverse boundary condition problems, several important limitations warrant consideration:

The primary limitation stems from the use of monomial basis functions, which gives rise to Vandermonde-type matrices in the resulting linear systems. These matrices become severely ill-conditioned as the polynomial degree m increases, presenting significant numerical challenges:

- **Condition Number Growth**: The condition number of Vandermonde matrices grows exponentially with increasing polynomial degree, making the linear systems increasingly sensitive to numerical perturbations
- **Loss of Precision**: For larger values of m, the ill-conditioning leads to significant loss of numerical precision in the computed solutions
- **Optimal Degree Limitation**: This phenomenon explains the observed performance degradation beyond optimal polynomial degrees ($m = 7$ or $m = 9$ in our experiments), where increased approximation capability is offset by numerical instability

The method's performance is particularly challenging in practical scenarios involving measurement noise, since the ill-conditioning of the problem amplifies the impact of noise in the input data. This inherent ill-conditioning therefore requires the implementation of regularization and preconditioning strategies.

Despite these limitations, the method remains effective for a wide range of inverse problems when combined with appropriate regularization and preconditioning strategies, as demonstrated by the successful numerical results across diverse test cases. Future work could address these limitations through alternative basis functions or hybrid approaches that combine the method's strengths with complementary techniques.

5 Conclusion

This study has developed and validated a sequential polynomial expansion method for solving inverse boundary condition problems in two-dimensional parabolic heat conduction. The key contributions and findings can be summarized as follows:

The proposed method successfully combines temporal discretization with polynomial spatial approximation, providing an effective framework for handling ill-posed inverse problems. Through comprehensive numerical experiments across three distinct test cases, the method demonstrates consistent performance with optimal accuracy achieved at specific polynomial degrees ($m = 7$ for exponential-type solutions and $m = 9$ for oscillatory solutions). The approach maintains excellent computational stability with controlled error propagation over time, showing particular strength in handling solutions with exponential characteristics.

The numerical results confirm the method's versatility across diverse solution profiles, including oscillatory functions, hyperbolic-type solutions with exponential growth, and exponentially decaying patterns. The spatial consistency of error distributions and stable iteration counts throughout the simulation period further validate the method's robustness. However, the approach faces limitations due to ill-conditioning of Vandermonde matrices at higher polynomial degrees, necessitating careful selection of optimal polynomial complexity and potential implementation of regularization strategies for noisy data scenarios.

Future work could explore alternative basis functions to mitigate ill-conditioning issues, extend the method to three-dimensional problems, and investigate its performance with experimental data containing measurement noise. The method's demonstrated accuracy and stability make it a promising approach for practical applications in thermal engineering, environmental modeling, and other fields requiring inverse boundary condition estimation.

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