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A NOVEL STATISTICAL APPROACH TO FUZZY MULTI-OBJECTIVE LINEAR FRACTIONAL PROGRAMMING FOR INDUSTRIAL PRODUCTION PLANNING

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Abstract. This paper presents an innovative statistical approach to solve fuzzy multi-objective linear fractional programming problems (FMOLFPP), where decision factors are represented as trapezoidal fuzzy numbers (TrFNs) without transforming to equivalent crisp problems. Each objective function of the FMOLFPP is solved individually, and optimal values are scalarized by statistical measures: mean, median, and standard deviation. Then, by the proposed statistical technique, the FMOLFPP is transformed into a fuzzy linear fractional programming problem (FLFPP), which is subsequently converted into an equivalent fuzzy linear programming problem (FLPP) using Charnes–Cooper variable transformation. The final solution is obtained using the conventional simplex method.To validate the effectiveness of the proposed approach, a numerical example based on a real-world production planning scenario is provided and a comparison with existing methods shows that our statistical framework is reliable.

Keywords: Multi-objective linear fractional programming, Linear programming problem, Trapezoidal fuzzy numbers, α-cut, Statistical measures.

AMS Subject Classification: 90C29, 90C32, 90C70, 90C30.

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1 Introduction

Linear fractional programming problems (LFPP), also known as ratio optimization, involve the ratio of two linear functions that are subject to limitations. This approach includes much more continuous applications in areas such as healthcare, finance, and industrial planning. In the field of production planning, LFPP is particularly effective when central goals (production efficiency, performance per hour) and profitability (benefit per cost) are of course expressed as conditions. A flexible framework with uncertainty due to fuzzy parameters makes it highly applicable in real manufacturing environments. Over time, various algorithms have been developed to tackle different LFPP challenges. Initially, research in LFPP focused on crisp environments. Charnes & Cooper (1962) introduced the foundational work of variable transformation method that significantly influenced the advancement of multi-objective linear programming problems (MOLPP). Zadeh (1965) introduction of fuzzy set theory, provided a significant breakthrough in addressing such challenges, leading to the development of methodologies for solving MOLFPP under uncertainty. Various researchers have contributed to this domain. Ganesan & Veeramani (2006) focused on fuzzy LPP (FLPP) involving symmetric TrFN, without converting them into crisp LPP. Their approach established fuzzy analogues of key LPP theorems, enabling direct solutions using primal and dual simplex methods. Veeramani & Sumathi (2014)tackled FLFPP using triangular fuzzy numbers(TFN) for costs, resources, and technical coefficients. Their approach transformed the FLFPP into MOFLFPP, which was then solved using the simplex method. Ebrahimnejad & Tavana (2014) developed a technique to translate FLPP into an equivalent crisp LPP, where the coefficients of the goal function and right-hand side values were expressed using TrFN. Deb & De (2014) introduce methods for solving FMOLFPP. In one approach, they use Zimmermann's min-operator, and in another, an additive weighted method—both transforming fuzzy models into crisp equivalents to derive Pareto optimal solutions by aggregating objectives based on their importance. Additionally, they proposes a fuzzy ranking-based transformation of FMOLFPP into a crisp MOLFPP, enabling its solution via the simplex method. Arya et al. (2018) introduced a fuzzy-based branch-and-bound approach to solve MOLFPP. Das et al. (2018) developed a novel method using the Charnes–Cooper transformation to efficiently solve FMOLFPP.

Similarly, Pramy & Islam (2017) introduced the GMIR method for handling fuzzy parameters. Pramanik et al. (2018) proposed a Taylor series for solving MOLFPP with fuzzy parameters, converting the fuzzy problem into a deterministic equivalent.

Further advancements include Loganathan & Ganesan (2021), who addressed MOLFPP solutions by considering all parameters, including objectives and constraints, as fuzzy numbers and solving the problem using the Gauss elimination method. Building upon this, Loganathan & Ganesan (2023) proposed an iterative method for solving FMOLFPP, where the fuzzy optimal solution of the reformulated problem represents the Pareto optimal solution of the original FMOLFPP. Vanaja et al. (2024) developed an exterior penalty method for solving fuzzy nonlinear programming problems using triangular fuzzy numbers, preserving fuzziness without crisp conversion. Borza & Rambely (2021) utilized the max-min technique for solving MOLFPP, and Sahoo et al. (2022) demonstrated that the efficient solution of MOLPP is also applicable to both intuitionistic MOLPP and intuitionistic MOLFPP. By employing Zimmermann's technique with suitable nonlinear membership functions, MOLPP is reduced to an LPP, which can be solved using any standard LPP method. Singh & Yadav (2022) presented a method for scalarizing FMOLFPP, transforming it into LFPP, then applied scalarization techniques, specifically the Gamma connective and minimum bounded sum operator methods to facilitating its solution. More recently, Maharana & Nayak (2024)proposed a Taylor series expansion combined with a weighted sum approach to address FMOLFPP with TrFN.

In recent decades, there has been a rising emphasis on statistical techniques for converting MOLFPP into LFPP and solving them using various methods. As a key contributions Sen (1983), introduced a rural development planning method tailored for MOLPP. Later Sen (2018) refined this approach by incorporating statistical techniques such as correlation, mean, median, optimal average, Sen (2019)followed by scalarizing techniques to improve solution efficiency. Sen (2020) proposed enhanced averaging techniques, refining traditional methods such as the mean, geometric mean, and harmonic mean. Other notable contributions include Sulaiman & Sahil (2010) who integrated mean and median values to improve solution robustness. Nahar & Alim (2015)proposed statistical averaging methods (arithmetic, geometric, harmonic) to convert MOLPP into LPP, demonstrating the superiority of the harmonic method. Nahar & Alim (2017) compared a new geometric average technique with a new arithmetic average technique for solving MOLFPP, Nahar et al. (2023) combined statistical averaging and fuzzy programming methods to improve solution accuracy and robustness in MOLPP. Huma & Modi (2017) introduced harmonic and advanced harmonic average techniques. Yesmin & Alim (2021) proposed advanced transformation techniques for solving MOLFPP, offering simpler solutions for complex scenarios.Mustafa & Sulaiman (2021) refined these approaches by introducing mean deviation techniques, while Abdalla & Abdullah (2022) developed skewness-based methods utilizing Galton's skewness and Pearson's second skewness coefficient to address MOLFPP with statistical depth. Further advancements include the work of Muruganandam & Ambika (2017) who introduced a harmonic mean technique for solving FMOLFPP. Fentaw & Akate (2024) who utilized mean,median values to transform MOLFPP into an equivalent LFPP, demonstrating their applicability in crisp environments. Finally, Singh & Yadav (2022) presented a method for scalarizing FMOLFPP, transforming it into LFPP using scalarization techniques to facilitate easier solutions. Table 1 provides an overview of the literature on MOLFPP and FMOLFPP, highlighting the novelty of recent approaches in comparison to existing research.

Table 1: Analysis of all literature review on MOLFPP and FMOLFPP

Author(s)	**Fuzzy**	**Approach**	**Statistical**
Borza & Rambely (2021)	✓	Max-min technique	×
Pramanik et al. (2018)	✓	Goal programming, Taylor's series	×
Arya et al. (2018)	✓	Branch and bound method	×
Parmy (2018)	✓	Graded Mean Integration, Guzel's proposal	×
Loganathan & Ganesan (2021)	✓	Gauss Elimination method	×
Maharana & Nayak (2024)	✓	Taylor's series expansion	×
Sen (1918)(2019)(2020)	×	Correlation, Scalarizing Techniques, Geometric, Harmonic mean	✓
Nahar & Alim (2017)	×	New Geometric	✓
Akhtar & Modi (2017)	×	Harmonic, Advanced harmonic avg.	✓
Mustafa & Sulaiman (2021)	×	Mean Deviation Techniques	✓
Abdalla & Abdullah (2022)	×	Galton, Pearson Skewness	✓
Getaye & Akate (2024)	×	New Mean Median Technique	✓
Muruganandam & Ambika (2017)	✓	Harmonic Mean using defuzzification	✓
Proposed Technique	✓	Hybrid Mean-Median-Standard deviation Technique and parametric form of TrFN (Without defuzzification)	✓

Research Gap and Motivation

- Researchers have made progress in applying statistical methods to MOLFPP. However, while various scalarization techniques have been explored from (Table 1). Most studies have predominantly focused on crisp parameters while giving little attention to fuzzy parameters, which are more suitable for capturing uncertainty in real-world situations.

- To fill this gap, we suggest a Hybrid Mean-Median-Standard Deviation Technique (HMMSDT) that is more advanced than standard scalarization. HMMSDT creates a statistically relevant single-objective function that better shows the trade-offs in multiple fuzzy fractional objectives by using both central tendency and variability without loss of fuzziness .

- Our contribution includes the formulation of the HMMSDT theorem and a supporting Pareto-optimality lemma, which provide a strong theoretical foundation for solving FMOLFPP under uncertainty. In addition, we compare the proposed method with existing statistical techniques, demonstrating its improved interpretability, computational efficiency, and effectiveness in handling ambiguity.

The paper's outline follows this structure: Section 2 presents the preliminary information following the introduction. Section 3 outlines the mathematical formulation, the proposed method, and the algorithm, along with a numerical example illustrating the methodology. Section 4 presents the results and discussion. Section 5 concludes the study and discusses future work, while part 5 includes the acknowledgments, followed by the references.

2 Preliminaries

To enhance understanding of the fundamental concepts, this section provides a brief overview of essential definitions, notations, and foundational theories relevant to the study.

Definition 1. Zadeh (1965) A fuzzy set $\tilde{\mathcal{A}}$ defined on the real number set $\mathbb{R}$ is characterized by a membership function $\mu_{\tilde{\mathcal{A}}} : \mathbb{R} \to [0,1]$ that satisfies the following conditions:

(i) Normality: There exists $x_0 \in \mathbb{R}$ such that $\mu_{\tilde{\mathcal{A}}}(x_0) = 1$

(ii) Convexity: For all $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0,1]$, the membership function satisfies: $\mu_{\tilde{\mathcal{A}}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\tilde{\mathcal{A}}}(x_1), \mu_{\tilde{\mathcal{A}}}(x_2)\}$.

(iii) Upper semi-continuity: The function $\mu_{\tilde{\mathcal{A}}}(x)$ is upper semi-continuous, meaning that for any sequence $x_n \to x$, $\limsup_{n\to\infty} \mu_{\tilde{\mathcal{A}}}(x_n) \leq \mu_{\tilde{\mathcal{A}}}(x)$.

(iv) Bounded support: The support of $\tilde{\mathcal{A}}$, defined as $\text{supp}(\tilde{\mathcal{A}}) = \{x \in \mathbb{R} \mid \mu_{\tilde{\mathcal{A}}}(x) > 0\}$, is a bounded subset of $\mathbb{R}$.

Definition 2. Deb & De (2014) A fuzzy set $\tilde{\mathcal{A}}$ defined on $\mathbb{R}$ is called a *trapezoidal fuzzy number* (TrFN) and is denoted by $\tilde{\mathcal{A}} = (a,b,c,d)$ if the membership function of $\tilde{\mathcal{A}}$ is given by:

$$\mu_{\tilde{\mathcal{A}}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & \text{if } a \leq x \leq b, \\ 1, & \text{if } b \leq x \leq c, \\ \dfrac{d-x}{d-c}, & \text{if } c \leq x \leq d, \\ 0, & \text{otherwise,} \end{cases}$$

where $a \leq b \leq c \leq d$ and $a,b,c,d \in \mathbb{R}$. When b=c, the TrFN reduces to a triangular fuzzy number (TFN).

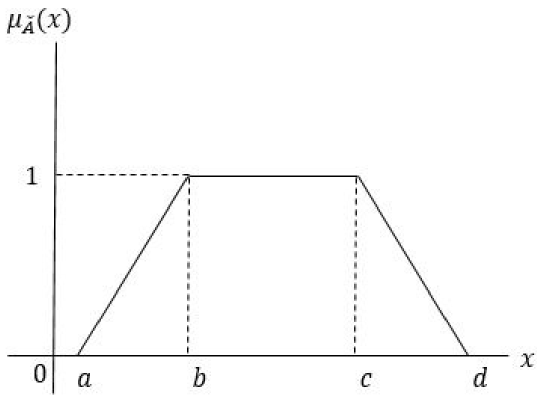

Figure 1: Trapezoidal Fuzzy Number

Definition 3. Dubois & Prade (1980) A TrFN $\tilde{\mathcal{A}}$ can be represented through its α-cut form as a pair of functions $(\underline{a}(\alpha), \overline{a}(\alpha))$, defined for $\alpha \in [0,1]$, and satisfying the following properties:

- $\underline{a}(\alpha)$ is a bounded, non-decreasing, and left-continuous function.
- $\overline{a}(\alpha)$ is a bounded, non-increasing, and left-continuous function.
- For all $\alpha \in [0,1]$, it holds that $\underline{a}(\alpha) \leq \overline{a}(\alpha)$.

This representation captures the interval-valued uncertainty associated with fuzzy numbers at each confidence level α.

Definition 4. Let $\tilde{\mathcal{A}} = (a, b, c, d)$ be a trapezoidal fuzzy number (TrFN), where $a \leq b \leq c \leq d$. Its α-cut representation is given by: $\underline{a}(\alpha) = a + (b-a)\alpha, \quad \overline{a}(\alpha) = d - (d-c)\alpha, \quad \alpha \in [0, 1]$

Alternatively, the fuzzy number $\tilde{\mathcal{A}}$ can be expressed in parametric form as: $\tilde{\mathcal{A}} = (a_0,\ a_L(\alpha),\ a_R(\alpha))$ where $a_0 = \frac{\underline{a}(1)+\overline{a}(1)}{2} = \frac{a_2+a_3}{2}, \quad a_L(\alpha) = a_0 - \underline{a}(\alpha), \quad a_R(\alpha) = \overline{a}(\alpha) - a_0$. Here, a_0 represents the location index, while $a_L(\alpha)$ and $a_R(\alpha)$ denote the left and right fuzziness index functions, respectively. .

Definition 5. Ma et al. (2006) A location index and asymmetric fuzziness index functions were used to provide a unified parametric representation of fuzzy numbers.

- Location indices are combined using classical arithmetic operations.
- Fuzziness indices are combined using lattice operations. Specifically, for any $a^*, b^* \in \mathcal{L}$, $a^* \vee b^* = \max\{a^*, b^*\}, \quad a^* \wedge b^* = \min\{a^*, b^*\}$.

Let $\tilde{\mathcal{A}} = (a_0, a_L, a_R)$ and $\tilde{\mathcal{B}} = (b_0, b_L, b_R)$ be fuzzy numbers in $\mathfrak{F}(\mathbb{R})$, where a_0 and b_0 denote the location indices, and a_L, a_R, b_L, b_R denote the left and right fuzziness indices, respectively. Then, for any arithmetic operation $* \in \{+, -, \times, \div\}$, the resulting fuzzy number is defined as:

$$\tilde{\mathcal{A}} * \tilde{\mathcal{B}} = (a_0 * b_0,\ \max\{a_L, b_L\},\ \max\{a_R, b_R\})\,.$$

In particular:

- Addition: $\tilde{\mathcal{A}} + \tilde{\mathcal{B}} = (a_0 + b_0,\ \max\{a_L, b_L\},\ \max\{a_R, b_R\})$
- Subtraction: $\tilde{\mathcal{A}} - \tilde{\mathcal{B}} = (a_0 - b_0,\ \max\{a_L, b_L\},\ \max\{a_R, b_R\})$
- Multiplication: $\tilde{\mathcal{A}} \times \tilde{\mathcal{B}} = (a_0 \times b_0,\ \max\{a_L, b_L\},\ \max\{a_R, b_R\})$
- Division: $\tilde{\mathcal{A}} \div \tilde{\mathcal{B}} = \left(\frac{a_0}{b_0},\ \max\{a_L, b_L\},\ \max\{a_R, b_R\}\right), \quad$ provided $b_0 \neq 0$.

Definition 6. Chen et al. (2006) The Graded Mean Integration Representation (GMIR) method is a widely used technique for ranking fuzzy numbers by mapping them onto the real number line, facilitating natural ordering. This method is particularly effective for TrFNs. The rank of a TrFN $\tilde{\mathcal{A}} = (a, b, c, d)$, satisfying $a \leq b \leq c \leq d$ is given by

$$R(\tilde{\mathcal{A}}) = \left(\frac{a + 2b + 2c + d}{6}\right)$$

This formula provides a representative real number for the trapezoidal fuzzy number, enabling straightforward comparison and ranking.If b=c then it becomes triangular fuzzy number.

Definition 7. Chen et al. (2006) , Deb & De (2014) Using the GMIR ranking function R, the ordering of two TrFNs $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ is determined as follows:

(i) $\tilde{\mathcal{A}} \succ \tilde{\mathcal{B}}$ if and only if $R(\tilde{\mathcal{A}}) > R(\tilde{\mathcal{B}})$.

(ii) $\tilde{\mathcal{A}} \prec \tilde{\mathcal{B}}$ if and only if $R(\tilde{\mathcal{A}}) < R(\tilde{\mathcal{B}})$.

(iii) $\tilde{\mathcal{A}} \sim \tilde{\mathcal{B}}$ if and only if $R(\tilde{\mathcal{A}}) = R(\tilde{\mathcal{B}})$.

This approach simplifies the process of comparing fuzzy numbers by converting them into their graded mean integration representations.

3 Mathematical Formulation

This section defines the structure of the FMOLFPP, incorporating fuzzy parameters and multiple fractional objectives to provide a clear framework for the proposed solution approach.

3.1 Multi Objective Linear Fractional Programming Problem (MOLFPP)

The general form of a MOLFPP is formulated, where the objective functions are in the form of the ratio of two linear functions. The problem consists of both maximization and minimization objectives in a fractional form.

$$\begin{aligned}
z_{1\max} &= \frac{\mathbf{c}_1^T\mathbf{x}+\alpha_1}{\mathbf{d}_1^T\mathbf{x}+\beta_1},\\
z_{2\max} &= \frac{\mathbf{c}_2^T\mathbf{x}+\alpha_2}{\mathbf{d}_2^T\mathbf{x}+\beta_2},\\
&\vdots\\
z_{r\max} &= \frac{\mathbf{c}_r^T\mathbf{x}+\alpha_r}{\mathbf{d}_r^T\mathbf{x}+\beta_r},\\
z_{r+1\min} &= \frac{\mathbf{c}_{r+1}^T\mathbf{x}+\alpha_{r+1}}{\mathbf{d}_{r+1}^T\mathbf{x}+\beta_{r+1}},\\
&\vdots\\
z_{n\min} &= \frac{\mathbf{c}_n^T\mathbf{x}+\alpha_n}{\mathbf{d}_n^T\mathbf{x}+\beta_n}.
\end{aligned} \tag{1}$$

subject to $A\mathbf{x}\{\leq,\geq,=\}\mathbf{b}$, $\mathbf{x}\geq 0$. Where $\mathbf{x}$ is an n-dimensional column vector of decision variables, A is an $(m\times n)$ matrix of constants, $\mathbf{b}$ is an m-dimensional vector of constraints, $\mathbf{c}_i,\mathbf{d}_i$ (for $i=1,2,\dots,n$) are n-dimensional vectors of constants, α_i,β_i (for $i=1,2,\dots,n$) are scalar constants.

3.2 Fuzzy Multi-Objective Linear Fractional Programming Problem

When a fuzzy number appears in at least one of the objective functions or constraints of an MOLFPP, it is called an FMOLFPP. Furthermore, fuzzy numbers may also be used as variables in an FMOLFPP. The problem is identified as an FMOLFPP if both the variables and the parameters are fuzzy numbers.

Let us consider the following form of MOLFPP with TrFN:

$$\begin{aligned}
\tilde{z}_{1\max} &= \frac{\tilde{\mathbf{c}}_1^T\tilde{\mathbf{x}}+\tilde{\alpha}_1}{\tilde{\mathbf{d}}_1^T\tilde{\mathbf{x}}+\tilde{\beta}_1}\\
\tilde{z}_{2\max} &= \frac{\tilde{\mathbf{c}}_2^T\tilde{\mathbf{x}}+\tilde{\alpha}_2}{\tilde{\mathbf{d}}_2^T\tilde{\mathbf{x}}+\tilde{\beta}_2}\\
&\dots\\
\tilde{z}_{r\max} &= \frac{\tilde{\mathbf{c}}_r^T\tilde{\mathbf{x}}+\tilde{\alpha}_r}{\tilde{\mathbf{d}}_r^T\tilde{\mathbf{x}}+\tilde{\beta}_r}\\
\tilde{z}_{r+1\min} &= \frac{\tilde{\mathbf{c}}_{r+1}^T\tilde{\mathbf{x}}+\tilde{\alpha}_{r+1}}{\tilde{\mathbf{d}}_{r+1}^T\tilde{\mathbf{x}}+\tilde{\beta}_{r+1}}\\
&\dots\\
\tilde{z}_{n\min} &= \frac{\tilde{\mathbf{c}}_n^T\tilde{\mathbf{x}}+\tilde{\alpha}_n}{\tilde{\mathbf{d}}_n^T\tilde{\mathbf{x}}+\tilde{\beta}_n}
\end{aligned} \tag{2}$$

subject to $\tilde{A}\tilde{\mathbf{x}}\{\leq, \geq, =\}\tilde{\mathbf{b}}, \quad \tilde{\mathbf{x}} \geq 0$. Where each fuzzy parameter follows a TrFNs representation: $\tilde{\mathbf{c}}_i = (c_i^L, c_i^M, c_i^N, c_i^U)$, $\tilde{\mathbf{d}}_i = (d_i^L, d_i^M, d_i^N, d_i^U)$, $\tilde{\mathbf{x}} = (x^L, x^M, x^N, x^U)$, $\tilde{\mathbf{x}}$ is an n-dimensional column vector of decision variables, $\tilde{A}$ is an $(m \times n)$ matrix of constants, $\tilde{\mathbf{b}}$ is an m-dimensional vector of constraints, $\tilde{\mathbf{c}}_i, \tilde{\mathbf{d}}_i$ (for $i = 1, 2, \ldots, n$) are n-dimensional vectors of constants, $\tilde{\alpha}_i, \tilde{\beta}_i$ (for $i = 1, 2, \ldots, n$) are scalar constants.

Definition 8. Farhana and Islam (2018), Loganathan & Ganesan (2021),Sama et al. (2024) Let $\tilde{\mathcal{F}}$ denote the set of all fuzzy feasible solutions of the given FMOLFPP(2). A solution $\tilde{x}^f \in \tilde{\mathcal{F}}$ is said to be a *fuzzy Pareto optimal solution* if there does not exist another solution $\tilde{x} \in \tilde{\mathcal{F}}$ such that:
For all maximization objective functions indexed by $i \in M$, $\tilde{z}_i(\tilde{x}) \succcurlyeq \tilde{z}_i(\tilde{x}^f)$ for all $i \in M$, with strict preference for at least one index $j \in M$, $\tilde{z}_j(\tilde{x}) \succ \tilde{z}_j(\tilde{x}^f)$. For all minimization objective $\tilde{z}_k(\tilde{x}) \preccurlyeq \tilde{z}_k(\tilde{x}^f)$ for all $k \in N$, with strict preference for at least one index $l \in N$, $\tilde{z}_l(\tilde{x}) \prec \tilde{z}_l(\tilde{x}^f)$. Here, M and N denote the index sets corresponding to the maximization and minimization objectives, respectively.

3.3 Statistical approach for solving MOLFPP

Several techniques have been proposed for solving MOLFPP. A summary of these existing methods is provided below.

Sen (1983) Suppose a single value is obtained for each objective function in the MOLFPP equation. By optimizing each objective function individually under the given constraints through variable transformation, we derive the following results:

$$\begin{aligned}
z_{1\max} &= \mathfrak{C}_1, \\
z_{2\max} &= \mathfrak{C}_2, \\
&\vdots \\
z_{r\max} &= \mathfrak{C}_r, \\
z_{r+1\min} &= \mathfrak{C}_{r+1}, \\
&\vdots \\
z_{n\min} &= \mathfrak{C}_n,
\end{aligned} \tag{3}$$

where $\mathfrak{C}_i$ $(i = 1, 2, \ldots, n)$ correspond to the optimal results of their respective objective functions, the levels of the decision variables may vary among optimal solutions due to conflicts between objectives.

Sen (2018) proposes a technique that identifies common sets of decision variables across objective functions, which is essential for selecting the best compromise solution for solving MOLFPP. The transformed objective function takes the form:

$$z_{\max} = \frac{\sum_{i=1}^{r} z_i - \sum_{i=r+1}^{n} z_i}{\text{mean or , median or,} |\mathfrak{C}_i|} \tag{4}$$

Where $|\mathfrak{C}_i|$ denotes the individual optimum of the i-th objective function, the mean and median represent the average and median of the optimal values across all individual optima. Utilizing the simplex method, this technique can be executed without compromising the predefined constraints.

Nahar & Alim (2017)introduced the new arithmetic average technique, which integrates objective functions using:

$$z_{\max} = \frac{\sum_{i=1}^{r} z_i - \sum_{i=r+1}^{n} z_i}{m} \tag{5}$$

Additionally, they proposed the new geometric average technique (NGA), which applies:

$$z_{\max} = \frac{\sum_{i=1}^{r} z_i - \sum_{i=r+1}^{n} z_i}{\text{NGA}} \tag{6}$$

where $m = \dfrac{m_1 + m_2}{2}$ and NGA $= \sqrt{m_1 m_2}$ with $m_1 = \min|\mathfrak{C}_i|, \quad i = 1, 2, \ldots, r$ and $m_2 = \min|\mathfrak{C}_i|, \quad i = r+1, r+2, \ldots, n.$

Huma & Modi (2017) proposed the advanced harmonic average technique, expressed as:

$$z_{\max} = \frac{\sum_{i=1}^{r} z_i - \sum_{i=r+1}^{n} z_i}{\text{AHav}} \tag{7}$$

where AHav $= \dfrac{2|m_1||m_2|}{|m_1| + |m_2|}$ with $m_1 = \min|\mathfrak{C}_i|, \quad i = 1, 2, \ldots, r$ and $m_2 = \max|\mathfrak{C}_i|, \quad i = r+1, r+2, \ldots, n.$

3.4 Proposed Technique

The FMOLFPP has been transformed into a scalar optimization problem using mean, median, and standard deviation. Presented below is a summary of the proposed methods for solving FMOLFPP based on statistical measures, the integrated objective function in our proposed approach is structured as follows:

$$z_{\max} = \frac{\sum_{i=1}^{r} z_i - \sum_{i=r+1}^{n} z_i}{\text{HMMSDT}} \tag{8}$$

here ,the Hybrid Mean-Median-Standard Deviation Technique (HMMSDT) serves as a framework for analyzing and optimizing problems using statistical measures.

$$\text{HMMSDT} = \left[\frac{\left(\min(\mu, m, \sigma) + \sqrt{\mu \cdot m \cdot \sigma}\right)}{\mu + m + \sigma} \right] \cdot \frac{1}{N} \tag{9}$$

where

- μ: Mean of $|\mathfrak{C}_i|$, calculated as $\mu = \dfrac{1}{n} \sum_{i=1}^{n} |\mathfrak{C}_i|$,
- m: Median of $|\mathfrak{C}_i|$, the middle value of the sorted sequence $|\mathfrak{C}_i|, |\mathfrak{C}_i|, \ldots, |\mathfrak{C}_i|$,
- σ: Standard deviation of $|\mathfrak{C}_i|$, given by $\sigma = \sqrt{\dfrac{1}{n} \sum_{i=1}^{n} (|\mathfrak{C}_i| - \mu)^2}$,
- N: Total number of objective functions in the optimization problem.
- $|\mathfrak{C}_i|$: Absolute value of the i-th objective function's coefficient.
  This formula uses the minimum of the three values to reduce the influence of outliers, adds the geometric mean $\mu \cdot m \cdot \sigma$ to reflect their interaction, and normalizes by the sum $\mu + m + \sigma$ to maintain proportionality. The result is then scaled by $\frac{1}{N}$ to adjust according to the problem context.

Theorem 1. The Hybrid Mean-Median-Standard Deviation Technique (HMMSDT)is defined as

$$\text{HMMSDT} = \left[\frac{\left(\min(\mu, m, \sigma) + \sqrt{\mu \cdot m \cdot \sigma}\right)}{\mu + m + \sigma} \right] \cdot \frac{1}{N}$$

where the mean (μ), median (m), and standard deviation (σ) are computed for a given dataset. This formula effectively integrates both central tendency and variability into a single scalar measure that is simple yet robust.The HMMSDT possesses the following fundamental properties: boundedness, monotonicity, invariance under scaling, and stability.

Proof:

1. Boundedness : the value of HMMSDT lies within the interval $[0, 1/N]$

$$0 \leq \text{HMMSDT} \leq \frac{1}{N}$$

Lower bound: Assume all differences $|\mathfrak{C}_i| \geq 0$, it follows that:

$$\mu \geq 0, \quad m \geq 0, \quad \sigma \geq 0$$

$$\sqrt{\mu m \sigma} \geq 0, \quad \min(\mu, m, \sigma) \geq 0$$

$$\mu + m + \sigma \geq 0$$

(Strictly positive unless all $|\mathfrak{C}_i| = 0$.)

So, both numerator and denominator of HMMSDT are non-negative. Hence: HMMSDT ≥ 0

If all $|\mathfrak{C}_i| = 0$, then $\mu = m = \sigma = 0$, and HMMSDT becomes:

$$\text{HMMSDT} = \left(\frac{0 + \sqrt{0}}{0}\right) \cdot \frac{1}{N} = 0$$

Upper bound: We now prove:

$$\frac{\min(\mu, m, \sigma) + \sqrt{\mu m \sigma}}{\mu + m + \sigma} \leq 1$$

Let

$$\mathfrak{g} = \min(\mu, m, \sigma), \quad \mathfrak{h} = \sqrt{\mu m \sigma}, \quad \mathfrak{k} = \mu + m + \sigma$$

Then we need to prove:

$$\frac{\mathfrak{g} + \mathfrak{h}}{s} \leq 1 \iff \mathfrak{g} + \mathfrak{h} \leq \mathfrak{k}$$

But since $\mathfrak{g} = \min(\mu, m, \sigma)$, we know:

$$\mathfrak{g} \leq \mu, \quad \mathfrak{g} \leq m, \quad \mathfrak{g} \leq \sigma \Rightarrow \mathfrak{g} \leq \frac{\mu + m + \sigma}{3} = \frac{\mathfrak{k}}{3}$$

Also, by the Arithmetic–Geometric Mean (AM–GM) inequality:

$$\sqrt[3]{\mu m \sigma} \leq \frac{\mu + m + \sigma}{3} = \frac{\mathfrak{k}}{3} \Rightarrow \mathfrak{h} = \sqrt{\mu m \sigma} \leq \mathfrak{k}$$

Hence

$$\mathfrak{g} + \mathfrak{h} \leq \mathfrak{k} \Rightarrow \frac{\mathfrak{g} + \mathfrak{h}}{\mathfrak{k}} \leq 1 \Rightarrow \text{HMMSDT} \leq \frac{1}{N}$$

Thus

$$0 \leq \text{HMMSDT} \leq \frac{1}{N}$$

Thus, HMMSDT is bounded within the interval $\left[0, \frac{1}{N}\right]$, where it aggregates the normalized spread of the differences $|\mathfrak{C}_i|$.

2. Monotonicity: HMMSDT is monotonic with respect to changes in the values of $|\mathfrak{C}_i|$.

Let all $|\mathfrak{C}_i|$ values increase. Then μ, m, σ all increase (properties of the mean, median, and standard deviation)

Claim: If each $|\mathfrak{C}_i|$ is increased (or remains unchanged), then HMMSDT does not decrease.

Let $|\mathfrak{C}_i| \to |\mathfrak{C}_i|' = |\mathfrak{C}_i| + \epsilon_i$, with $\epsilon_i \geq 0 \; \forall i$. Define the perturbed set as:

$$\{|\mathfrak{C}_i'|\}_{i=1}^N = \{|\mathfrak{C}_i| + \epsilon_i\}_{i=1}^N$$

Then,the mean μ' satisfies:

$$\mu' = \frac{1}{N}\sum_{i=1}^{N}(|\mathfrak{C}_i| + \epsilon_i) = \mu + \bar{\epsilon}, \quad \text{where } \bar{\epsilon} = \frac{1}{N}\sum_{i=1}^{N}\epsilon_i \geq 0 \Rightarrow \mu' \geq \mu$$

The median m' also increases or remains the same (monotonicity of order statistics under non-negative additive perturbations): since $|\mathfrak{C}_i|' \geq |\mathfrak{C}_i|$, the sorted list shifts right $\Rightarrow m' \geq m$. The standard deviation σ' satisfies:

$$\sigma'^2 = \frac{1}{N}\sum_{i=1}^{N}(|\mathfrak{C}_i| + \epsilon_i - \mu')^2 \geq \frac{1}{N}\sum_{i=1}^{N}(|\mathfrak{C}_i| - \mu)^2 = \sigma^2 \Rightarrow \sigma' \geq \sigma$$

Now define the function:

$$f(\mu, m, \sigma) = \frac{\min(\mu, m, \sigma) + \sqrt{\mu m \sigma}}{\mu + m + \sigma}$$

We want to show:

$$f(\mu', m', \sigma') \geq f(\mu, m, \sigma)$$

This is not trivial since both numerator and denominator increase. Assume $\mu, m, \sigma > 0$. Let:

$$\mathfrak{g} = \min(\mu, m, \sigma), \quad \mathfrak{h} = \sqrt{\mu m \sigma}, \quad \mathfrak{k} = \mu + m + \sigma$$

Then

$$f = \frac{\mathfrak{g} + \mathfrak{h}}{\mathfrak{k}}$$

Since

$$\mathfrak{g}' = \min(\mu', m', \sigma') \geq \mathfrak{g}, \quad \mathfrak{h}' = \sqrt{\mu' m' \sigma'} \geq \mathfrak{h}, \quad \mathfrak{k}' = \mu' + m' + \sigma' \geq \mathfrak{k}$$

We have

$$\frac{\mathfrak{g}' + \mathfrak{h}'}{\mathfrak{k}'} \geq \frac{\mathfrak{g} + \mathfrak{h}}{\mathfrak{k}} \quad \text{if and only if} \quad (\mathfrak{g}' + \mathfrak{h}')\mathfrak{k} \geq (\mathfrak{g} + \mathfrak{h})\mathfrak{k}'$$

Let

$$\Delta\mathfrak{g} = \mathfrak{g}' - \mathfrak{g} \geq 0, \quad \Delta\mathfrak{h} = \mathfrak{h}' - \mathfrak{h} \geq 0, \quad \Delta\mathfrak{k} = \mathfrak{k}' - \mathfrak{k} \geq 0$$

Then

$$(\mathfrak{g}' + \mathfrak{h}')\mathfrak{k} = (\mathfrak{g} + \Delta\mathfrak{g} + \mathfrak{h} + \Delta\mathfrak{h})\mathfrak{k} = (\mathfrak{g} + \mathfrak{h})\mathfrak{k} + (\Delta\mathfrak{g} + \Delta\mathfrak{h})\mathfrak{k}$$
$$(\mathfrak{g} + \mathfrak{h})\mathfrak{k}' = (\mathfrak{g} + \mathfrak{h})(\mathfrak{k} + \Delta\mathfrak{k}) = (\mathfrak{g} + \mathfrak{h})\mathfrak{k} + (\mathfrak{g} + \mathfrak{h})\Delta\mathfrak{k}$$

So the inequality becomes

$$(\mathfrak{g} + \mathfrak{h})\mathfrak{k} + (\Delta\mathfrak{g} + \Delta\mathfrak{h})\mathfrak{k} \geq (\mathfrak{g} + \mathfrak{h})\mathfrak{k} + (\mathfrak{g} + \mathfrak{h})\Delta\mathfrak{k} \Rightarrow (\Delta\mathfrak{g} + \Delta\mathfrak{h})\mathfrak{k} \geq (\mathfrak{g} + \mathfrak{h})\Delta\mathfrak{k}$$

This holds when

$$\frac{\Delta\mathfrak{g} + \Delta\mathfrak{h}}{\Delta\mathfrak{k}} \geq \frac{\mathfrak{g} + \mathfrak{h}}{\mathfrak{k}}$$

This is reasonable under small perturbations or when the numerator increases faster than the denominator, as is typical when all $|\mathfrak{C}_i|$ increase proportionally.Similarly, if all $|\mathfrak{C}_i|$ decrease, HMMSDT decreases. Hence, HMMSDT is monotonic under increase of data values.

3. Invariance under Scaling: Scale all $|\mathfrak{C}_i|$ values by a positive constant k Then
$\mu \to k\mu$, $m \to km$, $\sigma \to k\sigma$, Substituting in (9) we get

$$\text{HMMSDT} = \frac{\min(k\mu, km, k\sigma) + \sqrt{k\mu \cdot km \cdot k\sigma}}{k\mu + km + k\sigma} \cdot \frac{1}{N}$$

$$\text{HMMSDT} = \frac{k\left[\min(\mu, m, \sigma) + \sqrt{\mu \cdot m \cdot \sigma}\right]}{k(\mu + m + \sigma)} \cdot \frac{1}{N}$$

By canceling k,we get

$$\text{HMMSDT} = \frac{\min(\mu, m, \sigma) + \sqrt{\mu \cdot m \cdot \sigma}}{\mu + m + \sigma} \cdot \frac{1}{N}$$

Thus, HMMSDT is invariant under scaling.

4. Stability: Claim: HMMSDT is a continuous function of the mean μ, median m, and standard deviation σ, which are themselves continuous functions of the data. Let us define the HMMSDT core function

$$f(\mu, m, \sigma) = \frac{\min(\mu, m, \sigma) + \sqrt{\mu m \sigma}}{\mu + m + \sigma}$$

Now let

$$(\mu_n, m_n, \sigma_n) \to (\mu, m, \sigma) \quad \text{as} \quad n \to \infty,$$

i.e., all three parameters change slightly with the data. Then the following limits hold:

- $\min(\mu_n, m_n, \sigma_n) \to \min(\mu, m, \sigma)$,
- $\sqrt{\mu_n m_n \sigma_n} \to \sqrt{\mu m \sigma}$,
- $\mu_n + m_n + \sigma_n \to \mu + m + \sigma$.

Hence, $f(\mu_n, m_n, \sigma_n) \to f(\mu, m, \sigma)$, which implies that f is continuous in all its arguments. Furthermore, consider the component functions defined on the dataset $\{|\mathfrak{C}_i|\}$:

$$\mu = \frac{1}{N}\sum_{i=1}^{N} |\mathfrak{C}_i|, \quad m = \text{median}(|\mathfrak{C}_1|, \ldots, |\mathfrak{C}_N|), \quad \sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(|\mathfrak{C}_i| - \mu)^2}$$

The mean is a linear function, the median is a continuous function of order statistics, and the standard deviation is computed through continuous operations.
Therefore, small perturbations in the input data $\{|\mathfrak{C}_i|\}$ result in small changes in μ, m, and σ. Hence HMMSDT is a composition of continuous functions, and hence is itself a continuous function with respect to the data $\{|\mathfrak{C}_i|\}$. Thus, HMMSDT is stable under minor perturbations of $|\mathfrak{C}_i|$. and satisfies boundedness, monotonicity, invariance under scaling, and stability. Thus, it is a well-defined and reliable aggregation formula for optimization problems.

Lemma 1. If the FMOLFPP is solved using α-cut decomposition and the HMMSDT method applies statistical aggregation without defuzzification, then the resulting fuzzy solution satisfies fuzzy Pareto optimality condition

Proof: The HMMSDT method operates entirely in the fuzzy domain using α-cut representations, without defuzzification. At each α-level, it computes a compromise solution via consistent statistical aggregation. Since the fuzzy structure is preserved, the final solution is not strictly dominated in all objectives by any other feasible fuzzy solution, and hence satisfies fuzzy Pareto optimality condition.

Remark 1. Although HMMSDT does not generate the entire fuzzy Pareto front, it provides a single compromise solution that can be regarded as fuzzy Pareto efficient under suitable conditions such as convexity, consistent aggregation, and monotonic ranking. Hence, it effectively balances objectives while preserving fuzziness.

3.5 Algorithm for Solving FMOLFPP using Statistical approach

We propose an algorithm for solving the FMOLFPP based on statiatical approach as outlined below:

Step 1: Transform the FMOLFPP into parametric form of TrFN's.

Step 2: Convert the FMOLFPP to an equivalent FMOLPP using Charnes and Coopers variable transformation method. Define the transformation: $t_i = \frac{1}{\tilde{d}_i^T \tilde{x} + \tilde{\beta}_i}, \quad y_i = \tilde{x} \cdot t_i$. Then, each FLPP becomes linear $z_i(y_i, w_i) = \tilde{c}_i^T y_i + \tilde{\alpha}_i w_i$ and the corresponding transformed model becomes

maximize $\quad z_i(y_i, w_i) = \tilde{c}_i^T y_i + \tilde{\alpha}_i w_i, \quad i = 1, 2, \ldots, k$ subject to $\tilde{d}_i^T y_i + \tilde{\beta}_i w_i = 1$ $\tilde{A} y_i \lesssim \tilde{b} \cdot w_i$ $y_i \gtrsim \tilde{0}, \quad w_i > 0$

Step 3: Determine the optimal value of each objective function separately while adhering to the same constraints using the simplex method.

Step 4: Extract the mean,median and standard deviation from step 3 and substitute these values in HMMSDT

Step 5: Construct the single objective function using the formula.

$$z_{\max} = \frac{\sum_{i=1}^{r} z_i - \sum_{i=r+1}^{n} z_i}{\text{HMMSDT}}$$

Step 6: Optimize the function by reformulating the FLFPP into an FLPP using the variable transformation method while preserving the original constraints, and then determine the optimal solution.Furthermore, do a sensitivity analysis by varying the α-cut across a series of discrete values, namely α= 0, 0.25, 0.5, 0.75, 1. For each α-level, repeat the optimization procedure and record the resulting HMMSDT and corresponding objective value. This enables the assessment of the solution's stability and resilience over different degrees of fuzziness. The overall solution process for FMOLFPP is illustrated in the flowchart (Figure. 2) below.

3.6 Numerical Illustration

Example 1. An example taken from Sulaiman et al. (2014) and Huma & Modi (2017) is presented to demonstrate the technique for solving the FMOLFPP.

$$\tilde{z}_{1\,\max} = \frac{3x_1 - 2x_2}{x_1 + x_2 + 1}, \quad \tilde{z}_{2\,\max} = \frac{9x_1 + 3x_2}{x_1 + x_2 + 1}, \quad \tilde{z}_{3\,\max} = \frac{3x_1 - 5x_2}{2x_1 + 2x_2 + 2},$$
$$\tilde{z}_{4\,\min} = \frac{-6x_1 + 2x_2}{2x_1 + 2x_2 + 2}, \quad \tilde{z}_{5\,\min} = \frac{-3x_1 - x_2}{x_1 + x_2 + 1}$$

Subject to:

$$\begin{aligned} x_1 + x_2 &\leq 2, \\ 9x_1 + x_2 &\leq 9, \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

The trapezoidal fuzzy numbers (TrFNs) are defined as:

$$\tilde{1} = (0,\ 1,\ 1,\ 2), \quad \tilde{2} = (1,\ 2,\ 2,\ 3), \quad \tilde{3} = (2,\ 3,\ 3,\ 4),$$
$$\tilde{5} = (4,\ 5,\ 5,\ 6), \quad \tilde{6} = (5,\ 6,\ 6,\ 7), \quad \tilde{9} = (8,\ 9,\ 9,\ 10)$$

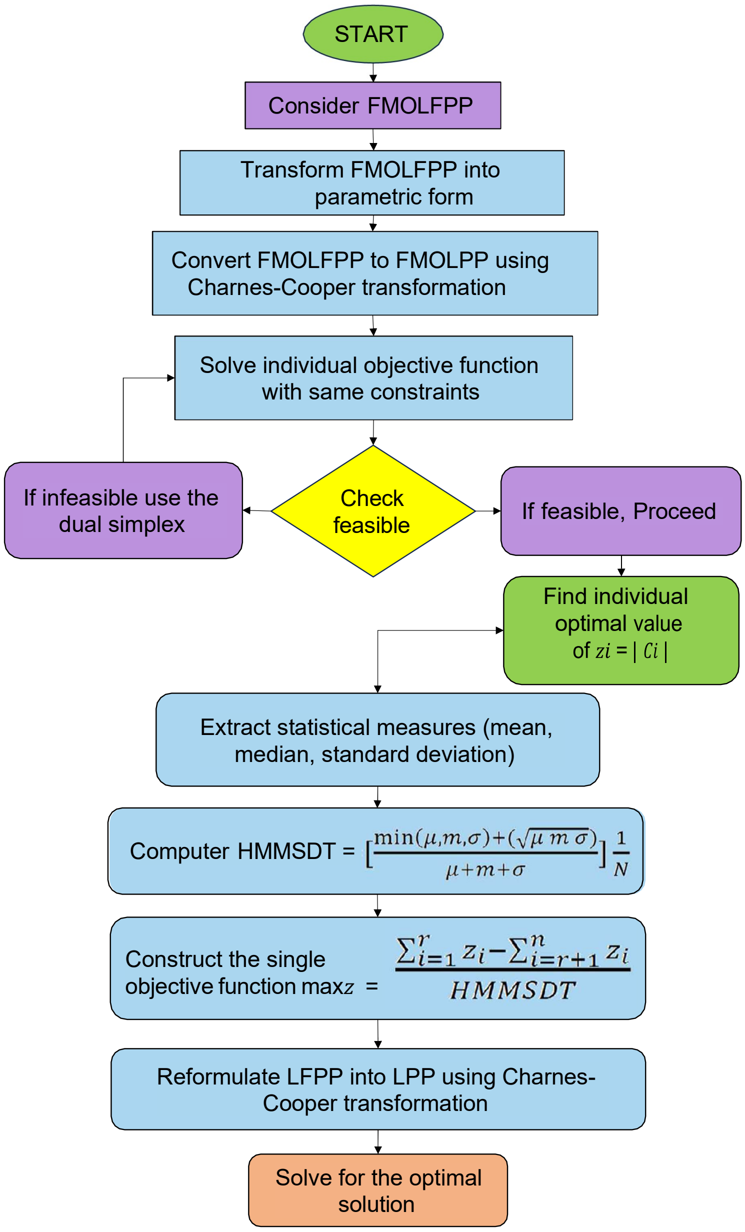

Figure 2: Flowchart of the proposed statistical approach

Solution: The FMOLFPP is expressed using fuzzy coefficients as:

$$\tilde{z}_{1\,\max}(\tilde{\mathbf{x}}) = \frac{(2,\ 3,\ 3,\ 4)x_1 - (1,\ 2,\ 2,\ 3)x_2}{(0,\ 1,\ 1,\ 2)x_1 + (0,\ 1,\ 1,\ 2)x_2 + (0,\ 1,\ 1,\ 2)},$$

$$\tilde{z}_{2\,\max}(\tilde{\mathbf{x}}) = \frac{(8,\ 9,\ 9,\ 10)x_1 + (2,\ 3,\ 3,\ 4)x_2}{(0,\ 1,\ 1,\ 2)x_1 + (0,\ 1,\ 1,\ 2)x_2 + (0,\ 1,\ 1,\ 2)},$$

$$\tilde{z}_{3\,\max}(\tilde{\mathbf{x}}) = \frac{(2,\ 3,\ 3,\ 4)x_1 - (4,\ 5,\ 5,\ 6)x_2}{(1,\ 2,\ 2,\ 3)x_1 + (1,\ 2,\ 2,\ 3)x_2 + (1,\ 2,\ 2,\ 3)},$$

$$\tilde{z}_{4\,\min}(\tilde{\mathbf{x}}) = \frac{-(5,\ 6,\ 6,\ 7)x_1 + (1,\ 2,\ 2,\ 3)x_2}{(1,\ 2,\ 2,\ 3)x_1 + (1,\ 2,\ 2,\ 3)x_2 + (1,\ 2,\ 2,\ 3)},$$

$$\tilde{z}_{5\,\min}(\tilde{\mathbf{x}}) = \frac{-(2,\ 3,\ 3,\ 4)x_1 - (0,\ 1,\ 1,\ 2)x_2}{(0,\ 1,\ 1,\ 2)x_1 + (0,\ 1,\ 1,\ 2)x_2 + (0,\ 1,\ 1,\ 2)}$$

Subject to

$$(0,\ 1,\ 1,\ 2)x_1 + (0,\ 1,\ 1,\ 2)x_2 \le (1, 2, 2, 3)$$
$$(8,\ 9,\ 9,\ 10)x_1 + (0,\ 1,\ 1,\ 2)x_2 \le (8,\ 9,\ 9,\ 10)$$
$$x_1,\ x_2 \ge 0$$

Step 1: Using α - cut and parametric form of TrFN ,we get objective functions as

$$\tilde{z}_{1\,\max}(\tilde{\mathbf{x}}, \alpha) = \frac{(3,\ 1{-}\alpha,\ 1{-}\alpha)x_1 - (2,\ 1{-}\alpha,\ 1{-}\alpha)x_2}{(1,\ 1{-}\alpha,\ 1{-}\alpha)x_1 + (1,\ 1{-}\alpha,\ 1{-}\alpha)x_2 + (1,\ 1{-}\alpha,\ 1{-}\alpha)}$$

$$\tilde{z}_{2\,\max}(\tilde{\mathbf{x}}, \alpha) = \frac{(9,\ 1{-}\alpha,\ 1{-}\alpha)x_1 + (3,\ 1{-}\alpha,\ 1{-}\alpha)x_2}{(1,\ 1{-}\alpha,\ 1{-}\alpha)x_1 + (1,\ 1{-}\alpha,\ 1{-}\alpha)x_2 + (1,\ 1{-}\alpha,\ 1{-}\alpha)}$$

$$\tilde{z}_{3\,\max}(\tilde{\mathbf{x}}, \alpha) = \frac{(3,\ 1{-}\alpha,\ 1{-}\alpha)x_1 - (5,\ 1{-}\alpha,\ 1{-}\alpha)x_2}{(2,\ 1{-}\alpha,\ 1{-}\alpha)x_1 + (2,\ 1{-}\alpha,\ 1{-}\alpha)x_2 + (2,\ 1{-}\alpha,\ 1{-}\alpha)}$$

$$\tilde{z}_{4\,\min}(\tilde{\mathbf{x}}, \alpha) = \frac{(-6,\ 1{-}\alpha,\ 1{-}\alpha)x_1 + (2,\ 1{-}\alpha,\ 1{-}\alpha)x_2}{(2,\ 1{-}\alpha,\ 1{-}\alpha)x_1 + (2,\ 1{-}\alpha,\ 1{-}\alpha)x_2 + (2,\ 1{-}\alpha,\ 1{-}\alpha)}$$

$$\tilde{z}_{5\,\min}(\tilde{\mathbf{x}}, \alpha) = \frac{(-3,\ 1{-}\alpha,\ 1{-}\alpha)x_1 - (1,\ 1{-}\alpha,\ 1{-}\alpha)x_2}{(1,\ 1{-}\alpha,\ 1{-}\alpha)x_1 + (1,\ 1{-}\alpha,\ 1{-}\alpha)x_2 + (1,\ 1{-}\alpha,\ 1{-}\alpha)}$$

Subject to

$$(1,\ 1{-}\alpha,\ 1{-}\alpha)x_1 + (1,\ 1{-}\alpha,\ 1{-}\alpha)x_2 \le (2,\ 1{-}\alpha,\ 1{-}\alpha)$$
$$(9,\ 1{-}\alpha,\ 1{-}\alpha)x_1 + (1,\ 1{-}\alpha,\ 1{-}\alpha)x_2 \le (9,\ 1{-}\alpha,\ 1{-}\alpha)$$
$$x_1,\ x_2 \ge 0$$

Step 2: Using the Charnes–Cooper variable transformation method, we convert the FMOLFPP into an FLPP. Solving it using the simplex method yields the results shown in Table 2. For

$$\tilde{z}_{1\,\max}(\tilde{y}, \alpha) = (3,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{y}_1 - (2,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{y}_2$$

Subject to

$$(1,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{y}_1 + (1,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{y}_2 + (1,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{w} = 1$$
$$(1,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{y}_1 + (1,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{y}_2 \le (2,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{w}$$
$$(9,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{y}_1 + (1,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{y}_2 \le (9,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{w}$$
$$\tilde{y}_1,\ \tilde{y}_2 \ge 0, \quad \tilde{w} > 0$$

we get

$$\tilde{y}_1 = (0.5,\ 1-\alpha,\ 1-\alpha), \quad \tilde{y}_2 = (0,\ 1-\alpha,\ 1-\alpha), \quad \tilde{w} = (0.5,\ 1-\alpha,\ 1-\alpha)$$

Next, we find the values of $\tilde{x}_1, \tilde{x}_2$ using the transformation $\tilde{x}_i = \dfrac{\tilde{y}_i}{\tilde{w}}$
$\tilde{x}_1 = (1,\ 1{-}\alpha,\ 1{-}\alpha)$; $\tilde{x}_2 = (0,\ 1{-}\alpha,\ 1{-}\alpha)$ and $\tilde{z}_{1\,\max}(\tilde{\mathbf{x}}, \alpha) = (1.5,\ 1{-}\alpha,\ 1{-}\alpha)$

For

$$\tilde{z}_{2\,\max}(\tilde{y}, \alpha) = (9,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{y}_1 + (3,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{y}_2$$

Subject to

$$(1,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{y}_1 + (1,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{y}_2 + (1,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{w} = 1$$
$$(1,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{y}_1 + (1,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{y}_2 \le (2,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{w}$$
$$(9,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{y}_1 + (1,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{y}_2 \le (9,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{w}$$

$$\tilde{y}_1,\ \tilde{y}_2 \geq 0, \quad \tilde{w} > 0$$

we get

$$\tilde{y}_1 = (0.5,\ 1-\alpha,\ 1-\alpha), \quad \tilde{y}_2 = (0,\ 1-\alpha,\ 1-\alpha), \quad \tilde{w} = (0.5,\ 1-\alpha,\ 1-\alpha)$$

Next, we find the values of $\tilde{x}_1, \tilde{x}_2$ using the transformation $\tilde{x}_i = \dfrac{\tilde{y}_i}{\tilde{w}}$
$\tilde{x}_1 = (1,\ 1-\alpha,\ 1-\alpha)$; $\tilde{x}_2 = (0,\ 1-\alpha,\ 1-\alpha)$ and $\tilde{z}_{2\max}(\tilde{\mathbf{x}}, \alpha) = (4.5,\ 1-\alpha,\ 1-\alpha)$

For

$$\tilde{z}_{3\max}(\tilde{y}, \alpha) = (3,\ 1-\alpha,\ 1-\alpha)\tilde{y}_1 - (5,\ 1-\alpha,\ 1-\alpha)\tilde{y}_2$$

Subject to

$$(2,\ 1-\alpha,\ 1-\alpha)\tilde{y}_1 + (2,\ 1-\alpha,\ 1-\alpha)\tilde{y}_2 + (2,\ 1-\alpha,\ 1-\alpha)\tilde{w} = 1$$

$$(1,\ 1-\alpha,\ 1-\alpha)\tilde{y}_1 + (1,\ 1-\alpha,\ 1-\alpha)\tilde{y}_2 \leq (2,\ 1-\alpha,\ 1-\alpha)\tilde{w}$$

$$(9,\ 1-\alpha,\ 1-\alpha)\tilde{y}_1 + (1,\ 1-\alpha,\ 1-\alpha)\tilde{y}_2 \leq (9,\ 1-\alpha,\ 1-\alpha)\tilde{w}$$

$$\tilde{y}_1,\ \tilde{y}_2 \geq 0, \quad \tilde{w} > 0$$

we get

$$\tilde{y}_1 = (0.5,\ 1-\alpha,\ 1-\alpha), \quad \tilde{y}_2 = (0,\ 1-\alpha,\ 1-\alpha), \quad \tilde{w} = (0.5,\ 1-\alpha,\ 1-\alpha)$$

Next, we find the values of $\tilde{x}_1, \tilde{x}_2$ using the transformation $\tilde{x}_i = \dfrac{\tilde{y}_i}{\tilde{w}}$
$\tilde{x}_1 = (1,\ 1-\alpha,\ 1-\alpha)$; $\tilde{x}_2 = (0,\ 1-\alpha,\ 1-\alpha)$ and $\tilde{z}_{3\max}(\tilde{\mathbf{x}}, \alpha) = (0.75,\ 1-\alpha,\ 1-\alpha)$

For

$$\tilde{z}_{4\min}(\tilde{y}, \alpha) = (-6,\ 1-\alpha,\ 1-\alpha)\tilde{y}_1 + (2,\ 1-\alpha,\ 1-\alpha)\tilde{y}_2$$

Subject to

$$(2,\ 1-\alpha,\ 1-\alpha)\tilde{y}_1 + (2,\ 1-\alpha,\ 1-\alpha)\tilde{y}_2 + (2,\ 1-\alpha,\ 1-\alpha)\tilde{w} = 1$$

$$(1,\ 1-\alpha,\ 1-\alpha)\tilde{y}_1 + (1,\ 1-\alpha,\ 1-\alpha)\tilde{y}_2 \leq (2,\ 1-\alpha,\ 1-\alpha)\tilde{w}$$

$$(9,\ 1-\alpha,\ 1-\alpha)\tilde{y}_1 + (1,\ 1-\alpha,\ 1-\alpha)\tilde{y}_2 \leq (9,\ 1-\alpha,\ 1-\alpha)\tilde{w}$$

$$\tilde{y}_1,\ \tilde{y}_2 \geq 0, \quad \tilde{w} > 0$$

we get

$$\tilde{y}_1 = (0.5,\ 1-\alpha,\ 1-\alpha), \quad \tilde{y}_2 = (0,\ 1-\alpha,\ 1-\alpha), \quad \tilde{w} = (0.5,\ 1-\alpha,\ 1-\alpha)$$

Next, we find the values of $\tilde{x}_1, \tilde{x}_2$ using the transformation $\tilde{x}_i = \dfrac{\tilde{y}_i}{\tilde{w}}$
$\tilde{x}_1 = (1,\ 1-\alpha,\ 1-\alpha)$; $\tilde{x}_2 = (0,\ 1-\alpha,\ 1-\alpha)$ and $\tilde{z}_{4\max}(\tilde{\mathbf{x}}, \alpha) = (-1.5,\ 1-\alpha,\ 1-\alpha)$

For

$$\tilde{z}_{5\min}(\tilde{y}, \alpha) = (-3,\ 1-\alpha,\ 1-\alpha)\tilde{y}_1 - (1,\ 1-\alpha,\ 1-\alpha)\tilde{y}_2$$

Subject to

$$(1,\ 1-\alpha,\ 1-\alpha)\tilde{y}_1 + (1,\ 1-\alpha,\ 1-\alpha)\tilde{y}_2 + (1,\ 1-\alpha,\ 1-\alpha)\tilde{w} = 1$$

$$(1,\ 1-\alpha,\ 1-\alpha)\tilde{y}_1 + (1,\ 1-\alpha,\ 1-\alpha)\tilde{y}_2 \leq (2,\ 1-\alpha,\ 1-\alpha)\tilde{w}$$

$$(9,\ 1-\alpha,\ 1-\alpha)\tilde{y}_1 + (1,\ 1-\alpha,\ 1-\alpha)\tilde{y}_2 \leq (9,\ 1-\alpha,\ 1-\alpha)\tilde{w}$$

$$\tilde{y}_1,\ \tilde{y}_2 \geq 0, \quad \tilde{w} > 0$$

Table 2: Results of example 1 after optimizing each objective function separately.

Objective Function	$z_i = \lvert\mathfrak{C}_i\rvert$	μ	m	σ
1	$(1.5,\ 1{-}\alpha,\ 1{-}\alpha)$	$(1.95,\ 1{-}\alpha,\ 1{-}\alpha)$	$(1.5,\ 1{-}\alpha,\ 1{-}\alpha)$	$(1.462,\ 1{-}\alpha,\ 1{-}\alpha)$
2	$(4.5,\ 1{-}\alpha,\ 1{-}\alpha)$			
3	$(0.75,\ 1{-}\alpha,\ 1{-}\alpha)$			
4	$(1.5,\ 1{-}\alpha,\ 1{-}\alpha)$			
5	$(1.5,\ 1{-}\alpha,\ 1{-}\alpha)$			

we get

$$\tilde{y}_1 = (0.5,\ 1-\alpha,\ 1-\alpha), \quad \tilde{y}_2 = (0,\ 1-\alpha,\ 1-\alpha), \quad \tilde{w} = (0.5,\ 1-\alpha,\ 1-\alpha)$$

Next, we find the values of $\tilde{x}_1, \tilde{x}_2$ using the transformation $\tilde{x}_i = \dfrac{\tilde{y}_i}{\tilde{w}}$
$\tilde{x}_1 = (1,\ 1{-}\alpha,\ 1{-}\alpha)$; $\tilde{x}_2 = (0,\ 1{-}\alpha,\ 1{-}\alpha)$ and $\tilde{z}_{5\,\max}(\tilde{\mathbf{x}}, \alpha) = (-1.5,\ 1{-}\alpha,\ 1{-}\alpha)$

Step 3: To construct a single objective function, we compute the HMMSDT value as follows:

$$\text{HMMSDT} = \frac{1}{5}\left[\frac{\begin{array}{c}\min\left((1.95,\ 1{-}\alpha,\ 1{-}\alpha),\ (1.5,\ 1{-}\alpha,\ 1{-}\alpha),\ (1.462,\ 1{-}\alpha,\ 1{-}\alpha)\right)\\ +\sqrt{(1.95,\ 1{-}\alpha,\ 1{-}\alpha)\cdot(1.5,\ 1{-}\alpha,\ 1{-}\alpha)\cdot(1.462,\ 1{-}\alpha,\ 1{-}\alpha)}\end{array}}{(1.95,\ 1{-}\alpha,\ 1{-}\alpha)+(1.5,\ 1{-}\alpha,\ 1{-}\alpha)+(1.462,\ 1{-}\alpha,\ 1{-}\alpha)}\right] = (0.1437,\ 1{-}\alpha,\ 1{-}\alpha)$$

Then, the combined objective function is:

$$\tilde{z}_{\max} = \frac{1}{(0.1437,\ 1{-}\alpha,\ 1{-}\alpha)}\left[\tilde{z}_{1\,\max}(\tilde{\mathbf{x}},\alpha) + \tilde{z}_{2\,\max}(\tilde{\mathbf{x}},\alpha) + \tilde{z}_{3\,\max}(\tilde{\mathbf{x}},\alpha) - \tilde{z}_{4\,\min}(\tilde{\mathbf{x}},\alpha) - \tilde{z}_{5\,\min}(\tilde{\mathbf{x}},\alpha)\right]$$

$$\tilde{z}_{\max} = \frac{1}{(0.1437,\ 1{-}\alpha,\ 1{-}\alpha)}\left[\frac{(19.5,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{x}_1 + (1.5,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{x}_2}{(1,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{x}_1 + (1,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{x}_2 + (1,\ 1{-}\alpha,\ 1{-}\alpha)}\right]$$

Subject to

$$(1,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{x}_1 + (1,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{x}_2 \le (2,\ 1{-}\alpha,\ 1{-}\alpha)$$

$$(9,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{x}_1 + (1,\ 1{-}\alpha,\ 1{-}\alpha)\tilde{x}_2 \le (9,\ 1{-}\alpha,\ 1{-}\alpha)$$

$$\tilde{x}_1, \tilde{x}_2 \ge 0$$

Converting the LFPP into an LPP using the variable transformation,
we get

$$\tilde{y}_1 = (0.5,\ 1-\alpha,\ 1-\alpha), \quad \tilde{y}_2 = (0,\ 1-\alpha,\ 1-\alpha), \quad \tilde{w} = (0.5,\ 1-\alpha,\ 1-\alpha)$$

By solving , we obtain the optimal solution $\tilde{x}_1 = (1,\ 1-\alpha,\ 1-\alpha)$, $\quad \tilde{x}_2 = (0,\ 1-\alpha,\ 1-\alpha)$, $\tilde{z}_{\max} = (67.8496,\ 1-\alpha,\ 1-\alpha)$

Converted to standard fuzzy form with left and right spreads as $\tilde{x}_1 = (0+\alpha,\ 1,\ 2-\alpha)$, $\tilde{x}_2 = (-1+\alpha,\ 0,\ 1-\alpha)$, $\tilde{z}_{\max} = (66.8496+\alpha,\ 67.8496,\ 68.8496-\alpha)$
For different values of $\alpha \in [0,1]$, the proposed method yields various fuzzy optimal solutions for the given FMOLFPP.

Table 3: Fuzzy Values of $\tilde{x}_1$, $\tilde{x}_2$, and $\tilde{z}_{\max}$ for different α

α	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{z}_{\max}$
0	(0, 1, 2)	(−1, 0, 1)	(66.8496, 67.8496, 68.8496)
0.25	(0.25, 1, 1.75)	(−0.75, 0, 0.75)	(67.0996, 67.8496, 68.5996)
0.5	(0.5, 1, 1.5)	(−0.5, 0, 0.5)	(67.3496, 67.8496, 68.3496)
0.75	(0.75, 1, 1.25)	(−0.25, 0, 0.25)	(67.5996, 67.8496, 68.0996)
1	(1, 1, 1)	(0, 0, 0)	(67.8496, 67.8496, 67.8496)

Step 4: Now, solving the problem using different existing techniques, we obtain the optimal solution for same optimal values The corresponding maximum objective values are as follows

1. Chandra Sen Mean Technique: $z_{\max} = (5,\ 1-\alpha,\ 1-\alpha)$
2. Chandra Sen Median Technique: $z_{\max} = (6.5,\ 1-\alpha,\ 1-\alpha)$
3. New Geometric Average Technique: $z_{\max} = (9.1924,\ 1-\alpha,\ 1-\alpha)$
4. New Arithmetic Average Technique: $z_{\max} = (8.6666,\ 1-\alpha,\ 1-\alpha)$
5. Advanced Harmonic Average Technique: $z_{\max} = (9.75,\ 1-\alpha,\ 1-\alpha)$

Table 4: Fuzzy Values of $z_{\max}$ for Different Techniques at Various α-Levels (Part 1)

α	Mean Technique	Median Technique	Geometric Avg
0	(4, 5, 6)	(5.5, 6.5, 7.5)	(8.1924, 9.1924, 10.1924)
0.25	(4.25, 5, 5.75)	(5.75, 6.5, 7.25)	(8.4424, 9.1924, 9.9424)
0.5	(4.5, 5, 5.5)	(6, 6.5, 7)	(8.6924, 9.1924, 9.6924)
0.75	(4.75, 5, 5.25)	(6.25, 6.5, 6.75)	(8.9424, 9.1924, 9.4424)
1	(5, 5, 5)	(6.5, 6.5, 6.5)	(9.1924, 9.1924, 9.1924)

Table 5: Fuzzy Values of $z_{\max}$ for Different Techniques at Various α-Levels (Part 2)

α	Arithmetic Avg	Harmonic Avg
0	(7.6666, 8.6666, 9.6666)	(8.75, 9.75, 10.75)
0.25	(7.9166, 8.6666, 9.4166)	(9, 9.75, 10.5)
0.5	(8.1666, 8.6666, 9.1666)	(9.25, 9.75, 10.25)
0.75	(8.4166, 8.6666, 8.9166)	(9.5, 9.75, 10)
1	(8.6666, 8.6666, 8.6666)	(9.75, 9.75, 9.75)

Example 2. Real life application to demonstrate the technique for solving the FMOLFPP from Maharana & Nayak (2024)

Three products, X, Y, and Z, are produced by a company at rates of 5, 8, and 9 units per hour, respectively. The time required to manufacture one unit of products X,Y, and Z is 4, 7, and 8 minutes. Additionally, the production includes a fixed batch of 7 units, each requiring 6 minutes to complete.The production costs per unit for X, Y, and Z are Rs.4, Rs.7, and Rs.8, respectively, while their corresponding profits per unit are Rs.6, Rs.8, and Rs.6.5. Furthermore, the company incurs a fixed profit of Rs.3 and a fixed manufacturing cost of Rs.6 throughout the process to support business growth. The company aims to optimize two performance criteria simultaneously:
Production efficiency: the ratio of total output to total production time.
Profitability: the ratio of total profit to total cost.
The raw material requirements per unit are 5.5, 3.5, and 5.75 tonnes for products X, Y, and Z respectively, with a total available supply capped at 6.75 tonnes. Due to uncertainties in labor availability, raw material supply, and other operational factors, parameters such as costs, profits, raw material usage, and production rates are represented as trapezoidal fuzzy numbers to appropriately capture the imprecision and vagueness inherent in the decision-making environment.

$$\tilde{z}_{1\max}(\tilde{\mathbf{x}}) = \frac{\tilde{5}\tilde{x}_1 + \tilde{8}\tilde{x}_2 + \tilde{9}\tilde{x}_3 + \tilde{7}}{\tilde{4}\tilde{x}_1 + \tilde{7}\tilde{x}_2 + \tilde{8}\tilde{x}_3 + \tilde{6}} \quad \text{(Production efficiency)}$$

$$\tilde{z}_{2\max}(\tilde{\mathbf{x}}) = \frac{\tilde{6}\tilde{x}_1 + \tilde{8}\tilde{x}_2 + \widetilde{6.5}\tilde{x}_3 + \tilde{3}}{\tilde{4}\tilde{x}_1 + \tilde{7}\tilde{x}_2 + \tilde{8}\tilde{x}_3 + \tilde{6}} \quad \text{(Profitability)}$$

$$\text{subject to:} \quad \widetilde{5.5}\tilde{x}_1 + \widetilde{3.5}\tilde{x}_2 + \widetilde{5.75}\tilde{x}_3 \preceq \widetilde{6.75}, \quad x_1, x_2, x_3 \geq 0.$$

where

$\tilde{3} = (2,3,3,4)$, $\tilde{3.5} = (2.5,3.5,3.5,4.5)$, $\tilde{4} = (3,4,4,5)$, $\tilde{5} = (4,5,5,6)$,
$\tilde{5.5} = (4.5,5.5,5.5,6.5)$, $\tilde{5.75} = (4.75,5.75,5.75,6.75)$, $\tilde{6} = (5,6,6,7)$, $\tilde{6.5} = (5.5,6.5,6.5,7.5)$,
$\tilde{6.75} = (5.75,6.75,6.75,7.75)$, $\tilde{7} = (6,7,7,8)$, $\tilde{8} = (7,8,8,9)$, $\tilde{9} = (8,9,9,10)$.

Solution : FMOLFPP is given as

$$\tilde{z}_{1\,\max}(\tilde{\mathbf{x}}) = \frac{(4,5,5,6)\tilde{x}_1 + (7,8,8,9)\tilde{x}_2 + (8,9,9,10)\tilde{x}_3 + (6,7,7,8)}{(3,4,4,5)\tilde{x}_1 + (6,7,7,8)\tilde{x}_2 + (7,8,8,9)\tilde{x}_3 + (5,6,6,7)} \quad \text{(Production efficiency)}$$

$$\tilde{z}_{2\,\max}(\tilde{\mathbf{x}}) = \frac{(5,6,6,7)\tilde{x}_1 + (7,8,8,9)\tilde{x}_2 + (5.5,6.5,6.5,7.5)\tilde{x}_3 + (2,3,3,4)}{(3,4,4,5)\tilde{x}_1 + (6,7,7,8)\tilde{x}_2 + (7,8,8,9)\tilde{x}_3 + (5,6,6,7)} \quad \text{(Profitability)}$$

subject to:

$$(4.5,5.5,5.5,6.5)\tilde{x}_1 + (2.5,3.5,3.5,4.5)\tilde{x}_2 + (4.75,5.75,5.75,6.75)\tilde{x}_3 \preceq (5.75,6.75,6.75,7.75),$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0$$

Step 1: Using α-cut and the parametric form of TrFN, the objective functions become:

$$\tilde{z}_{1\,\max}(\tilde{\mathbf{x}},\alpha) = \frac{(5,\ 1-\alpha,\ 1-\alpha)\tilde{x}_1 + (8,\ 1-\alpha,\ 1-\alpha)\tilde{x}_2 + (9,\ 1-\alpha,\ 1-\alpha)\tilde{x}_3 + (7,\ 1-\alpha,\ 1-\alpha)}{(4,\ 1-\alpha,\ 1-\alpha)\tilde{x}_1 + (7,\ 1-\alpha,\ 1-\alpha)\tilde{x}_2 + (8,\ 1-\alpha,\ 1-\alpha)\tilde{x}_3 + (6,\ 1-\alpha,\ 1-\alpha)}$$

$$\tilde{z}_{2\,\max}(\tilde{\mathbf{x}},\alpha) = \frac{(6,\ 1-\alpha,\ 1-\alpha)\tilde{x}_1 + (8,\ 1-\alpha,\ 1-\alpha)\tilde{x}_2 + (6.5,\ 1-\alpha,\ 1-\alpha)\tilde{x}_3 + (3,\ 1-\alpha,\ 1-\alpha)}{(4,\ 1-\alpha,\ 1-\alpha)\tilde{x}_1 + (7,\ 1-\alpha,\ 1-\alpha)\tilde{x}_2 + (8,\ 1-\alpha,\ 1-\alpha)\tilde{x}_3 + (6,\ 1-\alpha,\ 1-\alpha)}$$

Subject to

$$(5.5,\ 1-\alpha,\ 1-\alpha)\tilde{x}_1 + (3.5,\ 1-\alpha,\ 1-\alpha)\tilde{x}_2 + (5.75,\ 1-\alpha,\ 1-\alpha)\tilde{x}_3 \preceq (6.75,\ 1-\alpha,\ 1-\alpha) \quad \text{and}$$

$$x_1, x_2, x_3 \geq 0$$

Step 2:Using charnes and cooper variable transformation method, converting FMOLFPP into FLPP and using simplex method we get (Table 6),

$$\tilde{z}_{1\,\max}(\tilde{\mathbf{y}},\alpha) = (5,\ 1-\alpha,\ 1-\alpha)\tilde{y}_1 + (8,\ 1-\alpha,\ 1-\alpha)\tilde{y}_2 + (9,\ 1-\alpha,\ 1-\alpha)\tilde{y}_3 + (7,\ 1-\alpha,\ 1-\alpha)\tilde{w}$$

Subject to

$$(4,\ 1-\alpha,\ 1-\alpha)\tilde{y}_1 + (7,\ 1-\alpha,\ 1-\alpha)\tilde{y}_2 + (8,\ 1-\alpha,\ 1-\alpha)\tilde{y}_3 + (6,\ 1-\alpha,\ 1-\alpha)\tilde{w} = 1$$

$$(5.5,\ 1-\alpha,\ 1-\alpha)\tilde{y}_1 + (3.5,\ 1-\alpha,\ 1-\alpha)\tilde{y}_2 + (5.75,\ 1-\alpha,\ 1-\alpha)\tilde{y}_3 \leq (6.75,\ 1-\alpha,\ 1-\alpha)\tilde{w}$$

$$\tilde{y}_1,\ \tilde{y}_2,\ \tilde{y}_3 \geq 0, \quad \tilde{w} > 0$$

we get

$$\tilde{y}_1 = (0.1125,\ 1-\alpha,\ 1-\alpha), \quad \tilde{y}_2 = (0,\ 1-\alpha,\ 1-\alpha), \quad \tilde{y}_3 = (0,\ 1-\alpha,\ 1-\alpha),$$

$$\tilde{w} = (0.0917,\ 1-\alpha,\ 1-\alpha)$$

Next, we find the values of $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ using the transformation $\tilde{x}_i = \dfrac{\tilde{y}_i}{\tilde{w}}$: $\tilde{x}_1 = (1.2268,\ 1-\alpha,\ 1-\alpha)$ $\tilde{x}_2 = (0,\ 1-\alpha,\ 1-\alpha)$ $\tilde{x}_3 = (0,\ 1-\alpha,\ 1-\alpha)$ and $\tilde{z}_{1\,\max}(\tilde{\mathbf{x}},\alpha) = (1.2042,\ 1-\alpha,\ 1-\alpha)$

Similarly

$$\tilde{z}_{2\max}(\tilde{\mathbf{y}},\alpha) = (6,\ 1-\alpha, 1-\alpha)\tilde{y}_1 + (8,\ 1-\alpha,\ 1-\alpha)\tilde{y}_2 + (6.5,\ 1-\alpha,\ 1-\alpha)\tilde{y}_3 + (3,\ 1-\alpha,\ 1-\alpha)\tilde{w}$$

Subject to

$$(4,\ 1-\alpha,\ 1-\alpha)\tilde{y}_1 + (7,\ 1-\alpha,\ 1-\alpha)\tilde{y}_2 + (8,\ 1-\alpha,\ 1-\alpha)\tilde{y}_3 + (6,\ 1-\alpha,\ 1-\alpha)\tilde{w} = 1$$

$$(5.5,\ 1-\alpha,\ 1-\alpha)\tilde{y}_1 + (3.5,\ 1-\alpha,\ 1-\alpha)\tilde{y}_2 + (5.75,\ 1-\alpha,\ 1-\alpha)\tilde{y}_3 \leq (6.75,\ 1-\alpha,\ 1-\alpha)\tilde{w}$$

we get

$$\tilde{y}_1 = (0.1125,\ 1-\alpha,\ 1-\alpha),\quad \tilde{y}_2 = (0,\ 1-\alpha,\ 1-\alpha),\quad \tilde{y}_3 = (0,\ 1-\alpha,\ 1-\alpha),$$

$$\tilde{w} = (0.0917,\ 1-\alpha,\ 1-\alpha)$$

Next, we find the values of $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ using the transformation $\tilde{x}_i = \dfrac{\tilde{y}_i}{\tilde{w}}$: $\tilde{x}_1 = (1.2268,\ 1-\alpha,\ 1-\alpha)$ $\tilde{x}_2 = (0,\ 1-\alpha,\ 1-\alpha)$ $\tilde{x}_3 = (0,\ 1-\alpha,\ 1-\alpha)$ and $\tilde{z}_{2\max}(\tilde{\mathbf{x}},\alpha) = (0.95,\ 1-\alpha,\ 1-\alpha)$

Table 6: After optimizing each objective function individually, the solution of Example is presented.

Objective Function	$\tilde{z}_i = \lvert\mathfrak{C}_i\rvert$	μ	m	σ
1	$(0.1125,\ 1-\alpha,\ 1-\alpha)$			
2	$(0.95,\ 1-\alpha,\ 1-\alpha)$	$(0.5313,\ 1-\alpha,\ 1-\alpha)$	$(0.5313,\ 1-\alpha,\ 1-\alpha)$	$(0.4188,\ 1-\alpha,\ 1-\alpha)$

We are given the formula for HMMSDT:

$$\text{HMMSDT} = \frac{1}{2}\left[\frac{\begin{array}{c}\min\left((0.5313,\ 1-\alpha,\ 1-\alpha),\ (0.5313,\ 1-\alpha,\ 1-\alpha),\ (0.4188,\ 1-\alpha,\ 1-\alpha)\right)\\ +\sqrt{(0.5313,\ 1-\alpha,\ 1-\alpha)\cdot(0.5313,\ 1-\alpha,\ 1-\alpha)\cdot(0.4188,\ 1-\alpha,\ 1-\alpha)}\end{array}}{(0.5313,\ 1-\alpha,\ 1-\alpha)+(0.5313,\ 1-\alpha,\ 1-\alpha)+(0.4188,\ 1-\alpha,\ 1-\alpha)}\right] = (0.2575,\ 1-\alpha,\ 1-\alpha)$$

$$\tilde{z}_{\max} = \frac{1}{(0.2575,\ 1-\alpha,\ 1-\alpha)}\left[\tilde{z}_{1\max}(\tilde{\mathbf{x}},\alpha) + \tilde{z}_{2\max}(\tilde{\mathbf{x}},\alpha)\right]$$

$$\tilde{z}_{\max} = \frac{1}{(0.2575,\ 1-\alpha,\ 1-\alpha)}\left[\frac{(11,\ 1-\alpha,\ 1-\alpha)\tilde{x}_1 + (16,\ 1-\alpha,\ 1-\alpha)\tilde{x}_2 + (15.5,\ 1-\alpha,\ 1-\alpha)\tilde{x}_3 + (10,\ 1-\alpha,\ 1-\alpha)}{(4,\ 1-\alpha,\ 1-\alpha)\tilde{x}_1 + (7,\ 1-\alpha,\ 1-\alpha)\tilde{x}_2 + (8,\ 1-\alpha,\ 1-\alpha)\tilde{x}_3 + (6,\ 1-\alpha,\ 1-\alpha)}\right]$$

Subject to

$$(5.5,\ 1-\alpha,\ 1-\alpha)\tilde{x}_1 + (3.5,\ 1-\alpha,\ 1-\alpha)\tilde{x}_2 + (5.75,\ 1-\alpha,\ 1-\alpha)\tilde{x}_3 \preceq (6.75,\ 1-\alpha,\ 1-\alpha)$$

$$\text{and}\quad \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \geq 0$$

Converting the LFPP into an LPP using the Charnes–Cooper variable transformation method and solving using the simplex method, we obtain the optimal solution of FMOLFPP as $\tilde{x}_1 = (1.2268,\ 1-\alpha,\ 1-\alpha)$, $\tilde{x}_2 = (0,\ 1-\alpha,\ 1-\alpha)$, $\tilde{x}_3 = (0,\ 1-\alpha,\ 1-\alpha)$, $\tilde{z}_{\max} = (8.3658,\ 1-\alpha,\ 1-\alpha)$ Converted to standard fuzzy form with left and right spreads as $\tilde{x}_1 = (0.2268+\alpha,\ 1.2268,\ 2.2268-\alpha)$, $\tilde{x}_2 = (-1+\alpha,\ 0,\ 1-\alpha)$, $\tilde{x}_3 = (-1+\alpha,\ 0,\ 1-\alpha)$, $\tilde{z}_{\max} = (7.3658+\alpha,\ 8.3658,\ 9.3658-\alpha)$

Step 4: Now, solving the problem using different existing techniques, we obtain the optimal solution:

1. Chandra Sen -Mean Technique: $z_{\max} = (4.055,\ 1-\alpha,\ 1-\alpha)$

2. Chandra Sen -Median Technique: $z_{\max} = (4.055,\ 1{-}\alpha,\ 1{-}\alpha)$

3. New Arithmetic Average Technique: $z_{\max} = (8.1092,\ 1{-}\alpha,\ 1{-}\alpha)$

For different values of $\alpha \in [0, 1]$, the proposed method yields various fuzzy optimal solutions for the given FMOLFPP.

Table 7: Fuzzy Values of $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$, and $\tilde{z}_{\max}$ for different α

α	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{z}_{\max}$
0	(0.2268, 1.2268, 2.2268)	(−1, 0, 1)	(−1, 0, 1)	(7.3658, 8.3658, 9.3658)
0.25	(0.4768, 1.2268, 1.9768)	(−0.75, 0, 0.75)	(−0.75, 0, 0.75)	(7.6158, 8.3658, 9.1158)
0.5	(0.7268, 1.2268, 1.7268)	(−0.5, 0, 0.5)	(−0.5, 0, 0.5)	(7.8658, 8.3658, 8.8658)
0.75	(0.9768, 1.2268, 1.4768)	(−0.25, 0, 0.25)	(−0.25, 0, 0.25)	(8.1158, 8.3658, 8.6158)
1	(1.2268, 1.2268, 1.2268)	(0, 0, 0)	(0, 0, 0)	(8.3658, 8.3658, 8.3658)

The corresponding maximum objective values are existing technique as follows:

Table 8: Fuzzy Values of $z_{\max}$ for Different Techniques and α-Levels

α	Chandra Sen - Mean	Chandra Sen - Median	New Arithmetic Average
0	(3.055, 4.055, 5.055)	(3.055, 4.055, 5.055)	(7.1092, 8.1092, 9.1092)
0.25	(3.305, 4.055, 4.805)	(3.305, 4.055, 4.805)	(7.3592, 8.1092, 8.8592)
0.5	(3.555, 4.055, 4.555)	(3.555, 4.055, 4.555)	(7.6092, 8.1092, 8.6092)
0.75	(3.805, 4.055, 4.305)	(3.805, 4.055, 4.305)	(7.8592, 8.1092, 8.3592)
1	(4.055, 4.055, 4.055)	(4.055, 4.055, 4.055)	(8.1092, 8.1092, 8.1092)

Note: Since there are no minimization objectives, we take $m_2 = 0$. Consequently, methods such as the New Geometric Average (NGA) and Advanced Harmonic Average (AHav), which involve division by or multiplication with m_2, become undefined or inapplicable.

4 Result and Discussion

The proposed method offers significant improvements over existing techniques by delivering more accurate and dependable results, especially when evaluated against those in examples. This highlights the superior performance and effectiveness of the approach. The HMMSDT method further confirms its efficiency through its successful application to a real-world scenario in Example 2. Unlike earlier methods that convert fuzzy numbers into crisp equivalents, the proposed approach preserves the original fuzzy form, thereby maintaining the inherent uncertainty and offering a more realistic and flexible decision-making framework. This improvement is achieved using α-cut and parametric representation, which better capture the influence of uncertainty. Unlike the solutions mentioned before, the new method achieves better results that fit within the best ranges found in past research, showing that it is consistent, dependable, and strong. Figures 3 and 4 show how well the HMMSDT method works for solving FMOLFPP, highlighting its better ability to manage fuzzy data, produce higher quality solutions, and clearly evaluate the trade-offs between different goals.

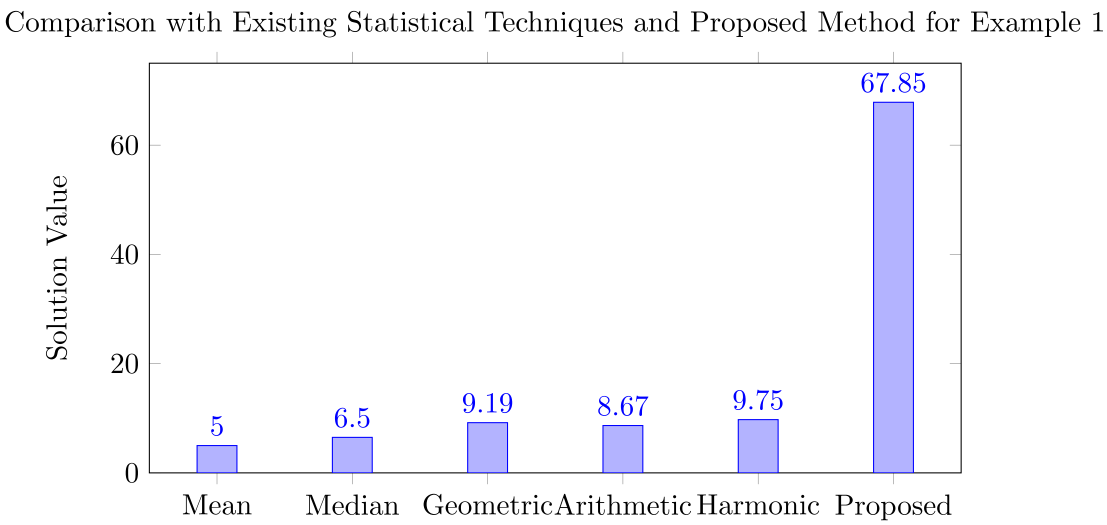

Figure 3: Comparison of Statistical Techniques and Proposed Method for Example 1

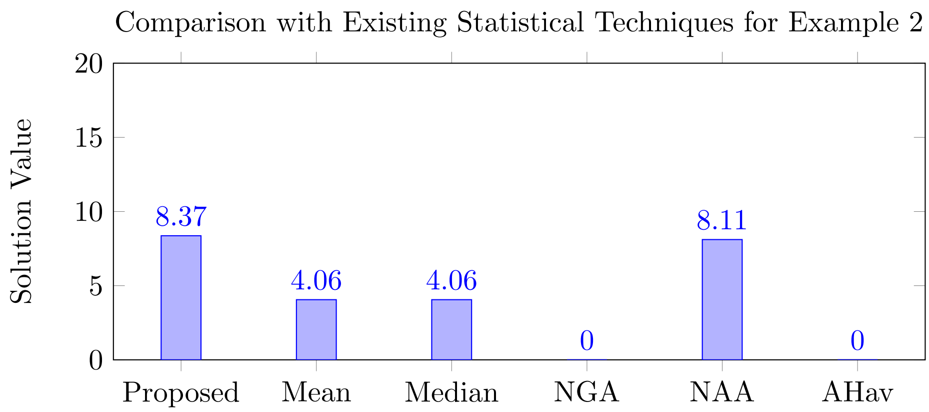

Figure 4: Comparison of Statistical Techniques and Proposed Method for Example 2

5 Conclusion

This paper introduces a solution framework for FMOLFPP utilising TrFN's, all procedures are conducted without transforming fuzzy parameters into precise values. The issue is reformulated into a parametric form and linearised by the variable transformation. Each target is optimized separately, and the outcomes are consolidated using statistical metrics — mean, median, and standard deviation. This statistical scalarization transforms the FMOLFP model into a singular deterministic format, preserving uncertainty and yielding a balanced optimum solution. Numerical examples were solved to illustrate the solution process and its underlying objective, with (Figure 3 and 4 and Table 3,4,5,7 and 8) presenting a comparison of the proposed techniques with previously studied methods. The proposed HMMSDT method yields a solution that satisfies fuzzy Pareto optimality through α-cut-based modeling and statistical scalarization without defuzzification. The results demonstrate that this approach not only preserves fuzzy uncertainty but also outperforms existing methods in terms of optimization quality. By effectively capturing the trade-offs between conflicting objectives, it enables more comprehensive and informed decision-making. Future research could extend this comparison by analyzing the proposed techniques alongside intuitionistic and neutrosophic methods, as well as additional approaches, to further validate the findings and enhance the results.

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