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A NUMERICAL METHOD FOR A NONLOCAL PARAMETER IDENTIFICATION PROBLEM

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Abstract. This paper addresses the inverse problem of identifying the time-dependent thermal conductivity in the one-dimensional parabolic heat equation from nonlocal overdetermination conditions. Such problems are typical in applications where only partial boundary data are available, leading to ill-posedness. We reformulate the problem after time discretization in such a way that the integral boundary condition is used to decouple the search for the thermal conductivity coefficient from the solution of the parabolic equation. Haar wavelets are used to approximate the solution of the modified Helmholtz equation obtained from time discretization. A system of algebraic equations is obtained via collocation, providing an efficient computational framework for solving the problem.

Keywords: Inverse problems, nonlocal condition, parabolic equation, Haar wavelet, collocation method.
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1 Introduction

Inverse problems involve determining unknown parameters in a system based on indirect observations or measurements. These problems arise across many fields, including medical imaging, geophysics, non-destructive testing, and fluid dynamics (Gorelick et al., 1983; Nachaoui, 1999; Chen & Wong, 1998; Huang & Chen, 2000; Ellabib et al., 2001; Loulou & Scott, 2006; Aster et al., 2012; Berdawood et al., 2021; Ellabib et al., 2021; Le & Nguyen, 2022; Nachaoui et al., 2022; Roy& Dhiman, 2023; Ellabib et al., 2024; Nachaoui, 2025).

In many practical scenarios, complete boundary data are unavailable. Measurements may be limited to specific portions of the boundary, while the rest remains inaccessible. This incomplete data regime further complicates the inverse problem, making it ill-posed in the sense of Hadamard (Hadamard, 1953; Dvalishvili et al., 2017). Even small perturbations in the data can lead to large deviations in the solution, highlighting the need for stable and accurate numerical methods (Nachaoui & Nassif, 1996; Jourhmane & Nachaoui, 1999; Huang & Chen, 2000; Kraus et al., 2001; Yarmukhamedov, 2003; Essaouini et al., 2004; Nachaoui, 2004; Regińska & Regiński, 2006; Shi et al., 2009; Qian et al., 2010; Kabanikhin, 2012; Mukanova, 2013; Lavrentiev, 2013; Hasan, 2015; Huang et al., 2017; Isakov, 2017; Qian & Feng, 2017; Vasil'ev et al., 2017; Hernandez-Montero et al., 2019; Aboud et al., 2021; Berdawood et al., 2021; Nachaoui & Salih, 2021; Reddy et al., 2021; Aboud et al., 2022; Ellabib et al., 2022; Berdawood et al., 2023; Nachaoui, 2023; Hadri et al., 2024).

In this work, we focus on an inverse coefficient identification problem where the unknown is the time-dependent thermal conductivity in a one-dimensional parabolic heat equation. The problem is defined with both initial and boundary conditions, as well as additional nonlocal condition. The nonlocal boundary conditions appear in the PDEs when the boundary data cannot be measured directly. Nonlocal PDEs are used to describe the dynamics of ground water (Bouziani, 1994; Pulkina, 1999). Some problems in visco-elasticity and food industry are also described in terms of nonlocal hyperbolic partial differential equations (Kabanikhin, 2002).

A feature of the proposed method is its efficient use of integral boundary conditions and a time-domain approximation to decouple the search for the thermal conductivity coefficient from the solution of the parbolic equation. Haar wavelets Nachaoui et al. (2018) introduced by Fu (1910), are used to approximate the solution of the modified Helmholtz equation obtained from time discretization.

Other basis such as polynomial-exponential basis was used in Klibanov (2017); Nguyen et al. (2023). It is important to note that well-known bases like Polynomial expansions Al-Humedi & Hasan. (2010); Nachaoui et al. (2023); Nachaoui & Sadiq (2025) polynomial-exponential basis Klibanov (2017); Nguyen et al. (2023), Legendre polynomials Lie (2013); Nachaoui & Gontier (2025) or trigonometric functions in the widely-used Fourier expansion Nachaoui & Salih (2021) may not be optimal. The readers can consult Nguyen, P. M. et al. (2021); Nachaoui & Gontier (2025); Nguyen et al. (2023), for a comparison of the performance of these bases in computation.

This paper presents a numerical technique based on recasting the inverse problem as an integral equation. In Section 2 we formulate the mathematical problem under investigation and then, in Section 3, we present the methods employed to solve the problem Section 4 is dedicated to the description of Haar wavelets and their application to solve our problem.

2 Mathematical Formulation

Let the solution domain be

$$D = \{(x,t) : 0 < x < h,\, 0 < t \leq T\},$$

where $h > 0$ and $T > 0$ are fixed. The governing equation is:

$$\frac{\partial u}{\partial t} = a(t)\frac{\partial^2 u}{\partial x^2}, \quad (x,t) \in D, \tag{1}$$

with unknowns $u(x,t)$ and $a(t) > 0$. Equation (1) is subject to:

Initial condition:

$$u(x,0) = \phi(x), \quad 0 \leq x \leq h. \tag{2}$$

Boundary conditions:

$$u(0,t) = \mu_1(t), \quad u(h,t) = \mu_2(t), \quad 0 \leq t \leq T. \tag{3}$$

Nonlocal condition:

$$\int_0^h u(x,t)dx = E(t), \quad 0 \leq t \leq T \tag{4}$$

where ϕ, μ_1, μ_2 and E are known functions. Further, we assume that this given data are sufficiently smooth in order to obtain a smooth solution u. It should be noted that $\phi(x)$ and $g(x)$ satisfy the following compatibility conditions

$$\phi(0) = \mu_1(0), \phi(h) = \mu_2(0),$$

$$\int_0^1 \phi(x)dx = E(0).$$

The existence, uniqueness and stability results for the given problem (1)–(4) that combine integral as well as Neumann conditions are discussed by Chen & Wong (2001). Gorelick et al. (2000) and Kabanikhin (2002) have also investigated hyperbolic partial differential equations with nonlocal boundary conditions.

If $a(t)$ is known, equations (1)–(3) constitute a direct problem for finding the unknown $u(x,t)$. When $a(t)$ is unknown, the problem becomes an inverse coefficient identification problem. The inverse problem requires solving equations (1)–(3) and (4).

3 Constructing of a Solution of the inverse Problem

Differentiating equation (4) with respect to t, we obtain:

$$\int_0^h \frac{\partial u}{\partial t}(x,t)dx = E'(t), \quad 0 \le t \le T. \tag{5}$$

From equation(1) we get

$$a(t)\int_0^h \frac{\partial^2 u}{\partial x^2}(x,t)\ dx = E'(t), \tag{6}$$

then

$$a(t)\left(\frac{\partial u}{\partial x}(h,t) - \frac{\partial u}{\partial x}(0,t)\right) = E'(t).$$

Let

$$\mu_3(t) = \left(\frac{\partial u}{\partial x}(h,t) - \frac{\partial u}{\partial x}(0,t)\right).$$

If $\mu_3(t)$ is known and $\mu_3(t) \neq 0$ then $a(t)$ can be obtained from the following formula:

$$a(t) = \frac{E'(t)}{\mu_3(t))}. \tag{7}$$

So if we can calculate the derivatives of u at 0 and at h, the calculation of $a(t)$ can be the same as the calculation of $u(x,t)$. Now, combining equation (7) with equation (2) we get

$$a(0) = \frac{E'(0)}{\phi'(h) - \phi'(0)}. \tag{8}$$

This means that the value of $a(t)$ at the initial time $t = 0$ is known. We will use this observation to construct our approximation scheme.

3.1 Temporal discretization

Let N be un intger. The temporal discretization is given by:

$$0 = t_0 < t_1 < t_2, \dots, t_n, \dots < t_N = T$$

where $n = 0, 1, 2, \dots, N$ and $t_n = n\delta t$ with $\delta t = \frac{T}{N}$.

Using a first order finite difference scheme for temporal discretization in equation (1) and approximating $a(t)$ using (7) at $t = (t_n)$, we obtain

$$\frac{u(x,t_{n+1}) - u(x,t_n)}{\delta t} = a(t_n)\,\frac{\partial^2 u}{\partial x^2}(x,t_{n+1}), \tag{9}$$

$$\delta t\, a(t_n)\,\frac{\partial^2 u}{\partial x^2}(x,t_{n+1}) - u(x,t_{n+1}) = -u(x,t_n),$$

then

$$a(t_n)\frac{\partial^2 u}{\partial x^2}(x,t_{n+1}) - \frac{u(x,t_{n+1})}{\delta t} = -\frac{u(x,t_n)}{\delta t}, \tag{10}$$

which is a modified Helmotz equation at each time step t_{n+1}.

3.2 Spatial discretization

Let $\varphi_i(x), i \geq 0$, denote the elements of a basis of $L^2(h,0)$ and assume that

$$\frac{\partial^2 u}{\partial x^2}(x,t_{n+1}) = \sum_i c_i\,\varphi_i(x), \tag{11}$$

where c_i are coefficients, depending on t_{n+1}, to be determined.

Integrating equation (11) twice from 0 to x, we get

$$u(x,t_{n+1}) - u(0,t_{n+1}) - x\,\frac{\partial u}{\partial x}(0,t_{n+1}) = \sum_i c_i \int_0^x \int_0^x \varphi_i(\zeta)\ d\zeta d\zeta. \tag{12}$$

For $x = h$, this last equation becomes

$$u(h,t_{n+1}) - u(0,t_{n+1}) - h\,\frac{\partial u}{\partial x}(0,t_{n+1}) = \sum_i c_i\,\varphi_i^2(h).$$

We can therefore express $\frac{\partial u}{\partial x}(0,t_{n+1})$ by:

$$\frac{\partial u}{\partial x}(0,t_{n+1}) = \frac{1}{h}\left(u(h,t_{n+1}) - u(0,t_{n+1}) - \sum_i c_i \int_0^h \int_0^h \varphi_i(\zeta)\ d\zeta d\zeta\right). \tag{13}$$

Substituting (13) in (12) we obtain

$$\begin{aligned} u(x,t_{n+1}) &= (1-\frac{x}{h})u(0,t_{n+1}) + \frac{x}{h}u(h,t_{n+1}) \\ &+ \sum_i c_i \left(\int_0^x \int_0^x \varphi_i(\zeta)\ d\zeta d\zeta - \frac{x}{h}\int_0^h \int_0^h \varphi_i(\zeta)\ d\zeta d\zeta\right). \end{aligned} \tag{14}$$

From the given boundary conditions (3), we get

$$\begin{aligned} u(x,t_{n+1}) &= (1-\frac{x}{h})\,\mu_1(t_{n+1}) + \frac{x}{h}\,\mu_2(t_{n+1}) \\ &+ \sum_i c_i \left(\int_0^x \int_0^x \varphi_i(\zeta)\ d\zeta d\zeta - \frac{x}{h}\int_0^h \int_0^h \varphi_i(\zeta)\ d\zeta d\zeta\right). \end{aligned} \tag{15}$$

Now, introducing spatial discretization (15) and (11) in (10), we obtain

$$\begin{aligned} & a(t_n)\sum_i c_i\,\varphi_i(x) - \frac{1}{\delta t}\sum_i c_i \left(\int_0^x \int_0^x \varphi_i(\zeta)\ d\zeta d\zeta - \frac{x}{h}\int_0^h \int_0^h \varphi_i(\zeta)\ d\zeta d\zeta\right) \\ & = -\frac{u(x,t_n)}{\delta t} + \frac{1}{\delta t}\left((1-\frac{x}{h})\,\mu_1(t_{n+1}) + \frac{x}{h}\,\mu_2(t_{n+1})\right). \end{aligned} \tag{16}$$

Thus, at $t = t_{n+1}$, we obtain the following discretized scheme,

$$\begin{aligned} & \sum_i c_i \left[a(t_n)\varphi_i(x) - \frac{1}{\delta t}\left(\int_0^x \int_0^x \varphi_i(\zeta)\ d\zeta d\zeta - \frac{x}{h}\int_0^h \int_0^h \varphi_i(\zeta)\ d\zeta d\zeta\right)\right] \\ & = -\frac{u(x,t_n)}{\delta t} + \frac{1}{\delta t}\left((1-\frac{x}{h})\,\mu_1(t_{n+1}) + \frac{x}{h}\,\mu_2(t_{n+1})\right). \end{aligned} \tag{17}$$

The evaluation of equation (17) at the collocation points $(x_i, i = 1, ..., 2M_x)$ gives:

$$\sum_j c_j \left[a(t_n)\varphi_j(x_i) - \frac{1}{\delta t}\left(\int_0^{x_i}\int_0^{x_i} \varphi_j(\zeta)\ d\zeta d\zeta - \frac{x_i}{h}\int_0^h\int_0^h \varphi_j(\zeta)\ d\zeta d\zeta\right)\right]$$
$$= -\frac{u(x_i,t_n)}{\delta t} + \frac{1}{\delta t}\left((1-\frac{x_i}{h})\,\mu_1(t_{n+1}) + \frac{x_i}{h}\,\mu_2(t_{n+1})\right), \tag{18}$$

which is a system of algebraic equations for the coefficients c_i.

$$\sum_j A_{ij}c_j = b_i, \qquad \text{for}\, i = 1,...,2M_x, \tag{19}$$

where

$$A_{ij} = \left[a(t_n)\varphi_j(x_i) - \frac{1}{\delta t}\left(\int_0^{x_i}\int_0^{x_i} \varphi_j(\zeta)\ d\zeta d\zeta - \frac{x_i}{h}\int_0^h\int_0^h \varphi_j(\zeta)\ d\zeta d\zeta\right)\right], \tag{20}$$
$$b_i = -\frac{u(x_i,t_n)}{\delta t} + \frac{1}{\delta t}\left((1-\frac{x_i}{h})\,\mu_1(t_{n+1}) + \frac{x_i}{h}\,\mu_2(t_{n+1})\right) \tag{21}$$

4 Haar wavelets approximation

Let J_x be a given maximum resolution level, and define the integer M_x by $M_x = 2^{J_x}$. Let $h_i^{J_x}(x), i = 1,\cdots,2M_x$ be the Haar wavelet functions defined on the interval (x_{Low}, x_{Upp}) for the resolution levels J_x as follows:

Let $\Delta_x = (x_{Upp} - x_{Low})/2M_x$. Now, define the dilation and translation parameters $j = 0,1,\cdots,J_x$ and $k = 0,1,...,m-1$ respectively, where $m = 2^j$. The wavelet number is given by $i = m+k+1$. The Haar wavelet functions $h_i^{J_x}(x), i = 1,\cdots,2M_x$ are given by

$$h_1^{J_x}(x) = \begin{cases} 1 & \text{when } x_{Low} \le\ x < x_{Upp} \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

For $i \ge 2$

$$h_i^{J_x}(x) = \begin{cases} 1 & \text{when } \beta_1(i) \le\ x < \beta_2(i) \\ -1 & \text{when } \beta_2(i) \le\ x < \beta_3(i) \\ 0 & \text{otherwise,} \end{cases} \tag{23}$$

where

$$\beta_1(i) = x_{Low} + 2k\zeta\Delta_x, \quad \beta_2(i) = x_{Low} + (2k+1)\zeta\Delta_x,$$
$$\beta_3(i) = x_{Low} + 2(k+1)\zeta\Delta_x, \quad \zeta = M_x/m.$$

4.1 Integrals Haar wavelets

Since we want to solve a problem which a second order partial differential equation we need the integrate twice the Haar wavelets functions thus we need the integrals given for $i \ge 2$ by :

$$P_{n,i}^{J_x}(x) = \begin{cases} h_i^{J_x}(x) & \text{when } n = 0 \text{ and } i = 1,2,...,2M_x, \\ \int_{x_{Low}}^x P_{n-1,i}^{J_x}(t)dt & \text{when } n \ge 1\text{and } i = 1,2,...,2M_x. \end{cases} \tag{24}$$

Using the Haar wavelet definition, we calculate:

$$P_{1,i}^{J_x}(x) = \int_{x_{Low}}^{x} h_i^{J_x}(t)dt, = \begin{cases} x-\beta_1(i) & \text{for } x\in[\beta_1(i),\beta_2(i)), \\ \beta_3(i)-x & \text{for } x\in[\beta_2(i),\beta_3(i)), \\ 0 & \text{elsewhere }, \end{cases} \tag{25}$$

and

$$P_{2,i}^{J_x}(x) = \int_{x_{Low}}^{x} P_{1,i}^{J_x}(t)dt, = \begin{cases} 0 & x<\beta_1(i), \\ \frac{(x-\beta_1(i))^2}{2} & x\in[\beta_1(i),\beta_2(i)), \\ (\beta_3(i)-\beta_2(i))^2-\frac{(\beta_3(i)-x)^2}{2} & x\in[\beta_2(i),\beta_3(i)), \\ (\beta_3(i)-\beta_2(i))^2 & x\geq\beta_3(i). \end{cases} \tag{26}$$

In case $i=1$ we have $\beta_1(1)=x_{Low}$, $\beta_2(1)=\beta_3(1)=x_{Upp}$ and

$$P_{1,1}^{J_x}(x) = (x-x_{Low}), \tag{27}$$

$$P_{2,1}^{J_x}(x) = \frac{1}{2}(x-x_{Low})^2 \tag{28}$$

The Haar wavelet functions defined in (22)-(23) form an $L^2(h,0)$ basis. In the last few years, Haar wavelet based collocation methods are extensively used for the numerical solution of partial differential equations. Because of various attractive properties of Haar wavelet such as closed form expression, compact support and orthonormality, it is widely used in various areas of science and engineering. We adopt this basis for the rest of our approximation.

4.2 The Haar wavelet based collocation method

Define the discretization of the intervals (x_{Upp}, x_{Low}) as follows:

$$x_r = x_{Low}+\frac{(r-0.5)(x_{Upp}-x_{Low})}{2M_x}, r=1,\cdots,2M_x; \tag{29}$$

Haar's wavelet approximation consists of replacing $\varphi_j(x)$ by Haar basis functions $h_j^{J_x}(x)$ in the (20) and (21) formulas, truncating the series to $2M_x$ terms, thus we obtain

$$A_{ij} = \left[a(t_n)h_j^{J_x}(x_i)-\frac{1}{\delta t}\left(\int_0^{x_i}\int_0^{x_i} h_j^{J_x}(\zeta)\,d\zeta d\zeta-\frac{x_i}{h}\int_0^{h}\int_0^{h} h_j^{J_x}(\zeta)\,d\zeta d\zeta\right)\right], \tag{30}$$

$$b_i = -\frac{u(x_i,t_n)}{\delta t}+\frac{1}{\delta t}\left((1-\frac{x_i}{h})\,\mu_1(t_{n+1})+\frac{x_i}{h}\,\mu_2(t_{n+1})\right) \tag{31}$$

which is equivalent to

$$A_{ij} = \left[a(t_n)h_j^{J_x}(x_i)-\frac{1}{\delta t}\left(P_{j2}(x_i)-\frac{x_i}{h}\,P_{j2}(h)\right)\right], \tag{32}$$

$$b_i = -\frac{u(x_i,t_n)}{\delta t}+\frac{1}{\delta t}\left((1-\frac{x_i}{h})\,\mu_1(t_{n+1})+\frac{x_i}{h}\,\mu_2(t_{n+1})\right). \tag{33}$$

The linear system defined by (32)-(33) is then a system of linear equations of order $2M_x$ making it possible to obtain the wavelet coefficients $c_i, i=1,\cdots,2M_x$ by solving (19) for each time step $t_{n+1}, n=0,\cdots,(N-1)$. The required solution is then obtained from (15).

5 Conclusion

In this paper, we developed a numerical strategy for solving an inverse coefficient identification problem in a one-dimensional parabolic heat equation with nonlocal integral boundary conditions. By leveraging a time discretization scheme and Haar wavelet basis for spatial approximation, we effectively decoupled the estimation of the unknown time-dependent thermal conductivity $a(t)$) from the direct solution of the heat equation. This formulation allowed us to transform the inverse problem into a well-posed system of equations. This approach leads to a system of algebraic equations solvable using standard numerical techniques. The method is computationally efficient and well-suited for problems with discontinuous or piecewise smooth data, making it a promising tool for practical inverse problems in science and engineering.

The choice of Haar wavelets provided a practical and flexible basis, especially suitable for problems involving sharp gradients or localized features, as often encountered in nonlocal formulations.

Future work will involve quantitative benchmarking against established iterative methods and exploring other wavelet families (e.g., Daubechies or spline wavelets) to further improve convergence and resolution. The adaptability of the framework to higher-dimensional or nonlinear inverse problems also offers a promising direction for extending this approach to more complex physical systems.

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