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A STUDY OF LIE POINT SYMMETRIES, OPTIMAL SYSTEM, ANALYTICAL SOLUTIONS AND CONVERSED VECTORS OF THE MIXED DOUBLY NONLINEAR DISPERSIVE BURGERS EQUATION

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Abstract. Burgers equation is a partial differential equation that is a convection-diffusion equation, which arises in many areas of applied mathematics including gas dynamics, nonlinear acoustics, fluid mechanics, and traffic flow. There are several forms of Burgers equations in the literature that are studied by researchers. In this study, using Lie group analysis, we explore a combination of two Burgers hierarchies which is called the mixed doubly nonlinear dispersive Burgers (mDDB) equation. Firstly, we compute the Lie point symmetries of the mDDB equation and then construct the optimal system of one-dimensional subalgebras. Using this optimal system we transform the mDDB equation into several nonlinear ordinary differential equations, which are then solved using various methods including the simplest equation method and the power series expansion method. Furthermore, we present the 3D, density, and 2D graphs of the derived solutions that demonstrate their dynamic behaviours. Finally, the conserved vectors of the mDDB equation are displayed using the multiplier technique and Ibragimov's method. These include the conserved energy and linear momentum.

Keywords: Mixed doubly nonlinear dispersive Burgers equation, Lie group analysis, analytical solutions, multiplier method, Ibragimov's method, conserved vectors.

AMS Subject Classification: 35B06, 35L65, 37J15, 37K05.

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1 Introduction

A nonlinear partial differential equation (NPDE) is a partial differential equation which contains nonlinear terms in it. Such equations describe many distinct physical phenomena of natural world. Compared to linear partial differential equations, NPDEs are far more difficult to solve because of their nonlinearity, which frequently results in the lack of explicit analytical solutions. Many physical systems, such as the nonlinear heat conduction equation arising from automated-vehicle traffic flow models (Theodosis et al., 2024), the 3D incompressible Navier-Stokes equations of fluid dynamics (Liu & Li, 2024), antispiral wave formation in reaction-diffusion systems (Gong & Christini, 2003) involve NPDEs. These equations are crucial for effectively simulating real-world phenomena because they frequently represent complex behaviour that linear models are unable to grasp.

Determining the analytical solutions for NPDEs provides a thorough understanding of the real-world appearance of the problem. Because NPDEs are so complicated to solve, it might be difficult to obtain their analytical solutions. Nonetheless, there are some special methods in the literature where special analytical solutions can be found. One method for locating precise answers is variables separation. This approach is effective for some NPDE types, particularly those that can be divided into more manageable ordinary differential equations (ODEs) by appropriately changing some variables (Polyanin & Zhurov, 2021). There are numerous other approaches to find special exact solutions of NPDEs. These include inverse scattering transform method (Ablowitz & Clarkson, 1991), the Hirota bilinear method (Li et al., 2019), (G'/G)-expansion technique (Wang et al., 2005), power series solution technique (Feng et al., 2017; Simbanefayi & Khalique, 2020), Darboux transformation approach (Zhang et al., 2020), Painlevé expansion technique (Weiss et al., 1985), bifurcation approach (Zhang & Khalique, 2018), homotopy perturbation (Chun & Sakthivel, 2018), extended homoclinic test technique (Darvishi & Najafi, 2018), Lie group analysis (Ovsiannikov, 1982; Olver, 1993; Ibragimov, 1999), the simplest equation method Kudryashov (2005); Zhao et al. (2013), and few others. For example, see (Khalique & Adeyemo, 2020; Kudryashov, 2019; Adeyemo et al., 2023; Alhasanat, 2023; Chen & Yan, 2005; Simbanefayi et al., 2023; Ma, 2015; Zhau & Han, 2017).

The study of integrable hierarchies is a significant and interesting topic in solitary wave theory. In Wazwaz (2010), the author investigated the Burgers hierarchy. The first and second elements in this hierarchy are

$$u_t + 2\beta u u_x + \beta u_{xx} = 0 \tag{1}$$

and

$$u_t + 3\alpha u^2 u_x + 3\alpha u_x^2 + 3\alpha u u_{xx} + \alpha u_{xxx} = 0, \tag{2}$$

respectively, where β and α are real constants. Equation (2) is also known as the Sharma-Tasso-Olver (STO) equation in the literature. See Yan & Lou (2020); Zhou & Zhuang (2022). The Sharma-Tasso-Olver-Burgers equation (Yan & Lou, 2020; Zhou & Zhuang, 2022), also known as the mixed doubly nonlinear dispersive Burgers (mDDB) equation (Alquran et al., 2023) reads

$$u_t + 3\alpha u^2 u_x + 3\alpha u_x^2 + 3\alpha u u_{xx} + 2\beta u u_x + \beta u_{xx} + \alpha u_{xxx} = 0 \tag{3}$$

with $u(t, x)$ being the unknown function. Clearly, for $\alpha = 0$, equation (3) reduces to the Burgers equation, while when $\beta = 0$, equation (3) represents the STO equation.

A few research that investigated certain physical characteristics and specific solutions of mDDB equation (3) has been recently published in the literature. For instance, Yan & Lou (2020) found kink, half periodic kink and breather solutions for mDDB equation (3) by incorporating the velocity resonant mechanism. The authors Zhou & Zhuang (2022) focused on the geometric properties of mDDB equation (3) and reported exact explicit kink wave solutions. Alquran et al. (2023) utilized different methods and obtained new explicit solutions with different physical waveforms, that included lump, kinky, and periodic solutions. Kudryashov's method and rational sine-cosine were used to derive kinky waves and singular periodic wave solutions. Moreover, Hirota bilinear form was employed to retrieve lump-type solutions of (3). Gomez & Hernandez (2017) studied equation (3) with variable coefficients, namely

$$\begin{aligned} &u_t(x,t) + \delta(t)[u(x,t)^3 + u(x,t)u_x(x,t) + u_{xx}(x,t)]_x \\ &+ \rho(t)[u(x,t)u_x(x,t) + u_{xx}(x,t)] = 0, \end{aligned} \tag{4}$$

where $\delta(t)$ and $\rho(t)$ are functions of the variable time, t. Travelling wave solutions of (4) were investigated by invoking two method; the improved tanh-coth technique and the exp-function technique. The solutions obtained by the two methods were then compared.

The remainder of the paper is structured as follows. Section 2 provides analytical solutions of (3) and it is divided into the following parts: firstly, we compute symmetries and use them to generate optimal system of one-dimensional subalgebras. Thereafter, we use the optimal systems to perform symmetry reductions and construct analytical solutions. The simplest equation and power series methods are the two approaches used to find solutions. In Section 3, we apply the multiplier approach and Ibragimov's method to construct the conserved vectors of equation (3). In the last Section 4, we summarize the work done in this paper.

2 Analytical solutions of (3)

We use Lie group analysis in conjunction with different techniques, such as the Riccati and Bernoulli equations as the simplest equations along with power series method to construct analytical solutions for the mDDB model (3).

2.1 Lie point symmetries of (3)

We consider the infinitesimal generator for the symmetry group as

$$\mathcal{R} = \Phi^1 \frac{\partial}{\partial t} + \Phi^2 \frac{\partial}{\partial x} + \Phi^3 \frac{\partial}{\partial u},$$

where the coefficient functions Φ^1, Φ^2 and Φ^3 depend on t, x and u. Invoking the invariance condition

$$\mathcal{R}^{[3]} \Delta|_{\Delta=0} = 0, \tag{5}$$

with $\Delta = u_t + 3\alpha u^2 u_x + 3\alpha u_x^2 + 3\alpha u u_{xx} + 2\beta u u_x + \beta u_{xx} + \alpha u_{xxx}$, we shall obtain the point symmetries of equation (3). Here $\mathcal{R}^{[3]}$ is the third extension of $\mathcal{R}$ given by (Ibragimov (1999))

$$\mathcal{R}^{[3]} = \mathcal{R} + \zeta_t \frac{\partial}{\partial u_t} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_{xx} \frac{\partial}{\partial u_{xx}} + \zeta_{xxx} \frac{\partial}{\partial u_{xxx}}, \tag{6}$$

with the ζ's given as

$$\zeta_t = D_t(\Phi^3) - u_t D_t(\Phi^1) - u_x D_t(\Phi^2), \tag{7}$$

$$\zeta_x = D_x(\Phi^3) - u_t D_x(\Phi^1) - u_x D_x(\Phi^2), \tag{8}$$

$$\zeta_{xx} = D_x(\zeta_x) - u_{tx} D_x(\Phi^1) - u_{xx} D_x(\Phi^2), \tag{9}$$

$$\zeta_{xxx} = D_x(\zeta_{xx}) - u_{txx} D_x(\Phi^1) - u_{xxx} D_x(\Phi^2) \tag{10}$$

in which total derivatives D_t and D_x are defined as

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + u_{txx} \frac{\partial}{\partial u_{xx}} + \cdots, \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + u_{xxx} \frac{\partial}{\partial u_{xx}} + \cdots. \end{aligned} \tag{11}$$

With the help of scientific program Maple, expanding (5) and separating on derivatives of u produces a linear homogeneous system of PDEs

$$\begin{aligned} &\Phi^1_{tt} = 0,\ \Phi^1_u = 0,\ \Phi^1_x = 0,\ \Phi^2_u = 0,\ 9\alpha\Phi^2_t + 2\beta^2\Phi^1_t = 0, \\ &3\Phi^2_x - \Phi^1_t = 0,\ 9\alpha\Phi^3 + (\beta + 3\alpha u)\Phi^1_t = 0. \end{aligned}$$

Solving the preceding equations yields

$$\Phi^1 = 9\alpha C_1 t + C_2,\ \Phi^2 = 3\alpha C_1 x - 2\beta^2 C_1 t + C_3,\ \Phi^3 = -\beta C_1 - 3\alpha C_1 u \tag{12}$$

with C_1, C_2, and C_3 being constants. Consequently, the symmetry group of (3) consists of three point symmetries, namely

$$\mathcal{R}_1 = \frac{\partial}{\partial t},\ \mathcal{R}_2 = \frac{\partial}{\partial x},\ \mathcal{R}_3 = 9\alpha t \frac{\partial}{\partial t} + \left(3\alpha x - 2\beta^2 t\right) \frac{\partial}{\partial x} - (\beta + 3\alpha u) \frac{\partial}{\partial u} \tag{13}$$

of which first two are translations in t and x, respectively.

One-parametric transformation groups associated with (13)

On solving Lie equations (Ibragimov (1999))

$$\frac{d\bar{t}}{dh} = \Phi^1,\ \bar{t}|_{h=0} = t,\quad \frac{d\bar{x}}{dh} = \Phi^2,\ \bar{x}|_{h=0} = x,\quad \frac{d\bar{u}}{dh} = \Phi^3,\ \bar{u}|_{h=0} = u,$$

where Φ^1, Φ^2 and Φ^3 depend on $(\bar{t}, \bar{x}, \bar{u})$, h is the group parameter, and for each $\mathcal{R}_i$, the group transformations T_{h_i} are given by

$$\begin{aligned} T_{h_1} &: (\bar{t}, \bar{x}, \bar{u}) = (t + h_1, x, u),\\ T_{h_2} &: (\bar{t}, \bar{x}, \bar{u}) = (t, x + h_2, u),\\ T_{h_3} &: (\bar{t}, \bar{x}, \bar{u}) = (e^{9\alpha h_3}t, xe^{3\alpha h_3} + (\beta^2/3\alpha)te^{3\alpha h_3} - (\beta^2/3\alpha)te^{9\alpha h_3},\\ &\qquad ue^{-3\alpha h_3} + (\beta/3\alpha)e^{-3\alpha h_3} - (\beta/3\alpha)). \end{aligned}$$

2.2 Optimal system of one-dimensional subalgebras

The idea of the optimal system of one-dimensional subalgebras of a particular Lie algebra for representing essential distinct invariant solutions was introduced by Ovsiannikov (1982). Using the Lie series

$$\mathrm{Ad}(\exp(\varepsilon\mathcal{R}_i))\mathcal{R}_j = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(\mathrm{ad}\mathcal{R}_i)^n(\mathcal{R}_j) = \mathcal{R}_j - \varepsilon[\mathcal{R}_i, \mathcal{R}_j] + \frac{\varepsilon^2}{2!}[\mathcal{R}_j, [\mathcal{R}_i, \mathcal{R}_j]] - \cdots,$$

we create the adjoint representation, where ε is a real number. The commutator of $\mathcal{R}_i$ and $\mathcal{R}_j$ is defined as

$$[\mathcal{R}_i, \mathcal{R}_j] = \mathcal{R}_i\mathcal{R}_j - \mathcal{R}_j\mathcal{R}_i.$$

Table 1. Commutator table of the Lie algebra of (3)

[,]	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$
$\mathcal{R}_1$	0	0	$9\alpha\mathcal{R}_1 - 2\beta^2\mathcal{R}_2$
$\mathcal{R}_2$	0	0	$3\alpha\mathcal{R}_2$
$\mathcal{R}_3$	$-9\alpha\mathcal{R}_1 + 2\beta^2\mathcal{R}_2$	$-3\alpha\mathcal{R}_2$	0

Table 2. Adjoint representation of subalgebras

Ad	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$
$\mathcal{R}_1$	$\mathcal{R}_1$	$\mathcal{R}_2$	$-9\alpha\varepsilon\mathcal{R}_1 + 2\beta^2\varepsilon\mathcal{R}_2 + \mathcal{R}_3$
$\mathcal{R}_2$	$\mathcal{R}_1$	$\mathcal{R}_2$	$-3\alpha\varepsilon\mathcal{R}_2 + \mathcal{R}_3$
$\mathcal{R}_3$	$e^{9\alpha\varepsilon}\mathcal{R}_1 - \frac{\beta^2 e^{3\alpha\varepsilon}(e^{6\alpha\varepsilon}-1)}{3\alpha}\mathcal{R}_2$	$e^{3\alpha\varepsilon}\mathcal{R}_2$	$\mathcal{R}_3$

We utilize Table 1 and Table 2 to compute an optimal system of one-dimensional subalgebras for equation ((3) Ovsiannikov (1982); Olver (1993)). Thus, the set

$$\{\mathcal{R}_1 + \nu\mathcal{R}_2, \mathcal{R}_2, \mathcal{R}_3\}$$

is an optimal system for equation (3), where ν is an arbitrary constant. Finally, it is important to note that the optimal system of subalgebras is not unique. It is determined by the selection of a representative from each class of related operators. As a result, the type of solutions included in an optimal system of invariant solutions is determined by the representatives chosen. The selection of representations has no impact on the number of invariant solutions in the optimal system, since the number of the classes of similar operators does not depend on the choice of representatives.

2.3 Symmetry reductions and analytical solutions

Now, we use the optimal system obtained in the previous section and perform symmetry reductions and compute analytical solutions of the mDDB equation (3).

Case 1. $\mathcal{R}_1 + \nu\mathcal{R}_2$

We consider the first member of the optimal system, namely $\mathcal{R} = \mathcal{R}_1 + \nu\mathcal{R}_2$, where ν is a constant. Associated Lagrangian system for symmetry $\mathcal{R}$ yields two invariants

$$f = x - \nu t, \;\; Q(f) = u(t,x). \tag{14}$$

Treating Q as the new dependent variable and f as the new independent variable, the mDDB equation (3) reduces to an ODE

$$3\alpha Q^2 Q' - \nu Q' + 3\alpha Q'^2 + 3\alpha QQ'' + 2\beta QQ' + \beta Q'' + \alpha Q''' = 0. \tag{15}$$

Solutions of (15) via the simplest equation approach

We solve the nonlinear third-order ODE (15) using the technique known as the simplest equation approach (Kudryashov, 2005; Zhao et al., 2013). The simplest equations used here are the Riccati and Bernoulli equations, given respectively by

$$R'(f) = mR^2(f) + lR(f) + n \tag{16}$$

and

$$R'(f) = lR^2(f) + mR(f), \tag{17}$$

where l, m and n are constants. We seek solutions to equation (15) of the kind

$$Q(f) = \sum_{i=0}^{M} A_i \left(R(f)\right)^i, \tag{18}$$

where $R(f)$ solves the Riccati equation (16) or the Bernoulli equation (17), $A_0, \ldots, A_M$ are parameters that need to be determined, and M is a positive integer that may be found by balancing the highest-order linear term in equation (15) with the highest-order nonlinear term.

The Riccati equation (16) solutions that we utilize in this context are provided by Zhao et al. (2013)

$$R(f) = -\frac{l}{2m} - \frac{\theta}{2m}\tanh\left[\frac{1}{2}\theta(f+S)\right] \tag{19}$$

and

$$R(f) = -\frac{l}{2m} - \frac{\theta}{2m}\tanh\left(\frac{\theta}{2}f\right) + \frac{\operatorname{sech}\left(\frac{\theta f}{2}\right)}{S\cosh\left(\frac{\theta f}{2}\right) - \frac{2m}{\theta}\sinh\left(\frac{\theta f}{2}\right)} \tag{20}$$

with $\theta^2 = l^2 - 4mn$ and S a constant of integration.

For the solutions of the Bernoulli equation (17), we utilize Zhao et al. (2013)

$$R(f) = \frac{-mc_1}{l\left(c_1 + \cosh\left(m\left(f+c_0\right)\right) - \sinh(m\left(f+c_0\right))\right)} \tag{21}$$

and

$$R(f) = \frac{-m(\cosh\left(m(f+c_0)\right) + \sinh\left(m(f+c_0)\right))}{l(c_2 + \cosh[m\left(f+c_0\right)) + \sinh\left(m(f+c_0)\right))}, \tag{22}$$

where c_0, c_1, and c_2 are constants.

Solutions of (15) using the Riccati equation

We find $M = 1$ by balancing the linear term of highest order with the nonlinear term. Then (18) is of the form

$$Q(f) = A_0 + A_1 R(f).$$

Inserting the above value of $Q(f)$ into (15), using the Riccati equation (16) and splitting on powers of $R(f)$, result in the algebraic system of equations given by

$$\begin{aligned}
&2\,\alpha\,A_1m^3+3\,\alpha\,A_1{}^2m^2+\,\alpha\,A_1{}^3m=0,\\
&12\,\alpha\,lm^2A_1+15\,\alpha\,lmA_1{}^2+3\,\alpha\,lA_1{}^3+6\,\alpha\,m^2A_0A_1+6\,\alpha\,mA_0A_1{}^2\\
&+2\,\beta\,m^2A_1+2\,\beta\,mA_1{}^2=0,\\
&\alpha\,A_1l^2n+3\,\alpha\,A_1A_0ln+2\,\alpha\,A_1mn^2+3\,\alpha\,A_1{}^2n^2+3\,\alpha\,A_1A_0{}^2n\\
&+\beta\,A_1ln+2\,\beta\,A_1A_0n-\nu\,A_1n=0,\\
&7\,\alpha\,l^2mA_1+6\,\alpha\,l^2A_1{}^2+9\,\alpha\,lmA_0A_1+6\,\alpha\,lA_0A_1{}^2+8\,\alpha\,m^2nA_1\\
&+12\,\alpha\,mnA_1{}^2+3\,\alpha\,mA_0{}^2A_1+3\,\alpha\,nA_1{}^3+3\,\beta\,lmA_1+2\,\beta\,lA_1{}^2\\
&+2\,\beta\,mA_0A_1-m\nu\,A_1=0,\\
&\alpha\,l^3A_1+3\,\alpha\,l^2A_0A_1+8\,\alpha\,lmnA_1+9\,\alpha\,lnA_1{}^2+3\,\alpha\,lA_0{}^2A_1+6\,\alpha\,mnA_0A_1\\
&+6\,\alpha\,nA_0A_1{}^2+\beta\,l^2A_1+2\,\beta\,lA_0A_1+2\,\beta\,mnA_1+2\,\beta\,nA_1{}^2-l\nu\,A_1=0.
\end{aligned}$$

By utilizing the Maple software, the solution set of the above system is

$$A_0=-\frac{1}{3\alpha}\left(\beta+3\alpha l\right),\;\; A_1=-2m,\;\; \nu=\frac{1}{3\alpha}\left(3\alpha^2l^2-12\alpha^2mn-\beta^2\right).$$

Therefore, the solutions for the mDDB equation (3) corresponding to the Riccati equation as simplest equation are

$$u(t,x)=-\frac{1}{3\alpha}\left(\beta+3\alpha l\right)-2m\left\{-\frac{l}{2m}-\frac{\theta}{2m}\tanh\left[\frac{1}{2}\theta(f+S)\right]\right\} \tag{23}$$

and

$$\begin{aligned}
u(t,x)=&-\frac{1}{3\alpha}\left(\beta+3\alpha l\right)-2m\left\{-\frac{l}{2m}-\frac{\theta}{2m}\tanh\left(\frac{1}{2}\theta f\right)\right.\\
&\left.+\frac{\operatorname{sech}\left(\frac{\theta f}{2}\right)}{S\cosh\left(\frac{\theta f}{2}\right)-\frac{2m}{\theta}\sinh\left(\frac{\theta f}{2}\right)}\right\},
\end{aligned} \tag{24}$$

where S is a constant of integration and $\theta^2=l^2-4mn$.

Solutions of (15) using the Bernoulli equation

As before, the balancing procedure yields $M=1$, and (18) takes the form

$$Q(f)=A_0+A_1R(f). \tag{25}$$

By substituting the value of $Q(f)$ from (25) into (15), utilizing the Bernoulli equation (17), and splitting on powers of $R(f)$, we get the following system of four algebraic equations in terms of A_0 and A_1:

$$\begin{aligned}
&2\,\alpha\,A_1l^3+3\,\alpha\,A_1{}^2l^2+\,\alpha\,A_1{}^3l=0,\\
&\alpha\,A_1m^3+3\,\alpha\,A_1A_0m^2+3\,\alpha\,A_1A_0{}^2m+\beta\,A_1m^2+2\,\beta\,A_1A_0m-\nu\,A_1m=0,\\
&12\,\alpha\,A_1l^2m+6\,\alpha\,A_1A_0l^2+15\,\alpha\,A_1{}^2lm+6\,\alpha\,A_1{}^2A_0l+3\,\alpha\,A_1{}^3m+2\,\beta\,A_1l^2\\
&+2\,\beta\,A_1{}^2l=0,\\
&7\,\alpha\,A_1lm^2+9\,\alpha\,A_1A_0lm+3\,\alpha\,A_1A_0{}^2l+6\,\alpha\,A_1{}^2m^2+6\,\alpha\,A_1{}^2A_0m+3\,\beta\,A_1lm\\
&+2\,\beta\,A_1A_0l+2\,\beta\,A_1{}^2m-\nu\,A_1l=0.
\end{aligned}$$

Using Maple, we solve the above system of equations and obtain

$$A_0 = -\frac{1}{3\alpha}(\beta + 3\alpha m),\ \ A_1 = -2l,\ \ \nu = \frac{1}{3\alpha}(3\alpha^2 m^2 - \beta^2).$$

Consequently, the solutions of the mDDB equation (3) are

$$u(t,x) = -\frac{1}{3\alpha}(\beta + 3\alpha m) - 2l\left\{\frac{-mc_1}{l\left(c_1 + \cosh\left(m\left(f + c_0\right)\right) - \sinh(m\left(f + c_0\right))\right)}\right\} \tag{26}$$

and

$$u(t,x) = -\frac{1}{3\alpha}(\beta + 3\alpha m) - 2l\left\{\frac{-m(\cosh\left(m(f + c_0)\right) + \sinh\left(m(f + c_0)\right))}{l(c_2 + \cosh[m\left(f + c_0\right)) + \sinh\left(m(f + c_0)\right))}\right\}, \tag{27}$$

where $f = x - \nu t$ and c_0, c_1, c_2 are constants.

Case 2. $\mathcal{R}_2$

We now consider the second member of the optimal sytem $\mathcal{R}_2 = \partial/\partial x$. Utilizing the associated Lagrangian system, we get t and $u = D(t)$ as the invariants. Using these invariants, the mDDB equation (3) is reduced to $D'(t) = 0$, which yields the solution of (3) as $u(t,x) = C_1$, where C_1 is a constant.

Case 3. $\mathcal{R}_3$

Finally, the last member of the optimal system, namely

$$\mathcal{R}_3 = 9\alpha t\frac{\partial}{\partial t} + \left(3\alpha x - 2\beta^2 t\right)\frac{\partial}{\partial x} - (\beta + 3\alpha u)\frac{\partial}{\partial u}$$

yields the invariant solution

$$u(t,x) = W(\lambda)/t^{\frac{1}{3}} - \beta/(3\alpha),\quad \lambda = (3\alpha x + \beta^2 t)/(3\alpha t^{\frac{1}{3}}).$$

The mDDB equation (3) reduces to the nonlinear ODE

$$9\alpha W^2 W' + 9\alpha W'^2 - \lambda W' - W + 9\alpha W W'' + 3\alpha W''' = 0. \tag{28}$$

Solutions of (28) via the power series method

Suppose that the solution of equation (28) can be expressed by

$$W(\lambda) = \sum_{m=0}^{\infty} N_m \lambda^m, \tag{29}$$

where $N_m (m = 1, 2, 3, \dots)$ are constants to be determined. Then we have

$$W'(\lambda) = \sum_{m=0}^{\infty}(m+1)N_{m+1}\lambda^m,\ \ W''(\lambda) = \sum_{m=0}^{\infty}(m+2)(m+1)N_{m+2}\lambda^m,$$
$$W'''(\lambda) = \sum_{m=0}^{\infty}(m+3)(m+2)(m+1)N_{m+3}\lambda^m.$$

Substituting the above values of $W(\lambda)$, $W'(\lambda)$, $W''(\lambda)$, and $W'''(\lambda)$ into equation (28), we get

$$\begin{aligned}
&9\alpha\left[\sum_{m=0}^{\infty} N_m\lambda^m\right]\left[\sum_{m=0}^{\infty} N_m\lambda^m\right]\left[\sum_{m=0}^{\infty}(m+1)N_{m+1}\lambda^m\right]\\
&+ 9\alpha\left[\sum_{m=0}^{\infty}(m+1)N_{m+1}\lambda^m\right]\left[\sum_{m=0}^{\infty}(m+1)N_{m+1}\lambda^m\right] - \sum_{m=0}^{\infty} mN_m\lambda^m\\
&- \sum_{m=0}^{\infty} N_m\lambda^m + 9\alpha\left[\sum_{m=0}^{\infty} N_m\lambda^m\right]\left[\sum_{m=0}^{\infty}(m+2)(m+1)N_{m+2}\lambda^m\right]\\
&+ 3\alpha\sum_{m=0}^{\infty}(m+3)(m+2)(m+1)N_{m+3}\lambda^m = 0.
\end{aligned} \tag{30}$$

Evaluating the above equation (30) on $m = 0, 1$, we obtain the values of N_3 and N_4 in terms of N_0, N_1 and N_2 as

$$\begin{aligned} N_3 = & \frac{1}{18\alpha}\left(N_0 - 9\alpha N_0^2 N_1 - 9\alpha N_1^2 - 18\alpha N_0 N_2\right), \\ N_4 = & \frac{1}{72\alpha}\left(2N_1 - 18\alpha N_1^2 N_2 - 36\alpha N_2^2 - 54\alpha N_1 N_3\right). \end{aligned}$$

Furthermore, the general formula associated with (30) is

$$\begin{aligned} N_{m+3} = & \frac{-1}{3\alpha(m+3)(m+2)(m+1)}\Bigg\{9\alpha\sum_{k=0}^{m}\sum_{i=0}^{k}(m-k+1)N_i N_{k-i} N_{m-k+1} - mN_m \\ & - N_m + 9\alpha\sum_{k=0}^{m}(m-k+2)(m-k+1)N_k N_{m-k+2} + 9\alpha\sum_{k=0}^{m-2}(m-k-1) \\ & \times (k+1)N_{k+1}N_{m-k-1}\Bigg\}, \end{aligned} \tag{31}$$

for $m \geq 0$. Without loss of generality, from equation (31), we have

$$|N_{m+3}| \leq M\left[\sum_{k=0}^{m}\sum_{i=0}^{k}|N_i N_{k-i} N_{m-k+1}| + |N_m| + \sum_{k=0}^{m}|N_k N_{m-k+2}| + \sum_{k=0}^{m-2}|N_{k+1}N_{m-k-1}|\right], \tag{32}$$

for $m = 0, 1, 2, \ldots$, where

$$M = \text{Max}\left\{\frac{9\alpha}{|(m+3)(m+2)(m+1)|}, \frac{-m-1}{|(m+3)(m+2)(m+1)|}\right\}.$$

Now, to prove the convergence of the power series solution of (29), we introduce a new series

$$\eta = T(z) = \sum_{m=0}^{\infty} q_m z^m \tag{33}$$

with $q_i = |N_i|$ $(i = 0, 1, \ldots)$ and

$$q_{m+3} = M\left[\sum_{k=0}^{m}\sum_{i=0}^{k}(q_i q_{k-i} q_{m-k+1}) + q_m + \sum_{k=0}^{m} q_k q_{m-k+2} + \sum_{k=0}^{m-2} q_{k+1} q_{m-k-1}\right], \tag{34}$$

for $m = 0, 1, 2, \ldots$. Thus, from (32) and (34), it becomes evident that

$$|N_m| \leq q_m, \quad m = 0, 1, 2, \ldots.$$

The above equation implies that equation (33) is a majorant series for equation (29). Lastly, we want to show whether the series $\eta = T(z)$ has a positive radius of convergence. The following result can be obtained using formal computation:

$$
\begin{aligned}
T(z) =& q_0 + q_1 z + q_2 z^2 + q_3 z^3 + q_4 z^4 + \sum_{m=2}^{\infty} q_{m+3} z^{m+3} \\
=& M \left[\sum_{m=2}^{\infty} \sum_{k=0}^{m} \sum_{i=0}^{k} q_i q_{k-i} q_{m-k+1} z^{m+3} + \sum_{m=2}^{\infty} q_m z^{m+3} + \sum_{m=2}^{\infty} \sum_{k=0}^{m} q_k q_{m-k+2} z^{m+3} \right. \\
& \left. + \sum_{m=2}^{\infty} \sum_{k=0}^{m-2} q_{k+1} q_{m-k-1} z^{m+3} \right] + q_4 z^4 + q_3 z^3 + q_2 z^2 + q_1 z + q_0 \\
=& M \left[\left(q_0^2 q_6 + 4\, q_0 q_1 q_5 + 4\, q_0 q_2 q_4 + 2\, q_0 q_3^2 + 3\, q_1^2 q_4 + 6\, q_1 q_2 q_3 + q_2^3 + q_0 q_7 \right. \right. \\
& \left. +2\, q_1 q_4 + q_1 q_6 + 2\, q_2 q_3 + 2\, q_2 q_5 + 2\, q_3 q_4 + q_5 \right) z^8 + \left(q_0^2 q_5 + 4\, q_0 q_1 q_4 + 4\, q_0 q_2 q_3 \right. \\
& \left. +3\, q_1^2 q_3 + 3\, q_1 q_2^2 + q_0 q_6 + 2\, q_1 q_3 + q_1 q_5 + q_2^2 + 2\, q_2 q_4 + q_3^2 + q_4 \right) z^7 \\
& + \left(q_0^2 q_4 + 4\, q_0 q_1 q_3 + 2\, q_0 q_2^2 + 3\, q_1^2 q_2 + q_0 q_5 + 2\, q_1 q_2 + q_1 q_4 + 2\, q_2 q_3 + q_3 \right) z^6 \\
& \left. + \left(q_0^2 q_3 + 4\, q_0 q_1 q_2 + q_1^3 + q_0 q_4 + q_1^2 + q_1 q_3 + q_2^2 + q_2 \right) z^5 \right] + q_4 z^4 + q_3 z^3 \\
& + q_2 z^2 + q_1 z + q_0.
\end{aligned}
$$

It is sufficient to show that implicit functional equation $\mathcal{X}(z,\eta)$ is analytic in the neighbourhood of $(0, q_0)$. Now, considering

$$
\begin{aligned}
\mathcal{X}(z,\eta) =& - M \left[\left(q_0^2 q_6 + 4\, q_0 q_1 q_5 + 4\, q_0 q_2 q_4 + 2\, q_0 q_3^2 + 3\, q_1^2 q_4 + 6\, q_1 q_2 q_3 + q_2^3 \right. \right. \\
& \left. +q_0 q_7 + 2\, q_1 q_4 + q_1 q_6 + 2\, q_2 q_3 + 2\, q_2 q_5 + 2\, q_3 q_4 + q_5 \right) z^8 + \left(q_0^2 q_5 + 4\, q_0 q_1 q_4 \right. \\
& \left. +4\, q_0 q_2 q_3 + 3\, q_1^2 q_3 + 3\, q_1 q_2^2 + q_0 q_6 + 2\, q_1 q_3 + q_1 q_5 + q_2^2 + 2\, q_2 q_4 + q_3^2 + q_4 \right) z^7 \\
& + \left(q_0^2 q_4 + 4\, q_0 q_1 q_3 + 2\, q_0 q_2^2 + 3\, q_1^2 q_2 + q_0 q_5 + 2\, q_1 q_2 + q_1 q_4 + 2\, q_2 q_3 + q_3 \right) z^6 \\
& \left. + \left(q_0^2 q_3 + 4\, q_0 q_1 q_2 + q_1^3 + q_0 q_4 + q_1^2 + q_1 q_3 + q_2^2 + q_2 \right) z^5 \right] - q_4 z^4 - q_3 z^3 \\
& - q_2 z^2 - q_1 z - q_0 - \eta,
\end{aligned}
$$

it is evident that $\mathcal{X}\left(z,\eta\right) = \mathcal{X}\left(0,q_0\right) = 0,\ \ \mathcal{X}'_{\eta}\left(0,q_0\right) = 1 \neq 0$. By applying the implicit function theorem, we claim that $\eta = T(z)$ is an analytic function near the point $(0, q_0)$, as described in Rudin (2015); Fichtenholz (1970). This suggest that the power series solution of equation (3) converges in the vicinity of point $(0, q_0)$ in the plane. Thus, the power series solution of the ODE (28) is

$$
W(\lambda) = N_0 + N_1 \lambda + N_2 \lambda^2 + N_3 \lambda^3 + N_4 \lambda^4 + \sum_{m=2}^{\infty} N_{m+3} \lambda^{m+3}. \tag{35}
$$

Therefore, keeping equation (35) in mind, the power series solution of the mDDB equation (3) is given by

$$
\begin{aligned}
u(t,x) = \frac{1}{t^{1/3}} \Bigg\{ & N_0 + N_1 \left(\frac{1}{3\alpha t^{\frac{1}{3}}} (3\alpha x + \beta^2 t) \right) + N_2 \left(\frac{1}{3\alpha t^{\frac{1}{3}}} (3\alpha x + \beta^2 t) \right)^2 \\
& + \frac{1}{18\alpha} \left(N_0 - 9\alpha N_0^2 N_1 - 9\alpha N_1^2 - 18\alpha N_0 N_2 \right) \left(\frac{1}{3\alpha t^{\frac{1}{3}}} (3\alpha x + \beta^2 t) \right)^3 \\
& + \frac{1}{72\alpha} \left(2N_1 - 18\alpha N_1^2 N_2 - 36\alpha N_2^2 - 54\alpha N_1 N_3 \right) \left(\frac{1}{3\alpha t^{\frac{1}{3}}} (3\alpha x + \beta^2 t) \right)^4 \\
& - \sum_{m=2}^{\infty} \Bigg[\frac{1}{3\alpha(m+3)(m+2)(m+1)} \Bigg(9\alpha \sum_{k=0}^{m} \sum_{i=0}^{k} (m-k+1) N_i N_{k-i} \\
& \times N_{m-k+1} - (m+1) N_m + 9\alpha \sum_{k=0}^{m} (m-k+2)(m-k+1) N_k N_{m-k+2} \\
& + 9\alpha \sum_{k=0}^{m-2} (m-k-1)(k+1) N_{k+1} N_{m-k-1} \Bigg) \Bigg] \left(\frac{1}{3\alpha t^{\frac{1}{3}}} (3\alpha x + \beta^2 t) \right)^{m+3} \Bigg\} \\
& - \frac{\beta}{3\alpha}.
\end{aligned}
$$

2.4 Results and discussion

In this section, we give different graphical representations to demonstrate the physical look of the solutions of the mixed doubly nonlinear dispersive Burgers equation (3) solutions. Thus, to view the dynamics of the solutions, we assign values to the constants in the solutions. Figure 1 depicts the dynamics of the solution (23) in 3D, 2D and density plots when the speed of the wave $\nu = -0.1$ and other parameters $\alpha = 1.2$, $\beta = 2$, $l = 30$, $m = -5$, $n = 5$, $S = 0$, $-5 \leq t, x \leq 5$.

Similarly, for the dynamics of the solution (24), in Figure 2, we assign different values $\nu = -0.1$, $\alpha = 1.2$, $\beta = 2$, $l = 30$, $m = -5$, $n = 5$, $S = -4$, $-10 \leq t \leq 10$, and $-5 \leq x \leq 10$.

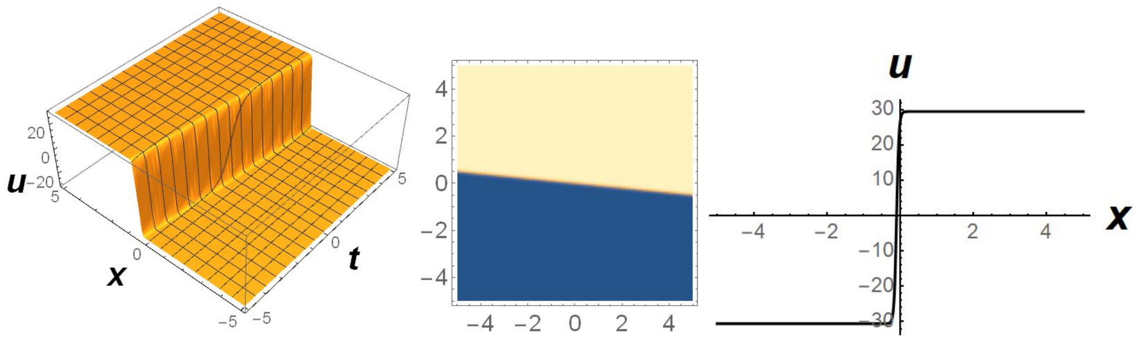

Figure 1: Kink shape graph of solution (23) in 3D, density and 2D for $-5 \leq t, x \leq 5$.

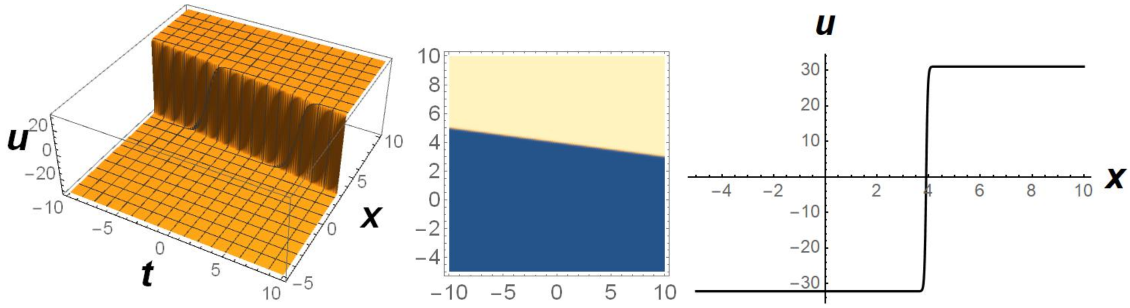

Figure 2: Profile of (24) in 3D, density and 2D for $-10 \le t \le 10$, $-5 \le x \le 10$.

Moreover, for Figure 3 representing solution (26), we assign $\alpha = 1.4$, $\beta = 3$, $\nu = -0.1$, $l = 0.5$, $m = 3$, $c_0 = 0$, $c_1 = 1$, $-5 \le t, x \le 5$, for 3D, density and 2D plots.

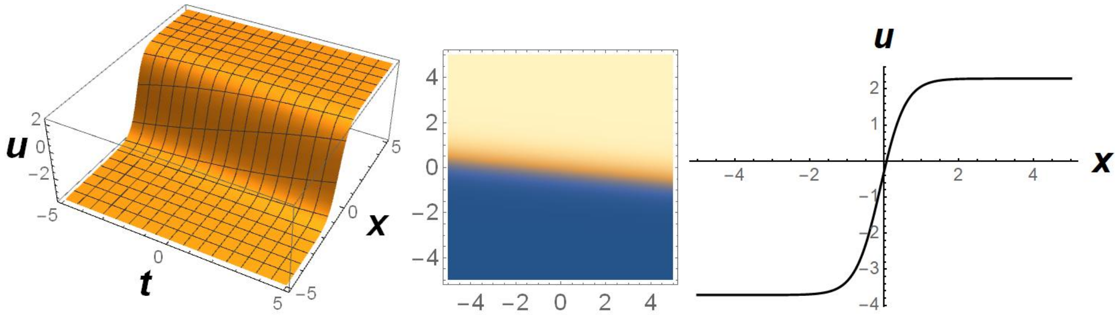

Figure 3: Profile of solution of (26) in 3D, density and 2D for $-5 \le t, x \le 5$.

Furthermore, for Figure 4 which represents solution (27), we choose $\alpha = 1.4$, $\beta = 3$, $\nu = -0.1$, $l = 0.5$, $m = 3$, $c_0 = 0$, $c_2 = 1$, $-10 \le t \le 10$, $-5 \le x \le 10$ for 3D, density and 2D plots.

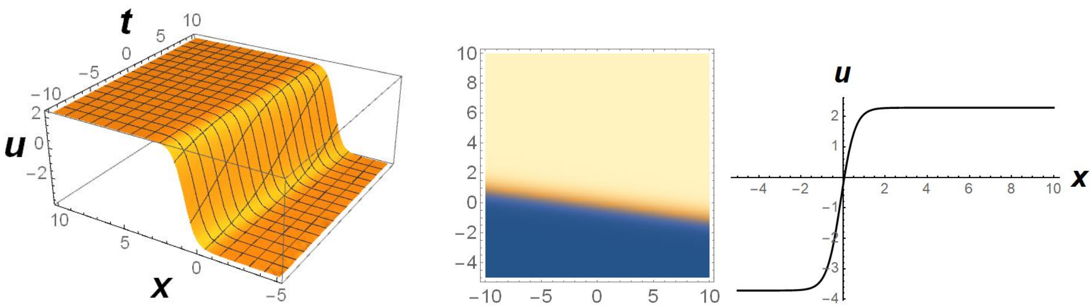

Figure 4: Profile of (27) in 3D, density and 2D for $-10 \le t \le 10$, $-5 \le x \le 10$.

3 Conservation laws for (3)

Conservation laws play a crucial role in solving DEs. Identifying these laws is usually the first step in deriving solutions to DEs. Also, the presence of numerous conservation laws in a PDE suggests that the PDE is integrable (Olver, 1993). Here we obtain conservation laws for the nDDB equation (3) by invoking two different techniques; the multiplier method (Olver, 1993) and the conservation theorem due to Ibragimov (Ibragimov, 2007).

3.1 Conserved vectors of (3) using multiplier method

We employ the multiplier technique to get conserved vectors of the nDDB equation (3). Accordingly, we compute the zeroth-order multipliers $\mathcal{I}(t,x,u)$ by using its determining equation

$$\frac{\delta}{\delta u}\left\{\mathcal{I}(t,x,u)\left(u_t+3\alpha u^2u_x+3\alpha u_x^2+3\alpha uu_{xx}+2\beta uu_x+\beta u_{xx}+\alpha u_{xxx}\right)\right\}=0, \tag{36}$$

where $\delta/\delta u$ is the Euler operator

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}-D_t\frac{\partial}{\partial u_t}-D_x\frac{\partial}{\partial u_x}+D_x^2\frac{\partial}{\partial u_{xx}}-D_x^3\frac{\partial}{\partial u_{xxx}}+\cdots,$$

and D_t and D_x stand for total derivatives. Expanding (36) and splitting on derivatives of u we obtain a system of PDEs, Solving this system, we obtain $\mathcal{I}=C_1$, where C_1 is a constant. Consequently, the multiplier is $\mathcal{I}=1$. A multiplier $\mathcal{I}$ for the mDDB equation (3) has the property that

$$\mathcal{I}[u_t+3\alpha u^2u_x+3\alpha u_x^2+3\alpha uu_{xx}+2\beta uu_x+\beta u_{xx}+\alpha u_{xxx}]=D_tT^t+D_xT^x, \tag{37}$$

where T^t represents the conserved density and T^x is the spatial fluxes. Hence, from the above equation, we obtain the following conserved vector:

$$\begin{aligned}T^t &=u,\\ T^x &=\alpha u^3+3\alpha uu_x+\beta u^2+\alpha u_{xx}+\beta u_x.\end{aligned}$$

Remark. It should be noted that the multiplier $\mathcal{I}=1$ reveals that the mDDB equation (3) is itself in conserved form. Moreover, it is imperative to emphasize that the first-order and second-order multipliers of the mDDB equation (3) also produce the same multiplier as the zeroth-order multiplier that is given above.

3.2 Conserved vectors of (3) using Ibragimov's method

We derive the conserved vectors of the mDDB equation (3) using the Ibragimov's approach (Ibragimov, 2007). With this approach, a distinct connection between every infinitesimal generator and a conserved quantity is produced. We first write down the adjoint equation of (3), which is

$$F^*\equiv\frac{\delta}{\delta u}\left[v\left\{u_t+3\alpha u^2u_x+3\alpha u_x^2+3\alpha uu_{xx}+2\beta uu_x+\beta u_{xx}+\alpha u_{xxx}\right\}\right]=0, \tag{38}$$

where the new variable v depends on t and x and

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}-D_t\frac{\partial}{\partial u_t}-D_x\frac{\partial}{\partial u_x}-D_x^2\frac{\partial}{\partial u_{xx}}-D_x^3\frac{\partial}{\partial u_{xxx}}+\cdots$$

is the Euler-Lagrange operator with the total derivatives

$$\begin{aligned}D_t &= \frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+v_t\frac{\partial}{\partial v}+u_{tt}\frac{\partial}{\partial u_t}+v_{tt}\frac{\partial}{\partial v_t}+u_{tx}\frac{\partial}{\partial u_x}+v_{tx}\frac{\partial}{\partial v_x}+\cdots,\\ D_x &= \frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+v_x\frac{\partial}{\partial v}+u_{xx}\frac{\partial}{\partial u_x}+v_{xx}\frac{\partial}{\partial v_x}+u_{xt}\frac{\partial}{\partial u_t}+v_{xt}\frac{\partial}{\partial v_t}+\cdots.\end{aligned} \tag{39}$$

Expanding equation (38) we get the adjoint equation as

$$3\alpha uv_{xx}-2\beta uv_x-3\alpha v_xu^2+6\alpha u_{xx}v-v_t+6\alpha u_xv_x-\alpha v_{xxx}+\beta v_{xx}=0. \tag{40}$$

The formal Lagrangian for equation (3) and its adjoint equation (40) is

$$\mathcal{L}=v\left\{u_t+3\alpha u^2u_x+3\alpha u_x^2+3\alpha uu_{xx}+2\beta uu_x+\beta u_{xx}+\alpha u_{xxx}\right\}. \tag{41}$$

The following generator is obtained by extending the Lie symmetries of (3) to the recently introduced variable $v(t,x)$:

$$\mathcal{K} = \Phi^1 \frac{\partial}{\partial t} + \Phi^2 \frac{\partial}{\partial x} + \Phi^3 \frac{\partial}{\partial u} - \eta_1 \frac{\partial}{\partial v},$$

where $\eta_1 = v[\lambda + D_t(\Phi^1) + D_x(\Phi^2)]$ and the value of λ is determined from $\mathcal{R}^{[2]}(F) = \lambda(F)$. The following formulae (Ibragimov, 2007) will be used to calculate the conserved vectors corresponding to the symmetry generators based on the Lagrangian (41):

$$C^i = \xi^i Ł + W^\alpha \left(\frac{\partial Ł}{\partial u_i^\alpha} - D_k \frac{\partial Ł}{\partial u_{ik}^\alpha} \right) + D_k(W^\alpha) \frac{\partial Ł}{\partial u_{ik}^\alpha}, \tag{42}$$

where W^α is the Lie characteristic function given by $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$ (Ibragimov, 2007). Using the formula (42), we compute the conserved vectors corresponding to the three Lie symmetries of (3).

Case 1. We first take the symmetry $\mathcal{R}_1 = \partial/\partial t$ into consideration. The energy conserved quantity vector (T_1^t, T_1^x) associated to symmetry $\mathcal{R}_1$ is given by

$$\begin{aligned}
T_1^t = & 3\alpha u_x u^2 v + 3\alpha u_{xx} uv + \alpha u_{xxx} v + 2\beta u_x uv + \beta u_{xx} v, \\
T_1^x = & -3\alpha u_t u^2 v + 3\alpha u_t v_x u - 3\alpha uv u_{tx} + 3\alpha u_t u_x v - \alpha v u_{txx} - 2\beta u_t uv - \beta v u_{tx} \\
& - \alpha u_t v_{xx} + \alpha v_x u_{tx} + \beta u_t v_x.
\end{aligned}$$

Case 2. For the second symmetry $\mathcal{R}_2 = \partial/\partial x$, the associated momentum conserved vector (T_2^t, T_2^x) is given by

$$\begin{aligned}
T_2^t = & u_x - v, \\
T_2^x = & 3\alpha u_x^2 v + 3\alpha u_x v_x u + u_t v - \alpha u_x v_{xx} + \alpha u_{xx} v_x + \beta u_x v_x.
\end{aligned}$$

Case 3. The symmetry $\mathcal{R}_3 = 9\alpha t\partial/\partial t + (3\alpha x - 2\beta^2 t)\partial/\partial x - (\beta + 3\alpha u)\partial/\partial u$ yields the associated conserved vector (T_3^t, T_3^x) as

$$\begin{aligned}
T_3^t = & 27\alpha^2 t u_x u^2 v + 27\alpha^2 t u_{xx} uv + 9\alpha^2 t u_{xxx} v + 18\alpha\beta t u_x uv + 9\alpha\beta t u_{xx} v - 3\alpha x u_x v \\
& + 2\beta^2 t u_x v - 3\alpha uv - \beta v, \\
T_3^x = & 9\alpha^2 x u_x^2 v - 9\alpha^2 u_x uv + 9\alpha^2 v_x u^2 + 9\alpha^2 x u_x v_x u - 9\alpha^2 u_{xx} v - 3\alpha^2 v_{xx} u \\
& - 27\alpha^2 t u_t u^2 v + 27\alpha^2 t u_t u_x v + 27\alpha^2 t u_t v_x u - 27\alpha^2 t uv u_{tx} - 9\alpha^2 t v u_{txx} \\
& - 6\alpha\beta^2 t {u_x}^2 v - 6\alpha\beta^2 t u_x v_x u - 3\alpha\beta u_x v + 6\alpha\beta v_x u - 18\alpha\beta t u_t uv - 9\alpha\beta t v u_{tx} \\
& + 3\alpha x u_t v - 2\beta^2 t u_t v - 9\alpha^2 u^3 v - 9\alpha\beta u^2 v - 2\beta^2 uv - 9\alpha^2 t u_t v_{xx} + 9\alpha^2 t v_x u_{tx} \\
& - 2\alpha\beta^2 t u_{xx} v_x + 2\alpha\beta^2 t u_x v_{xx} + 9\alpha\beta t u_t v_x - 2\beta^3 t u_x v_x + 6\alpha^2 u_x v_x + 3\alpha^2 x u_{xx} v_x \\
& - 3\alpha^2 x u_x v_{xx} + 3\alpha\beta x u_x v_x - \alpha\beta v_{xx} + \beta^2 v_x.
\end{aligned}$$

4 Concluding remarks

To summarize the work done in this paper, we discovered new exact solutions to the mixed doubly nonlinear dispersive Burgers equation (3) by utilizing Lie group analysis as well as the simplest equation method and the power series expansion method. We demonstrated that this equation has several forms of solutions that included kink-shaped solutions. Moreover, to demonstrate the dynamics of the solution profiles, a graphical depiction of the discovered solutions for acceptable parameter values was also given. Futhermore, using the general multilier technique, we found out that the mixed doubly nonlinear dispersive Burgers equation (3) is itself in conserved form. Moreover, the conserved quantity of energy and linear momentum amongst others were extracted from the mixed doubly nonlinear dispersive Burgers equation (3) using Ibragimov's method. Lastly, it should be noted that the novel solutions developed in this paper may have a substantial influence on future research.

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