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APPROXIMATION BY MICCHELLI COMBINATION OF HALF BERNSTEIN POLYNOMIALS

Hadeel O. Muslim[1*], **Waffa F. Keidan**[1], **Amal K. Hassan**[2]

[1]Department of Mathematics, College of Sciences, Diyala University, Diyala, Iraq
[2]Department of Mathematics, College of Science, University of Basra, Iraq

Abstract. This paper improves the rate of convergence of the half Bernstein polynomial of positive and linear operators $\mathcal{H}_{n,2k}(g;x)$ by applying the technique of iterative combination (Micchelli combination). It turns out that this method improves the order of approximations to $O(n^{-k})$. Next, using this sequence, we investigate the Voronovskaja asymptotic expression in both ordinary and instantaneous approximating for smooth functions. The mistake caused by this approximation is then estimated by means of the modulus for continuity. Next, in order for approximation assessment functions, we provide numerical illustrations via this iterative combination of the sequence $\mathcal{H}_{n,2k}(g;x)$. Lastly, we contrast the repeated combination's outcomes with the numerical outcomes obtained from the sequence $\mathcal{H}_{n,2k}(g;x)$ without iterative combination.

Keywords: Iterative combinations, Micchelli combination, ordinary approximation, simultaneous approximation.
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Corresponding author: Hadeel, Muslim, Department of Mathematics, College of Sciences, Diyala University, Diyala, Iraq, e-mail: *hadeel.o.alkhaled@uodiyala.edu.iq, hadeel.oalkh@gmail.com*
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1 Introduction

Voronovskaja (1932) has demonstrated that approximates ordering by the sequence of Bernstein $B_n(f;x)$ is $O(n^{-1})$. Kelisky & Rivlin (1967) studied the convergence of the iterates $B_n^k(f;x) = B_n(B_n^{k-1}(f);x)$ as $k \to \infty$, $n \to \infty$ and k independent of n.

Karlin & Ziegler (1970) presented a study of how $k(n)$-th repeats of approximations sequences behave when n and $k(n)$ decrease toward infinity. Following this work, Micchelli (1973) presented a novel approach to use the recurrent combinations $L_{n,k}(f(t);x) = (I - (I - B_n)^k)f(t);x$, with $k \in \mathbb{N}$, to increase the order of the Bernstein sequence. For certain sequences, Agrawal (1992); Agrawal & Gupta (1990), and Agrawal & Kasana (1984) used the Micchelli approach to increase their orders in both the ordinary and contemporaneous approximations. After then, several attempts are made to speed up the procedure of approximation for different sequences. However, there are no numerical examples in any of these research (Khosravian-Arab et al., 2018; Micchelli, 1973; Mohammad & Muslim, 2022; Sinha et al., 2009; Micchelli, 1973; Gasimov & Napoles-Valdés, 2022; Stan, 2014; Mohammad & Muslim, 2019,0,0). Many researchers have recently tried to improve the degree of approximation, some of them Jabbar & Hassan (2024); Kaur & Goyal (2024); Mohammad & Hassan (2021); Jabber & Hassan (2025).

For $f \in C_\alpha[0+\sigma, 1-\sigma]$, the sequence $\mathcal{H}_n(g;)$ is defined as

$$\mathcal{H}_{n,2k}(g(t);x) = 2\sum_{k=0}^{\lfloor n/2 \rfloor} \ell_{n,2k}(x) g\left(\frac{2k}{n}\right),$$

where $\ell_{n,2k}(x) = \binom{n}{2k}(1-x)^{n-2k}x^{2k}$, $x \in [0+\sigma, 1-\sigma]$.

We define the iterates of the sequence $\mathcal{H}_{n,2k}(g(t);x)$ as

$$\mathcal{H}^0_{n,2k} = I \quad \text{and} \quad \mathcal{H}^r_{n,2k} = \mathcal{H}_{n,2k}(\mathcal{H}^{r-1}_{n,2k}), \quad r \in \mathbb{N}.$$

Suppose that $k \in \mathbb{N}$ and the Micchelli combination of the sequences $\mathcal{H}_{n,2k}(g(t);x)$ is denoted by $M_{n,2k} : C_\alpha[0+\sigma, 1-\sigma] \to C^\infty[0+\sigma, 1-\sigma]$ and is defined as

$$M_{n,2k}(g(t);x) = \Big(I - (I - \mathcal{H}_{n,2k})^k\Big)(g(t);x), \tag{1}$$

which expands as

$$= \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} \mathcal{H}^r_{n,2k}(g(t);x).$$

We assume that $0 < a_1 < a_2 < b_2 < b_1 < \infty$ and $I_i = [a_i, b_i]$, $i = 1, 2$.

The main goal of using the Micchelli combination is to improve the order of approximation to $H_{n,2k}(f;x)$.

This paper shows that the k-th iterative combination of the sequence $\mathcal{H}_{n,2k}(g(t);x)$ has an order of approximation $O(n^{-k})$, and this is an improvement of $\mathcal{H}_{n,2k}(g;x)$, which has an order of approximation of $O(n^{-1})$. Additionally, we provide a Voronovskaja equation for smooth functions in both ordinary and simultaneous approximations via the sequence $M_{n,2k}(g(t);x)$.

The mistake resulting from this approximate result is then estimated by means of the modulus of continuity. Lastly, we estimate three test functions using a few mathematical samples $g(t) = \sin(10t)e^{-t}$ by the sequences $M_{n,2k}(g(t);x)$ and $\mathcal{H}_{n,2k}(g(t);x)$. In addition, we determine the maximum number of errors that can occur via these approximations and compare the numerical results obtained.

2 Preliminary Results

Here, we give the following:

Lemma 1. *(Zygmund, 1935) Let $1 \le p < \infty$, $f \in L_p[a,b]$, $g \in AC[a,b]$ (the space of absolutely continuous functions on $[a,b]$) and $g^{(k+1)} \in L_p[a,b]$. Then:*

$$\|g^{(j)}\|_{L_p[a,b]} \le C_j \le \Big\{\|g^{(k+1)}\|_{L_p[a,b]} + \|g\|_{L_p[a,b]}\Big\}, \quad j = 1, 2, ..., k,$$

where C_j's are certain constants depending on $j, k, p, a,$ and b.

Lemma 2. *(Mohammad & Muslim, 2024) For the function $Y_{n,m}(x)$ one has*

(i) $Y_{n,0}(x) = 1 + (1-2x)^n$;

(ii) $Y_{n,1}(x) = 2x(x-1)(1-2x)^{n-1}$.

(iiI) $Y_{n,2}(x) = x(1-x)\frac{1}{n}\left[1 + (1-2x)^{n-2} + x^2(1-2x)^{n-2}\left[1 + (1-2x)^2\right] + 2x(1-2x)^{n-1}\right]$.

Now, $Y_{n,m}(x)$ has the following recurrence relations

$$nY_{n,m+1}(x) = x(1-x)m\,Y_{n,m-1}(x) + x(1-x)Y'_{n,m}(x), \quad m \ge 2. \tag{2}$$

Further, one gets

1) $Y_{n,m}(x)$ is a polynomial in x of degree m for sufficiently large n.

2) For every $x \in [0+\sigma, 1-\sigma]$, $Y_{n,m}(x) = O\left(n^{-\lfloor m+1/2\rfloor}\right)$.

Lemma 3. *(Mohammad & Muslim, 2024) Suppose that $[a,b] \subset (0,\infty)$ and that δ and α are both positive real characters. Therefore, if $\lambda > 0$, we get:*

$$\sup_{x\in[a,b]}\left|\sum_{k=0}^{\lfloor n/2\rfloor} \ell_{n,2k}(x)t^{\alpha}\right| = O(n^{-\beta}).$$

The demonstration for this lemma is straightforward and relies on lemma 2 (ii), Sturm's inequality, and Taylor's growth.

Definition 1. *(Mohammad & Muslim, 2024)*

For every $m \in \mathbb{N}^0$, the m-th rank $\mathcal{Y}^{(p)}_{n,m}(x)$ of sequence $\mathcal{H}^{(p)}_{n,2k}(x)$ where $p \in \mathbb{N}$ is defined as

$$\mathcal{Y}^{(p)}_{n,m}(x) = \mathcal{H}^{(p)}_{n,2k}((x-x)^m; x). \tag{3}$$

We denote $\mathcal{Y}^{(1)}_{n,m}(x) = Y_{n,m}(x)$.

Lemma 4. *For $p \in \mathbb{N}$ and $x \in [0+\sigma, 1-\sigma]$, we possess*

$$\mathcal{Y}^{(p+1)}_{n,m}(x) = \sum_{j=0}^{m}\binom{m}{j}\sum_{i=0}^{m-j}\binom{m-j}{i} Y_{n,i+j}(x)\frac{D^i}{i!}\left(\mathcal{Y}^{(p)}_{n,m-j}(x)\right), \tag{4}$$

where $D^i \equiv \frac{d^i}{dx^i}$.

Proof. By the above definition, we have

$$\begin{aligned}\mathcal{Y}^{(p+1)}_{n,m}(x) &= \mathcal{H}_{n,2k}\left(\mathcal{Y}^{(p)}_{n,m}((t-x)^m; u); x\right)\\ &= \mathcal{H}_{n,2k}\left(\mathcal{H}^{(p)}_{n,2k}((t-u+u-x)^m; u); x\right).\end{aligned}$$

$$\sum_{j=0}^{m}\binom{m}{j}\mathcal{H}_{n,2k}((t-u)^j\mathcal{H}_{n,2k}((u-x)^{m-j}; u); x).$$

Since $\mathcal{H}_{n,2k}((t-u)^m; u)$ is a polynomial in u of degree $< m-j$, for sufficiently large n, Taylor's expansions allow us to state:

$$\mathcal{H}_{n,2k}((t-u)^m; u) = \sum_{i=0}^{m-j}\frac{(u-x)^i}{i!}D^i\left(\mathcal{Y}^{(p)}_{n,m-j}(x)\right).$$

Hence, (3) is immediate. □

Lemma 5. *For $x \in [0+\sigma, 1-\sigma]$, we have:*

$$\mathcal{Y}^{(p)}_{n,m}(x) = O\left(n^{-\lfloor\frac{m+1}{2}\rfloor}\right). \tag{5}$$

Proof. For p, we demonstrate via induction. For $p = 1$, the outcome of Lemma 2 is true.

We will prove the outcome for $p+1$ if it is accurate for p.

Presently, $\mathcal{Y}^{(p)}_{n,m-j}(x) = O\left(n^{-\lfloor\frac{m+1}{2}\rfloor}\right)$ and $\mathcal{Y}^{(p)}_{n,m-j}(x)$ represents a degree polynomial in $x < m-j$. Consequently:

$$D^i\left(\mathcal{Y}^{(p)}_{n,m-j}(x)\right) = O\left(n^{-\lfloor\frac{m+1}{2}\rfloor}\right), \quad \text{for every } i.$$

Now, by using Lemma 4, we get

$$\mathcal{Y}_{n,m}^{(p+1)}(x) = O\left(\sum_{j=0}^{m}\sum_{i=0}^{m-j} n^{-\lfloor\frac{m-j+1}{2}\rfloor} n^{-\lfloor\frac{i+1}{2}\rfloor}\right)$$

$$= O\left(\sum_{j=0}^{m}\sum_{i=0}^{m-j} n^{-\lfloor\frac{m+i+1}{2}\rfloor}\right)$$

$$= O\left(n^{-\lfloor\frac{m+1}{2}\rfloor}\right).$$

The outcome is thus valid for $p+1$. Thus, using the induction theory, we arrive at the result (5). □

Lemma 6. *From the moment of m-th rank, $m \in \mathbb{N}$ for operators $M_{n,2k}$ defined in (1), we find that:*

$$M_{n,2k}((t-x)^m; x) = O(n^{-k}).$$

Proof. The proof is done via induction on k. Lemma 2 yields a true conclusion for $k = 1$. We will show the lemma at $k+1$ if it is accurate for k.

$$M_{n,2k+1}((t-x)^m; x) = \sum_{p=1}^{2k+1} (-1)^{p+1} \binom{2k+1}{p} \mathcal{Y}_{n,m}^{(p)}((t-x)^m; x)$$

$$= \sum_{p=1}^{2k+1} (-1)^{p+1} \binom{2k+1}{p} \mathcal{Y}_{n,m}^{(p)}(x) + \sum_{p=1}^{2k+1} (-1)^{p+1} \binom{2k+1}{p} \left(\frac{2k}{p-1}\right) \mathcal{Y}_{n,m}^{(p)}(x),$$

$$:= \Sigma_1 + \Sigma_2.$$

Clearly,

$$\Sigma_1 = M_{n,2k}((t-x)^m; x).$$

Now, using Lemma 2.7, we have:

$$\Sigma_2 = \sum_{p=0}^{2k} (-1)^{p+1} \binom{2k}{p} \mathcal{Y}_{n,m}^{(p+1)}(x)$$

$$= \mathcal{Y}_{n,m}(x) - \left\{ \sum_{j=1}^{m-1} \binom{m}{j} \sum_{i=0}^{m-j} \binom{m-j}{i} \mathcal{Y}_{n,i+j}(x) \frac{D^i}{i!} (M_{n,2k}((t-x)^{m-j}; x)) \right.$$

$$\left. + \sum_{i=1}^{m} \binom{m}{i} \frac{D^i}{i!} (M_{n,k}((t-x)^m; x)) + M_{n,k}((t-x)^m; x) \right\}.$$

Therefore,

$$\Sigma_1 + \Sigma_2 = \sum_{j=1}^{m-1} \binom{m}{j} \sum_{i=0}^{m-j} \binom{m-j}{i} \mathcal{Y}_{n,i+j}(x) \frac{D^i}{i!} (M_{n,k}((t-x)^{m-j}; x)).$$

$$+ \sum_{i=1}^{m} \binom{m}{i} \frac{D^i}{i!} (M_{n,k}((t-x)^m; x)) = O(n^{-k+1}).$$

□

3 Ordinary Approximation

Firstly, we must determine the sequence's asymptotic expression $M_{n,2k}$.

Theorem 1. *Let $g \in C_\alpha[0+\sigma, 1-\sigma]$. If $g^{(2k)}(x)$ exists at a point $x \in [0+\sigma, 1-\sigma]$, then*

$$\lim_{n\to\infty} n^k \left\{M_{n,2k}(g;x) - g(x)\right\} = \sum_{i=1}^{4k} \frac{g^{(i)}(x)}{i!} Q(i,k,x), \tag{6}$$

and

$$\lim_{n\to\infty} n^k \left\{M_{n,2k+1}(g;x) - g(x)\right\} = 0. \tag{7}$$

involving the polynomials $Q(i,k,x)$ in x having a maximum rank of $i-1$.

Additionally, if $g^{(2k-1)}$ occurs and is completely continuous in the range $[0,b]$, with $g^{(2k)} \in L_\infty[0,b]$, so whatever $[c,d] \subseteq (0,b)$ possesses:

$$\|M_{n,2k}(g;x) - g(x)\|_{[c,d]} \le Cn^{-k}\left(\|g\|_{L_\alpha} + \|g^{(2k)}\|_{L_\infty[0,b]}\right). \tag{8}$$

Proof. By Taylor's expansion of g, we get

$$g(t) = \sum_{j=0}^{4k} \frac{g^{(j)}(x)}{j!}(t-x)^j + \varepsilon(t)(t-x)^{4k},$$

where $\varepsilon(t) \to 0$ as $t \to x$ and $\|\varepsilon(t)\|_{[c,d]} \le Ct^a$ for some $C > 0$. Therefore

$$n^k\{M_{n,2k}(g;x) - g(x)\} = n^k \sum_{j=1}^{4k} \frac{g^{(j)}(x)}{j!} M_{n,2k}((t-x)^j;x) + n^k M_{n,2k}(\varepsilon(t)(t-x)^{4k};x),$$

$$:= \Sigma_1 + \Sigma_2.$$

Using Lemma 6, we obtain:

$$\Sigma_1 = n^k \sum_{j=1}^{4k} \frac{g^{(j)}(x)}{j!} M_{n,2k}((t-x)^j;x),$$

$$= n^k \sum_{j=1}^{4k} \frac{g^{(j)}(x)}{j!} Q(j,k,x) + O(1).$$

Since $\varepsilon(t,x) \to 0$ as $t \to x$, thus there is a $\delta > 0$ that corresponds to $|\varepsilon(t,x)| < \varepsilon$ whenever $x - t < \delta$ for a given $\varepsilon > 0$. Assume the given interval is $(x-\delta, x+\delta)$, characterizing function equals $\Theta_\delta(t)$. Thus:

$$|\Sigma_2| \le n^k \sum_{p=1}^{k} \binom{k}{p} \mathcal{H}^p_{n,2k}(\varepsilon(t,x)(t-x)^{2k}\Theta_\delta(t);x) + n^k \sum_{p=1}^{k} \binom{k}{p} \mathcal{H}^p_{n,2k}((\varepsilon(t,x)(t-x)^{2k}(1-\Theta_\delta(t));x),$$

$$:= J_1 + J_2.$$

To estimate J_1, we apply Lemma 6. We have

$$J_1 \le \varepsilon n^k \sum_{p=1}^{k} \binom{k}{p} \mathcal{H}^p_{n,2k}((t-x)^{2k};x),$$

$$< \varepsilon n^k O(n^{-k}) = \varepsilon O(1).$$

For arbitrary $\lambda > 0$, applying Lemma 3, we have

$$|J_2| \leq n^k \sum_{p=1}^{k} \binom{k}{p} W_p^n \left(C\varepsilon(t-x)^{2k}(1-\Theta_\delta(t)); x \right) < CO(n^{-\lambda}) = O(1).$$

Given the arbitrary nature of $\varepsilon > 0$, it implies that $\Sigma_2 \to 0$ with $n \to \infty$.
We get (6) via the combination of the estimations Σ_1 and Σ_2.
The subsequent fact will be used to demonstrate formula (7) in a similar way

$$M_{n,2k+1}((t-x)^j; x) = O(n^{-(k+1)}), \quad \forall j \in \mathbb{N}.$$

We will now demonstrate (8). Let ψ serve as a property function over this $[0, b]$. Therefore

$$M_{n,2k}(g(t); x) - g(x) = M_{n,2k}(\psi(t)(g(t) - g(x)); x) + M_{n,2k}((1 - \psi(t))(g(t) - g(x)); x),$$

$$:= \Sigma_3 + \Sigma_4.$$

The estimate of Σ_4 can be found by the same way to the estimate of J_2. Thus, we have, for all $x \in [c, d]$:

$$\Sigma_4 \leq Cn^{-k} \|g\|_{L_\alpha}.$$

For $t \in [0, b]$ and $x \in [c, d]$, by our hypothesis of f, we can write f as:

$$g(t) - g(x) = \sum_{i=1}^{2k-1} \frac{g^{(i)}(x)}{i!}(t-x)^i + \frac{1}{(2k-1)!} \int_x^t (t-w)^{2k-1} g^{(2k)}(w)\, dw.$$

Thus:

$$\Sigma_3 = \sum_{i=1}^{2k-1} \frac{g^{(i)}(x)}{i!} M_{n,2k}(\psi(t)(t-x)^i; x).$$

$$-\frac{1}{(2k-1)!} M_{n,2k} \left(\psi(t) \int_x^t (t-w)^{2k-1} g^{(2k)}(w) dw; x \right). \tag{9}$$

$$= \sum_{i=1}^{2k-1} \frac{g^{(i)}(x)}{i!} \left\{ M_{n,2k}((t-x)^i; x) + M_{n,2k}((\psi(t) - 1)(t-x)^i; x) \right\}$$

$$-\frac{1}{(2k-1)!} M_{n,2k} \left(\psi(t) \int_x^t (t-w)^{2k-1} g^{(2k)}(w) dw; x \right).$$

$$:= \sum_{i=1}^{2k-1} \frac{g^{(i)}(x)}{i!} \left\{ J_3 + J_4 \right\} + J_5.$$

By Lemma 6, we get

$$J_5 = O(n^{-k}) \text{ uniformly for } x \in [c, d].$$

Following the J_2 approximation and considering that which $x \in [c, d]$

$$J_4 = O(n^{-k}).$$

By the equations (1), (4), we have

$$J_5 = \|g^{(2k)}\|_{L_\infty[0,b]} O(n^{-k}).$$

Combining the estimates $J_3 - J_5$ in (9), we obtain

$$\|\Sigma_3\| \leq Cn^{-k} \left(\sum_{i=1}^{2k-1} \|g^{(i)}\|_{[c,d]} + \|g^{(2k)}\|_{L_\infty[0,b]} \right).$$

Lastly, we get the desired outcome by using Lemma 1. □

Theorem 2. *Let $g \in C_\alpha[0+\sigma, 1-\sigma]$ is an integer with $1 \le q \le 4k$. If $g^{(q)}$ existing and is ongoing on $(a-\eta, b+\eta) \subset (0,\infty)$, $\eta > 0$, thus for sufficiently large n*

$$|M_{n,2k}(g(t);x) - g(x)|_{[c,a,b]} \le \max\left\{C_1 n^{-\frac{q}{2\omega}}(g^{(q)}, n^{-\frac{1}{2}}), C_2 n^{-k}\right\}, \tag{10}$$

where $C_1 = C_1(k,p,g)$, $C_2 = C_2(k,p,g)$.

Proof. By our hypothesis

$$g(t) = \sum_{i=0}^{q} \frac{g^{(i)}(x)}{i!}(t-x)^i + \frac{g^{(q)}(\xi) - g^{(q)}(x)}{q!}(t-x)^q \chi(t) + h(t,x)(1-\chi(t)),$$

where the interval $(a-\eta, b+\eta)$ has a characteristic function $\chi(t)$, while ξ lies between t, x.

Assuming $x \in [a,b]$ and $t \in (a-\eta, b+\eta)$, we obtain

$$g(t) = \sum_{i=0}^{q} \frac{g^{(i)}(x)}{i!}(t-x)^i + \frac{g^{(q)}(\xi) - g^{(q)}(x)}{q!}(t-x)^q.$$

For $t \in [0,x] \cap (a-\eta, b+\eta)$ with $x \in [a,b]$, we specify

$$h(t,x) = g(t) - \sum_{i=0}^{q} \frac{g^{(i)}(x)}{i!}(t-x)^i.$$

Now,

$$\begin{aligned} M_{n,2k}(g(t);x) &= \sum_{i=0}^{q} \frac{g^{(i)}(x)}{i!} M_{n,2k}((t-x)^i;x) \\ &+ M_{n,2k}\left(\frac{g^{(q)}(\xi) - g^{(q)}(x)}{q!}(t-x)^q \chi(t); x\right) \\ &+ M_{n,2k}(h(t,x)(1-\chi(t));x). \end{aligned}$$

$$:= \Sigma_1 + \Sigma_2 + \Sigma_3.$$

By using Lemma 6, we get

$$\Sigma_1 = g(x) + O(n^{-k}) \text{ uniformly in } x \in [a,b].$$

Since $g^{(q)}$ is continuous in $[0,\infty)$, we obtain $|g^{(q)}(\xi) - g^{(q)}(x)| \le (1 + |t-x|t^\delta)\omega(g^{(q)}, \delta)$ for any $\delta > 0$.

Using Schwartz inequality and Lemma 5, we obtain

$$\begin{aligned} |\Sigma_2| &\le \sum_{p=1}^{k} \binom{k}{p} \mathcal{H}_{n,2k}^{p}\left(\frac{|g^{(q)}(\xi) - g^{(q)}(x)|}{q!}|t-x|^q \chi(t); x\right) \\ &\le \omega(g^{(q)}, \delta)\frac{1}{q!}\sum_{p=1}^{k} \binom{k}{p} \mathcal{H}_{n,2k}^{p}\left(\left(1 + \frac{|t-x|}{\delta}\right)|t-x|^q; x\right) \\ &\le \omega(g^{(q)}, \delta)\frac{1}{q!}\sum_{p=1}^{k} \binom{k}{p}\left[\mathcal{H}_{n,2k}^{p}((t-x)^q;x) + \mathcal{H}_{n,2k}^{p}((t-x)^{q+1+\delta^{-1}};x)\right]. \end{aligned}$$

Choosing $\delta = n^{-1/2}$, we get

$$|\Sigma_2| \le \omega(g^{(q)}, n^{-1/2}) O(n^{-k}) \text{ uniformly in } x \in [a,b].$$

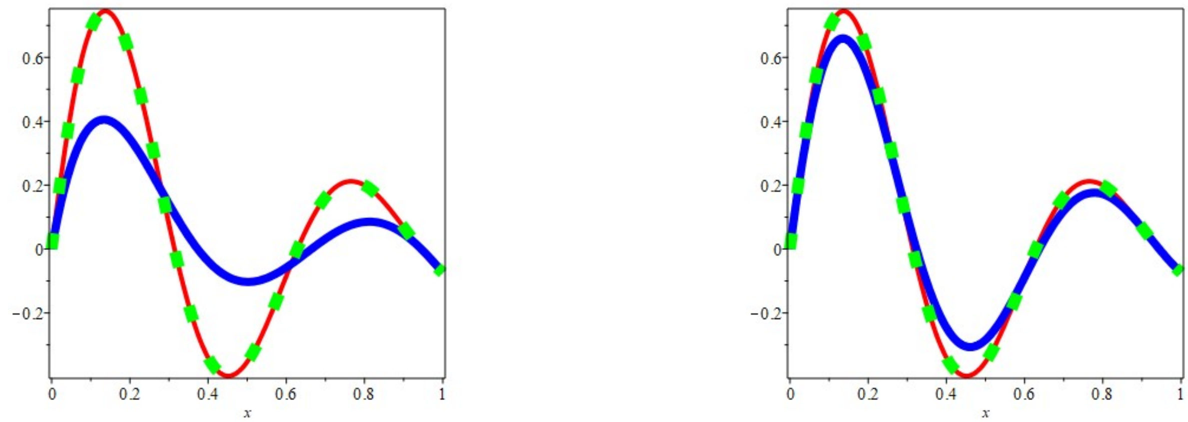
Utilizing Lemma 3, we've obtained the function $h(t,x)$ over $x \in [a,b]$ constrained by Ce^{at} with some constant $C > 0$. Thus, $\Sigma_3 = O(n^{-\lambda})$, $\lambda > 0$ uniformly on $[a,b]$. Choosing $\lambda > k$, we get $\Sigma_3 = O(n^{-k})$ uniformly in $x \in [a,b]$. □

4 Numerical Examples

In this part, we introduce a numerical example of the Micchelli combination sequence for positive linear operators $M_{n,2k}(\cdot;x)$, which is defined in (1) for 3-iterates. We describe the convergence of this sequence to the test function $g(x) = \sin(10x)e^{-2x}$, $x \in [0+\rho, 1-\rho]$. Then, we compare the error function between the sequence $M_{n,2k}(\cdot;x)$ and $\mathcal{H}_{n,2k}(g;x)$ for some values of $n = 10, 60$, respectively.

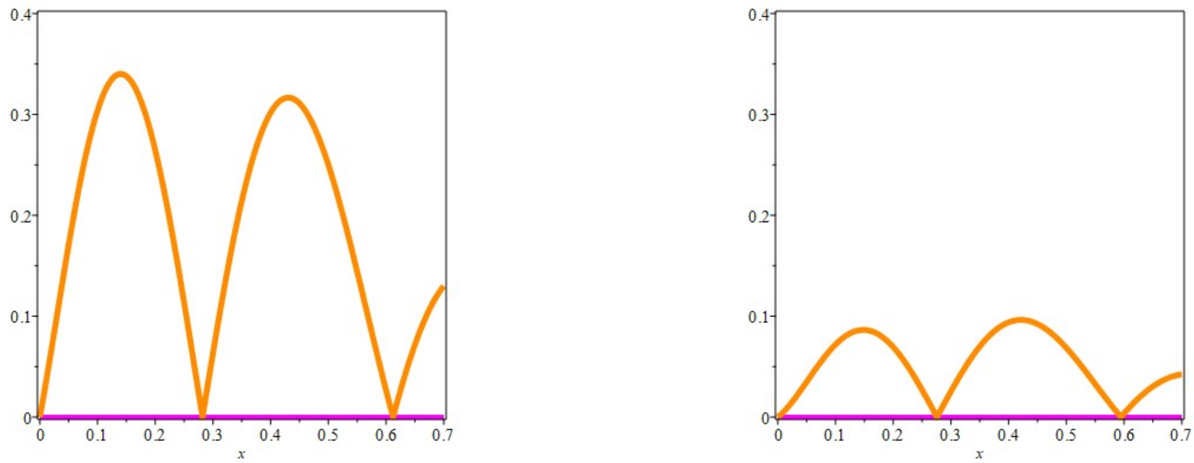
Example 1:

For $n = 10; 60$, respectively, we compare the convergence of the Micchelli combination sequence $M_{n,2k}(g;x)$ and $\mathcal{H}_{n,2k}(g;x)$ to the test function $g(x) = \sin(10x)e^{-2x}$ and the error functions for these sequences at 3-iterates.



(A) (B)

Figure 1: The comparison between the convergent polynomials $M_{n,2k}(g;x)$ (Green) and $\mathcal{H}_{n,2k}(g;x)$ (Blue) to the function $g(x)$ (Red) for 3-iterates.



(A) (B)

Figure 2: The Comparison for the error functions between $M_{n,2k}(g;x)$ (Pink) and $\mathcal{H}_{n,2k}(g;x)$ (Orange) to the function $g(x)$ for 3-iterates.

5 Conclusions

From the above numerical example for the Micchelli combination $M_{n,2k}(\cdot;x)$ for 3-iterates and the sequence $\mathcal{H}_{n,2k}(g;x)$ to approximate the test function $g(x) = \sin(10x)e^{-2x}$, in cases $n = 10, 60$, respectively, we get that:

The sequence $M_{n,2k}(g(t);x), k = 3$ (see the green point in Figure 1) gives better results with the least error (As shown in Figure 2) than the sequence $\mathcal{H}_{n,2k}(g;x)$ (see the Blue Curve in Figure 1) for all values of n. We also got a better approximation order for $M_{n,2k}(g(t);x)$ equal to $O(n^{-k})$ compared the sequence $\mathcal{H}_{n,2k}(g;x)$ which is equal to $O(n^{-1})$. So, we recommend using the Micchelli combination of the sequence $\mathcal{H}_{n,2k}(g;x)$ as well as we need to find an approximation for a function in the space $C_\alpha[0+\rho, 1-\rho]$. It can also be used in applied mathematics, especially in differential equations and numerical analysis problems.

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