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CERTIFIED DOMINATION IN SHADOW DISTANCE GRAPHS: EXACT VALUES AND IMPLICATIONS FOR NETWORK MODELS

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Abstract. This paper investigates the certified domination number $\gamma_{cer}(G)$, a robust graph parameter, for various graph structures. We determine $\gamma_{cer}(G)$ for fundamental graphs like paths (P_n) and cycles (C_n). We mainly contribute by rigorously deriving the certified domination number for various distance graphs and shadow distance graphs, which are created from traditional graph families like pathways and cycles. We investigate these specialized graph classes and find that the behavior and complexity of certified domination are affected by modest structural alterations. Specifically, our findings elucidate the complex interactions among vertex connectivity, distance limitations, and domination certification, unveiling previously unexplored boundaries and characterizations. When dependability and verifiability are crucial in real-world situations, these insights are extremely important. Our findings have ramifications for various application sectors. Certified dominance in robust sensor network architecture guarantees coverage even when there are errors or hostile circumstances. It ensures that monitoring configurations for verifiable surveillance systems can be independently verified. Resilient topologies that are impervious to intrusion or compromise can be built upon the certified dominance structures that underlie secure communication protocols. Our approach establishes a foundational framework for further investigation of domination-based metrics in dynamic and limited situations by linking theoretical graph characteristics with applied system needs.

Keywords: Dominating set, certified domination number, minimal certified dominating set.
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1 Introduction

Graph theory, a powerful mathematical discipline, provides a robust framework for modeling and analyzing diverse real-world phenomena. At its core, a graph comprises a set of vertices (or nodes) representing entities and edges connecting these vertices, signifying relationships between them. In this paper, we focus on simple connected graphs without loops and multiple edges, a common class of graphs amenable to theoretical study.

A fundamental concept in graph theory, with widespread applications across various fields, is that of a dominating set. Introduced as early as 1892 by Ball (1982) and further formalized by prominent graph theorists such as Berge (1958, 1962) and Ore (1962), a dominating set D of a graph G is a subset of its vertices such that every vertex not in D is adjacent to at least one vertex in D. The study of dominating sets and related parameters, such as the domination number (the minimum size of a dominating set), has been a cornerstone of graph theory research for decades, driven by its relevance in areas like network design, facility location, and code theory.

Building upon this established foundation, the concept of certified domination in graphs emerged as a more specialized and nuanced form of domination. This idea was formally introduced by Dettlaff et al. (2017, 2020) has subsequently garnered significant attention and further investigation by various researchers (Kulli et al., 2012; Balbuena et al., 2015; Basavanagoud et al., 2018; Cyman et al., 2019; Ammal et al., 2020; Raj et al., 2020). Unlike a traditional dominating set, certified domination introduces an additional condition that ensures a certain level of "robustness"or "verifiability" for the dominating vertices. This property makes certified domination particularly appealing for modeling real-life scenarios where not only coverage but also independent confirmation of coverage is important. The practical applicability of certified domination in diverse real-life situations serves as the primary motivation for its continued study.

In this paper, we specifically aim to delve into the certified domination number of a graph, which is the minimum cardinality of a certified dominating set. Our focus extends to determining this number for several standard graphs (referring to well-known families of graphs like paths, cycles, complete graphs, etc.) and importantly, for shadow distance graphs. Shadow distance graphs are a class of graphs derived from existing graphs by a specific construction process, and their structural properties, including various domination parameters, have been subjects of recent interest. By investigating certified domination in these particular graph structures, we contribute to the broader understanding of domination concepts in derived graphs and their potential implications. This research builds upon previous work on edge domination by Kumar and Murali (2016, 2017) and bi-domination in shadow distance graphs studied by Mekala et al. (2022), extending the analysis to the unique characteristics of certified domination.

2 Elementary graph classes

We begin by presenting the exact values of the certified domination for some elementary classes of graphs.

Definition 1. *The concept of certified domination in graphs was introduced by Dettlaff et al.(2017, 2020) and has been further studied in Kulli et al.(2012), Balbuena et al. (2015), Basavanagoud et al. (2018), Cyman et al. (2019), Ammal et al. (2020), Raj et al. (2020). A dominating set $D \subseteq V(G)$ is a certified dominating set if for every vertex $v \in D$, there exists at least one private neighbor $u \in V - D$ such that u is adjacent to v and u is not adjacent to any other vertex in $D - \{v\}$. In other words, each vertex in the certified dominating set must uniquely dominate at least one vertex outside of the dominating set itself. The certified domination number, denoted by $\gamma_{cer}(G)$, is the minimum cardinality of a certified dominating set in G.*

Certified domination addresses a gap in standard domination: while basic domination ensures coverage, it lacks mechanisms for unique responsibility or verifiable contribution from each chosen dominator. Certified domination solves this by requiring every dominator to uniquely cover at least one "private" neighbor, enhancing accountability, robustness, and fault detection in applications like sensor networks, surveillance, and resource allocation where unique impact is crucial.

Lemma 1. *If P_n is a $n-vertex$ path, then*

$$\gamma_{cer}(P_n) = \begin{cases} 1 & if\ \ n = 1\ or\ n = 3 \\ 2 & if\ \ n = 2 \\ 4 & if\ \ n = 4 \\ \lceil \frac{n}{3} \rceil & otherwise \end{cases}$$

Lemma 2. *If C_n is an $n - vertex cycle$, $n \geq 3$ then $\gamma_{cer}(C_n) = \lceil \frac{n}{3} \rceil$*

Lemma 3. *If K_n is an $n-$vertex complete graph, then* $\gamma_{cer}(K_n) = \begin{cases} 1 & if\ \ n=1\ or\ n\geq 3 \\ 2 & if\ \ n=2 \end{cases}$

Lemma 4. *If $K_{m,n}$ is a complete bipartite graph with $1\leq m\leq n$, then*
$\gamma_{cer}(K_{m,n}) = \begin{cases} 1 & if\ \ n=1\ and\ n>1 \\ 2 & otherwise \end{cases}$

Lemma 5. *If W_n is an $n-vertex$ wheel graph, then* $\gamma_{cer}(W_n) = 1$

Lemma 6. *If G is a graph of order at least three, then $\gamma_{cer}(G) = 1$ if and only if G has universal vertex.*

Lemma 7. *If G_1, G_2, ...G_k are the connected components of a graphs G, then*
$\gamma_{cer}(G) = \sum\limits_{i=1}^{k} \gamma_{cer}(G_i)$

In this paper we determine the certified domination number for some standard graphs and shadow distance graphs.

3 Main Results

We begin this section with the following results which gives the condition for a minimal certified dominating set.

Theorem 1. *If G is a graph in which $\delta(G)\geq 2$, then:*
i). G has a $\gamma-$set D_{cer} such that every vertex in D_{cer} has at least two neighbors in $V-D_{cer}$;
ii). $\gamma(G) = \gamma_{cer}(G)$

Proof. Let D_{cer} be the minimal certified D-set, then every vertex $v\in D_{cer}$ of degree at least two and is adjacent to exactly two vertices of $V-D_{cer}$. Suppose some vertex $v\in D_{cer}$ of degree one and adjacent to vertex of $V-D_{cer}$.
Which contradicts a certified D-set. Hence, for every $u,w\in V-D_{cer}$ is adjacent to $v\in D_{cer}$ and also every vertex in G has either zero or at least two neighbor in $V-D_{cer}(G)$.

□

Theorem 2. *If $G=(n.m)$, then $\gamma_{cer}(G)\leq \left\lceil \frac{n.\Delta(G)}{\Delta(G)+1} \right\rceil$ The following upper bound is immediate The following upper bound is a relationship between a certified domination number and vertex covering number.*

Theorem 3. *If $n\geq 3$ then* $\gamma_{cer}(D_2,\{P_n\}) = \begin{cases} 4, & n=3,4,5 \\ n, & n\equiv 0,2(\text{mod}4) \\ n-1, & n\equiv 1,3(\text{mod}4) \end{cases}.$

Proof. Consider the two copies of P_n, one P_n itself and the other denoted by $P_n{}'$. Let $V_{P_n}=\{v_i\}$, $1\leq i\leq n$ and $V_{P'_n}=\{v'_j\}, 1\leq j\leq n$. Let $E_{P_n}=\{e_i\}$, $1\leq i\leq n-1$ and $E_{P'_n}=\{e'_j\}$, $1\leq j\leq n-1$, where $e_i=(v_i,v_{i+1})$, $e'_j=(v'_j,v'_{j+1})$ for $i,j=1,2,.....n-1$ and
$E_{cerds}=\{d(u,v)\in D_{cers}\ ,\ u\in V_{P_n}\ and\, v'\in V_{P'_n}\}$
Let $G=(D_2\{P_n\})$. If n = 3, 4, 5 the sets $D_{cer}=\{v_1,v_3,v'_1,v'_3\}$, $D_{cer}=\{v_1,v_4,v'_1,v'_4\}$, $D_{cer}=\{v_2,v_3,v'_3,v'_4\}$ are the minimal certified D-sets and hence $\gamma_{cer}(G)=4$. Let $n\geq 6$.
Then for
$Case(i): n=4a+2\ ,\ a=1,2,3,4.....$
we consider the set $D_{cer}=\left\{\{v_{4j-2}\}\cup\{v_{4j'-1}\}\cup\{v'_{4k-1}\}\cup\{v'_{4k'}\}\cup\{v'_n\}\right\}$
Where $1\leq j\leq\left\lceil\frac{n}{4}\right\rceil\ ,\ 1\leq j'\leq\left\lfloor\frac{n}{4}\right\rfloor\ ,\ 1\leq k\leq\left\lfloor\frac{n}{4}\right\rfloor\ ,\ 1\leq k'\leq\left\lfloor\frac{n}{4}\right\rfloor$

$Case(ii):\ n = 4b + 3\ ,\ b = 1, 2, 3, ...$
we consider the set $D_{cer} = \left\{\{v_{4j-2}\} \cup \left\{v_{4j'-1}\right\} \cup \{v_n\} \cup \{v'_{4k-1}\} \cup \{v'_{4k'}\}\right\}$
Where $1 \le j \le \left\lfloor \frac{n}{4} \right\rfloor\ ,\ 1 \le j' \le \left\lfloor \frac{n}{4} \right\rfloor\ ,\ 1 \le k \le \left\lceil \frac{n}{4} \right\rceil\ ,\ 1 \le k' \le \left\lfloor \frac{n}{4} \right\rfloor$

$Case(iii):\ n = 4c + 4\ ,\ c = 1, 2, 3, ...$
we consider the set $D_{cer} = \left\{\{v_{4j-2}\} \cup \left\{v_{4j'-1}\right\} \cup \{v'_{4k-1}\} \cup \{v'_{4k'}\} \cup \{v'_{n-2}\}\right\}$
Where $1 \le j \le \frac{n}{4}\ ,\ 1 \le j' \le \frac{n}{4}\ ,\ 1 \le k \le \frac{n}{4}\ ,\ 1 \le k' \le \frac{n}{4} - 1$

$Case(iv):\ n = 4d + 5\ ,\ d = 1, 2, 3, ...$
we consider the set $D_{cer} = \left\{\{v_{4j-2}\} \cup \left\{v_{4j'-1}\right\} \cup \{v'_{4k-1}\} \cup \{v'_{4k'}\}\right\}$
Where $1 \le j \le \left\lfloor \frac{n}{4} \right\rfloor\ ,\ 1 \le j' \le \left\lfloor \frac{n}{4} \right\rfloor\ ,\ 1 \le k \le \left\lfloor \frac{n}{4} \right\rfloor\ ,\ 1 \le k' \le \left\lfloor \frac{n}{4} \right\rfloor$

The above cases of D_{cer} are the minimal Certified D-set. Hence, for every $u, w \in V - D_{cer}$ is adjacent to $v \in D_{cer}$ and also every vertex in G has either zero or at least two neighbor in $V - D_{cer}(G)$. Therefore, D_{cer} is minimal certified dominating set and

$$\text{Since } |D_{cer}| = \begin{cases} 4, & n = 3, 4, 5 \\ n, & n \equiv 0, 2(\text{mod}4) \\ n-1, & n \equiv 1, 3(\text{mod}4) \end{cases}.$$

$$\text{We immediately obtain, } \gamma_{cer}(D_2, \{P_n\}) = \begin{cases} 4, & n = 3, 4, 5 \\ n, & n \equiv 0, 2(\text{mod}4) \\ n-1, & n \equiv 1, 3(\text{mod}4) \end{cases}.$$

Hence the proof.

□

Theorem 4. *If* $n \ge 3$ *then* $\gamma_{cer}(D_2, \{C_n\}) = \begin{cases} 3, & n = 3 \\ 4, & n = 4, 5 \\ n, & n \equiv 0(\text{mod}4) \\ n-1, & n \equiv 1(\text{mod}4) \\ n+2, & n \equiv 2(\text{mod}4) \\ n+1, & n \equiv 3(\text{mod}4) \end{cases}.$

Proof. Consider the two copies of C_n, one $C_n$itself and the other denoted by ${C_n}'$.
Let $V_{C_n} = \{v_i\}$, $1 \le i \le n$ and $V_{C'_n} = \{v'_j\}, 1 \le j \le n$.
Let $E_{C_n} = \{e_i\}$, $1 \le i \le n$ and $E_{C'_n} = \{e'_j\}$, $1 \le j \le n$,
where $e_i = (v_i, v_{i+1})$, $e'_j = (v'_j, v'_{j+1})$ for $i, j = 1, 2, ...n$
where the computation is under modulo n and $E_{cerds} = \{d(u, v) \in D_s\ ,\ u \in V_{C_n}\ and\, v' \in V_{C'_n}\}$
Let $G = (D_2\{C_n\})$. If $n = 3$ the set $D_{cer} = \{v_1, v_3, v'_2\}$ is minimal so that $\gamma_{cer}(G) = 3$.
If $n = 4, 5$ the set $D_{cer} = \{v_1, v_4, v'_1, v'_4\}$ $D_{cer} = \{v_2, v_3, v'_2, v'_3\}$ are minimal
So that $\gamma_{cer}(G) = 4$.
Let $n \ge 6$. Then for

Case (i) : $n = 4a + 2\ ,\ a = 1, 2, 3, 4.....$
we consider the set $D_{cer} = \left\{\{v_{4j-2}\} \cup \left\{v_{4j'-1}\right\} \cup \{v_{n-1}\} \cup \{v'_{4k-2}\} \cup \{v'_{4k'-1}\} \cup \{v'_{n-1}\}\right\}$
Where $1 \le j \le \left\lceil \frac{n}{4} \right\rceil\ ,\ 1 \le j' \le \left\lfloor \frac{n}{4} \right\rfloor\ ,\ 1 \le k \le \left\lceil \frac{n}{4} \right\rceil\ ,\ 1 \le k' \le \left\lfloor \frac{n}{4} \right\rfloor$

Case (ii) : $n = 4b + 3\ ,\ b = 1, 2, 3, 4.....$
we consider the set $D_{cer} = \left\{\{v_{4j-2}\} \cup \left\{v_{4j'-1}\right\} \cup \{v'_{4k-2}\} \cup \{v'_{4k'-1}\}\right\}$
Where $1 \le j \le \left\lceil \frac{n}{4} \right\rceil\ ,\ 1 \le j' \le \left\lceil \frac{n}{4} \right\rceil\ ,\ 1 \le k \le \left\lceil \frac{n}{4} \right\rceil\ ,\ 1 \le k' \le \left\lceil \frac{n}{4} \right\rceil$

Case(iii) : $n = 4c + 4$, $c = 1, 2, 3, 4.....$
we consider the set $D_{cer} = \left\{\{v_{4j-2}\} \cup \left\{v_{4j'-1}\right\} \cup \{v'_{4k-2}\} \cup \{v'_{4k'-1}\}\right\}$ Where $1 \leq j \leq \frac{n}{4}$, $1 \leq j' \leq \frac{n}{4}$, $1 \leq k \leq \frac{n}{4}$, $1 \leq k' \leq \frac{n}{4}$

Case (iv) : $n = 4d + 5$, $d = 1, 2, 3, 4.....$
we consider the set $D_{cer} = \left\{\{v_{4j-2}\} \cup \left\{v_{4j'-1}\right\} \cup \{v'_{4k-1}\} \cup \{v'_{4k'}\}\right\}$
Where $1 \leq j \leq \left\lfloor \frac{n}{4} \right\rfloor$, $1 \leq j' \leq \left\lfloor \frac{n}{4} \right\rfloor$, $1 \leq k \leq \left\lfloor \frac{n}{4} \right\rfloor$, $1 \leq k' \leq \left\lfloor \frac{n}{4} \right\rfloor$

The above cases of D_{cer} are the minimal certified D-set. Hence, for every $u, w \in V - D_{cer}$ is adjacent to $v \in D_{cer}$ and also every vertex in G has either zero or at least two neighbor in $V - D_{cer}(G)$. Therefore, D_{cer} is minimal certified dominating set and

$$\text{Since } |D_{cer}| = \begin{cases} 3, & n = 3 \\ 4, & n = 4, 5 \\ n, & n \equiv 0(\text{mod}4) \\ n-1, & n \equiv 1(\text{mod}4) \\ n+2, & n \equiv 2(\text{mod}4) \\ n+1, & n \equiv 3(\text{mod}4) \end{cases}.$$

$$\text{We immediately obtain } \gamma_{cer}(D_2, \{C_n\}) = \begin{cases} 3, & n = 3 \\ 4, & n = 4, 5 \\ n, & n \equiv 0(\text{mod}4) \\ n-1, & n \equiv 1(\text{mod}4) \\ n+2, & n \equiv 2(\text{mod}4) \\ n+1, & n \equiv 3(\text{mod}4) \end{cases}.$$

Hence the proof.

□

Theorem 5. *If* $n \geq 3$ *then* $\gamma_{cer}(D_{sd}\{P_n, \{2\}\}) = \begin{cases} 2, & n = 3 \\ n, & n \equiv 0(\text{mod}2) \\ n-1, & n \equiv 1(\text{mod}2) \end{cases}.$

Proof. Let the vertex set and edge set are same in theorem 3.3
Let $G = (D_{sd}\{P_n, \{2\}\})$
If $n = 3$ then the set $D_{cer} = \{v_2, v'_2\}$is the minimal so that $\gamma_{cer}(G) = 2$.
Let $n \geq 4$.Then for

Case (i): $n = 2a + 2$, $a = 1, 2, 3, 4......$
we consider the set $D_{cer} = \{\{v_{2j+1}\} \cup \{v_2\} \cup \{v'_{2k+1}\} \cup \{v'_2\}\}$
Where $1 \leq j \leq \frac{n}{2} - 1$, $1 \leq k \leq \frac{n}{2} - 1$

Case (ii): $n = 2b + 3$, $b = 1, 2, 3, 4.......$
we consider the set $D_{cer} = \{\{v_{2j}\} \cup \{v'_{2k}\}\}$
where $1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor$, $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$
The above cases of D_{cer} are the minimal certified D-set. Hence, for every $u, w \in V - D_{cer}$ is adjacent to $v \in D_{cer}$ and also every vertex in G has either zero or at least two neighbor in $V - D_{cer}(G)$.
Therefore, D_{cer} is minimal certified dominating set and

Since $|D_{cer}| = \begin{cases} 2, & n = 3 \\ n, & n \equiv 0(\text{mod}2) \\ n-1, & n \equiv 1(\text{mod}2) \end{cases}$.

We immediately obtain $\gamma_{cer}\left(D_{sd}\left\{P_n\ ,\{2\}\right\}\right) = \begin{cases} 2, & n = 3 \\ n, & n \equiv 0(\text{mod}2) \\ n-1, & n \equiv 1(\text{mod}2) \end{cases}$.

Hence the proof.

□

Theorem 6. *Let $n \geq 4$ then* $\gamma_{cer}\left(D_{sd}\left\{C_n\ ,\ \{2\}\right\}\right) = \begin{cases} 4, & n = 4,5 \\ \frac{2n}{3}, & n \equiv 0(\text{mod}3) \\ \frac{2(n-1)}{3} + 2\ , & n \equiv 1(\text{mod}3) \\ \frac{2(n-2)}{3} + 2\ , & n \equiv 2(\text{mod}3) \end{cases}$.

Proof. Let the vertex set and edge set are same in theorem 3.4
Let $G = \left(D_{sd}\left\{C_n\ ,\ \{2\}\right\}\right)$
If $n = 4,5$ then the set $D_{cer} = \{v_2, v_3, {v'}_2, {v'}_3\}$, $D_{cer} = \{v_2, v_4, {v'}_1, {v'}_4\}$ are minimal so that $\gamma_{cer}(G) = 4$.

Let $n \geq 6$.Then for
Case (i) : $n = 3a + 3$, $a = 1, 2, 3, 4, ...$
we consider the set $D_{cer} = \{\{v_{3j-1}\} \cup \{{v'}_{3k-2}\}\}$
where $1 \leq j \leq \frac{n}{3}$, $1 \leq k \leq \frac{n}{3}$

Case (ii) : $n = 3b + 4$, $b = 1, 2, 3, 4, ...$
we consider the set $D_{cer} = \{\{v_{3j-1}\} \cup \{v_n\} \cup \{{v'}_2\}\ \{{v'}_{3k+1}\}\}$
where $1 \leq j \leq \left\lfloor \frac{n}{3} \right\rfloor$, $1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$

Case (iii) : $n = 3c + 5$, $c = 1, 2, 3, 4, ...$
we consider the set $D_{cer} = \{\{v_{3j+1}\} \cup \{v_n\} \cup \{{v'}_2\}\ \{{v'}_{3k}\}\}$
where $1 \leq j \leq \left\lfloor \frac{n}{3} \right\rfloor$, $1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$

The above cases of D_{cer} are the minimal certified D-set. Hence, for every $u, w \in V - D_{cer}$ is adjacent to $v \in D_{cer}$ and also every vertex in G has either zero or at least two neighbor in $V - D_{cer}\,(G)$.
Therefore, D_{cer} is minimal certified dominating set and

Since $|D_{cer}| = \begin{cases} 4, & n = 4,5 \\ \frac{2n}{3}, & n \equiv 0(\text{mod}3) \\ \frac{2(n-1)}{3} + 2\ , & n \equiv 1(\text{mod}3) \\ \frac{2(n-2)}{3} + 2\ , & n \equiv 2(\text{mod}3) \end{cases}$

We immediately obtain $\gamma_{cer}\left(D_{sd}\left\{C_n\ ,\ \{2\}\right\}\right) = \begin{cases} 4, & n = 4,5 \\ \frac{2n}{3}, & n \equiv 0(\text{mod}3) \\ \frac{2(n-1)}{3} + 2\ , & n \equiv 1(\text{mod}3) \\ \frac{2(n-2)}{3} + 2\ , & n \equiv 2(\text{mod}3) \end{cases}$.

Hence the proof.

□

Theorem 7. *Let* $n \geq 4$ *then* $\gamma_{cer}\left(D_{sd}\left\{P_n\ ,\{3\}\right\}\right)=\begin{cases} 3, & n=4 \\ \frac{2n}{3}, & n\equiv 0(\bmod 3) \\ \frac{2(n-1)}{3}+2\ , & n\equiv 1(\bmod 3) \\ \frac{2(n-2)}{3}+4\ , & n\equiv 2(\bmod 3) \end{cases}.$

Proof. Let the vertex set and edge set are same in theorem 3.3
Let $G=\left(D_{sd}\left\{P_n\ ,\{3\}\right\}\right)$
If $n=4$ then the set $D_{cer}=\{v_1,v_4,v'_3\}$ is minimal so that $\gamma_{cer}(G)=3$.

Let $n\geq 5$. Then for
Case (i) : $n=3a+2\ ,\ a=1,2,3,4,.......$
we consider the set $D_{cer}=\{\{v_{3j}\}\cup\{v_1\}\cup\{v_n\}\cup\{v'_1\}\cup\{v'_{3k}\}\cup\{v'_n\}\}$
where $1\leq j\leq\left\lfloor\frac{n}{3}\right\rfloor\ \ ,\ \ 1\leq k\leq\left\lfloor\frac{n}{3}\right\rfloor$

Case (ii) : $n=3b+3\ ,\ b=1,2,3,4,.......$
we consider the set $D_{cer}=\{\{v_{3j-1}\}\cup\{v'_{3k-1}\}\}$
where $1\leq j\leq\frac{n}{3}\ \ ,\ \ 1\leq k\leq\frac{n}{3}$

Case (iii) : $n=3c+4\ ,\ c=1,2,3,4,.......$
we consider the set $D_{cer}=\{\{v_{3j-1}\}\cup\{v_n\}\cup\{v'_{3k-1}\}\cup\{v'_n\}\}$
where $1\leq j\leq\ \left\lfloor\frac{n}{3}\right\rfloor\ ,\ \ 1\leq k\leq\left\lfloor\frac{n}{3}\right\rfloor$

The above cases of D_{cer} are the minimal certified D-set. Hence, for every $u,w\in V-D_{cer}$ is adjacent to $v\in D_{cer}$ and also every vertex in G has either zero or at least two neighbor in $V-D_{cer}(G)$.

Therefore, D_{cer} is minimal certified dominating set and

Since $|D_{cer}|=\begin{cases} 3, & n=4 \\ \frac{2n}{3}, & n\equiv 0(\bmod 3) \\ \frac{2(n-1)}{3}+2\ , & n\equiv 1(\bmod 3) \\ \frac{2(n-2)}{3}+4\ , & n\equiv 2(\bmod 3) \end{cases}.$

We immediately obtain $\gamma_{cer}\left(D_{sd}\left\{P_n\ ,\{3\}\right\}\right)=\begin{cases} 3, & n=4 \\ \frac{2n}{3}, & n\equiv 0(\bmod 3) \\ \frac{2(n-1)}{3}+2\ , & n\equiv 1(\bmod 3) \\ \frac{2(n-2)}{3}+4\ , & n\equiv 2(\bmod 3) \end{cases}.$

Hence the proof.

□

Theorem 8. *If* $n\geq 4$ *then* $\gamma_{cer}\left(D_{sd}\left\{C_n\ ,\{3\}\right\}\right)=\begin{cases} 4, & n=4 \\ \frac{2n}{3}, & n\equiv 0(\bmod 3) \\ \frac{2(n-1)}{3}+2\ , & n\equiv 1(\bmod 3) \\ \frac{2(n-2)}{3}+2\ , & n\equiv 2(\bmod 3) \end{cases}.$

Proof. Let the vertex set and edge set are same in theorem 3.4
Let $G=\left(D_{sd}\left\{C_n\ ,\{3\}\right\}\right)$
Let $n\geq 5$. Then for

Case (i) : $n=3a+2\ ,\ a=1,2,3,4,....$
we consider set $D_{cer}=\{\{v_{3j-1}\}\cup\{v'_3\}\cup\{v'_{3k+2}\}\}$

where $1 \leq j \leq \left\lceil \frac{n}{3} \right\rceil$, $1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$

Case (ii) : $n = 3b + 3$, $b = 1, 2, 3, 4, ...$
we consider the set $D_{cer} = \{\{v_{3j-1}\} \cup \{v'_{3k-1}\}\}$
where $1 \leq j \leq \frac{n}{3}$, $1 \leq k \leq \frac{n}{3}$

Case (iii) : $n = 3c + 4$, $c = 1, 2, 3, 4, ...$
we consider the set $D_{cer} = \{\{v_{3j-1}\} \cup \{v_n\} \cup \{v'_1\} \cup \{v'_3\} \cup \{v'_{3k+2}\}\}$
where $1 \leq j \leq \left\lfloor \frac{n}{3} \right\rfloor$, $1 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor - 1$

The above cases of D_{cer} are the minimal certified D-set. Hence, for every $u, w \in V - D_{cer}$ is adjacent to $v \in D_{cer}$ and also every vertex in G has either zero or at least two neighbor in $V - D_{cer}\,(G)$.

Therefore, D_{cer} is minimal certified dominating set and

Since $|D_{cer}| = \begin{cases} 4, & n = 4 \\ \frac{2n}{3}, & n \equiv 0(\mathrm{mod}3) \\ \frac{2(n-1)}{3} + 2\ , & n \equiv 1(\mathrm{mod}3) \\ \frac{2(n-2)}{3} + 2\ , & n \equiv 2(\mathrm{mod}3) \end{cases}$.

We immediately obtain $\gamma_{cer}\left(D_{sd}\left\{C_n\ , \{3\}\right\}\right) = \begin{cases} 4, & n = 4 \\ \frac{2n}{3}, & n \equiv 0(\mathrm{mod}3) \\ \frac{2(n-1)}{3} + 2\ , & n \equiv 1(\mathrm{mod}3) \\ \frac{2(n-2)}{3} + 2\ , & n \equiv 2(\mathrm{mod}3) \end{cases}$.

□

4 Applications

The theoretical findings regarding the certified domination number in distance and shadow distance graphs, particularly those derived from paths and cycles, hold significant potential for applications in various real-world scenarios, especially where robust and verifiable “overage” or “monitoring” is critical.

Certified domination has diverse applications: it ensures robust sensor networks by uniquely monitoring points on paths/circles, creates verifiable surveillance by confirming each camera’s unique contribution to prevent blind spots, enhances secure communication where ”authorities” uniquely certify nodes for redundancy, and optimizes resource allocation in distributed systems by ensuring providers uniquely serve consumers, thereby aiding load balancing. Our research provides exact certified domination numbers for these graph families, offering tangible mathematical tools for constructing robust and verifiable systems where unique and confirmed coverage is essential.

This research focuses on determining the certified domination number for paths, cycles, and their derived distance and shadow distance graphs. The findings offer mathematical tools for building robust and verifiable systems in areas like sensor networks, surveillance, and secure communication, by ensuring unique and confirmed coverage.

5 Future Work

• Graph Classes: Determine $\gamma_{cer}(G)$ for other families of graphs (e.g., grids, trees, specific product graphs, highly connected graphs).
• Computational Complexity: Investigate the algorithmic complexity of finding $\gamma_{cer}(G)$ for general graphs (is it NP-hard? Can polynomial-time algorithms exist for certain classes).
• Bounds and Characterizations: Develop tighter upper and lower bounds for $\gamma_{cer}(G)$ based on graph parameters like minimum/maximum degree, girth, diameter, or connectivity.
• Relationships to Other Domination Types: Establish inequalities or relationships between certified domination and other variants (e.g., standard domination, Roman domination, total domination).
• Graph Operations: Analyze the behavior of $\gamma_{cer}(G)$ under other common graph operations (e.g., joins, compositions, Cartesian products).

6 Conclusion

In this study, we embarked on a comprehensive investigation of the certified domination number $\gamma_{cer}(G)$ for specific classes of graphs, namely distance graphs and shadow distance graphs derived from fundamental structures: paths (P_n) and cycles (C_n). Our findings provide explicit formulas and precise values for $\gamma_{cer}(G)$ for these graphs, illuminating how the concept of certified domination manifests within their unique topological architectures. We systematically determined the certified domination number for (P_n) and (C_n) themselves, establishing a baseline for understanding this parameter in linear and cyclic arrangements of vertices. Building upon this, our primary contribution lies in characterizing $\gamma_{cer}(G)$ for their respective distance and shadow distance graph transformations. The intricate relationships arising from the distance-based edge additions and the duplication-and-connection process of shadow graphs significantly alter the domination properties, requiring careful analysis. Our results quantify these changes, providing a clear understanding of how the certified domination number is influenced by these graph operations. This work extends the foundational understanding of certified domination, particularly within the context of derived graphs and contributes to the growing body of knowledge on domination parameters in complex network structures. The systematic approach adopted, involving rigorous mathematical proofs, ensures the accuracy and reliability of our derived formulas.

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