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# DETERMINATION THE RIGHT HAND SIDE OF THE LINEAR EQUATION OF OSCILLATIONS OF PLATE-LIKE CONSTRUCTIONS

**H.F. Guliyev**[1*], **Kh.I. Seyfullayeva**[2]

[1]Institute of Mathematics and Mechanics, Ministry of Science and Education, B. Vahabzade 9, AZ1148, Baku, Azerbaijan
[2]Sumgayit State University, AZ5008, Sumgayit, Azerbaijan

**Abstract.** The paper considers the inverse problem of determining the right side of the linear equation of vibrations of plate structures. Dynamics of plate-like constructions, for example of bridges, roof coverings and of other structural elements can be controlled by the resistance to seismic loads and optimal displacement equation. This time an optimal control method is used to optimize the behavior of the construction, i.e. with the aim of achieving maximum stability with minimum energy consumption. In the paper first, the considered problem is reduced to the corresponding optimal control problem. Further, using the standard procedures the existence theorem for the optimal control is proved. Then the differentiability of the constructed functional is investigated, and the necessary and sufficient optimality conditions are derived.

**Keywords**: plate-like constructions, equation of vibrations, inverse problem, existence theorem, optimal control, optimality conditions.

**AMS Subject Classification:** 03C07, 03C60.

**Corresponding author:** Hamlet Guliyev, Institute of Mathematics and Mechanics, Ministry of Science and Education, B. Vahabzade 9, AZ1148, Baku, Azerbaijan, e-mail: *hamletquliyev51@gmail.com*

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# 1 Introduction

It is known that vibration processes of a plate-like constructions are described by second order partial differential equations (Arman, 1977). Therefore, it is important to study the existence of the solution of a boundary value problem for this equation. Furthermore, the study of inverse problems for partial differential equations, especially for the equation of vibrations of a plate-like construction is of theoretical and practical interest (Kabanikhin, 2009).

One of the approaches for solving inverse problems for partial differential equations is the optimization method or variational method. The essence of this method is that the inverse problem is reduced to an optimal control problem and the newly obtained problem is solved by the methods of the optimal control theory. In Arman (1977), the differential equation of a plate-like construction was derived and this equation is a hyperbolic equation. Then, a surface integral was considered, the problem of finding the minimal optimal thickness of the plate was studied and a necessary condition for the optimality in an optimal control problem was obtained only for an appropriate elliptic type equation. In this paper under consideration the inverse problem of

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finding the right hand side of the equation for nonhomogeneous case of the hyperbolic equation obtained in Arman (1977) was reduced to an inverse optimal control problem, a theorem on the existence of an optimal control was proved, necessary and sufficient condition for the optimality was derived.

The study of inverse optimization problems began at the end of the XX century (Alifanov et al., 1988), and these problems are currently intensively studied. It should be noted that some related problems were considered in the papers Guliyev & Seyfullayeva (2017); Li et al. (2002); Bauer et al. (2022); Burczyński et al. (2020); Altenbach & Eremeyev (2023); Aridogan & Basdogan (2015). In Guliyev & Seyfullayeva (2017), the inverse problem on defining the right hand side for an equation of vibrations of an elastic plate is solved by the optimization method. In Li et al. (2002) the problem on defining damaged parts of a plate-like construction was considered, numerical modeling and experiments were carried. Such problems arise in machine building, in aeronautics, in designing constructions and they are very important for safety of these constructions. In Bauer et al. (2022) it is noted that new processing are required for meeting the growing demand to plate materials and it is necessary to combine different materials for improving the properties of materials when designing constructions.

This time it is necessary to give a certain structure to the material and its components. The interaction between materials and their structure can be on different scales. They can be applied in various fields. In above mentioned works the optimal control problems have not been solved from mathematical point of view.

In Burczyński et al. (2020); Altenbach & Eremeyev (2023); Aridogan & Basdogan (2015) numerical modeling of constructions was considered. In Seyfullaeva (2012), an optimal control problem for an equation of vibrations of a plate-like constructions was studied, i.e. in the problem under consideration, the existence of an optimal control is proved and a necessary condition for optimality is found. Dynamics of plate-like constructions, for example of bridges, roof coverings and of other structural elements can be controlled by the resistance to seismic loads and optimal displacement equation. This time an optimal control method is used to optimize the behavior of the construction, i.e. with the aim of achieving maximum stability with minimum energy consumption.

Furthermore, plate-like systems can be used for the robots to move and for stabilization processes. There it is intendend to control the movements or maneuver of the robot, to build a control management strategy according to it, for example, to optimize the movements of flexible elastic plates manipulated by robots. Dynamics of plate like structures plays an important role in various mechanical systems, for example in flying or floating objects. Optimal control corresponding to this equations can be used to optimize motion, especially in aerodynamic and hydrodynamic mode. Optimal control problems can be applied taking into account the behavior of plate-like structure in the human body or prostheses. For example, these problems can be used to increase patient mobility by controlling the movement of medical devices and prostheses.

More stable systems can be built through optimized control equations for the safety of buildings and other infrastructure against vibration or impacts.

## 2 Formulation of the problem

It needs to find the pair of functions $(u, \upsilon) \in W_2^1(Q) \times L_2(\Omega)$ from the relations

$$\frac{\rho}{G} h \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(h\frac{\partial u}{\partial y}\right) = \upsilon(x,y) f(t),\ (x,y,t) \in Q, \tag{1}$$

$$u(x,y,0) = \varphi_0(x,y),\ \frac{\partial u(x,y,0)}{\partial t} = \varphi_1(x,y),\ (x,y) \in \Omega, \tag{2}$$

$$\begin{aligned} &u(0,y,t)=0, u(a,y,t)=0,\ \ 0\le y\le b,\ \ 0\le t\le T,\\ &u(x,0,t)=0, u(x,b,t)=0,\ \ 0\le x\le a,\ \ 0\le t\le T, \end{aligned} \tag{3}$$

$$\int_0^T K(x,y,t)u(x,y,t)dt=\chi(x,y),\ (x,y)\in\Omega. \tag{4}$$

Note that equation (1) is the equation of vibrations of a plate-like construction (Arman, 1977). Where $(x,y)\in\Omega=\{(x,y):0<x<a,\ 0<y<b\}$, $t\in(0,T)$, $Q=\Omega\times(0,T)$, $a$, $b$, $T$ are given positive numbers, $\rho(x,y)$ is a dense of the mass at the point $(x,y)$, $G$- voltage, $h(x,y)$ is the heath thickness of the plate in the point $(x,y)$, $0<\nu\le h(x,y)\le\mu$, $\nu$, $\mu$-given positive numbers, $u(x,y,t)$-is deflection of the plate in the point $(x,y)$ at the moment $t$, $f(t)\in L_2(0,T)$, $\varphi_0(x,y)\in \overset{o}{W}{}_2^1(\Omega)$, $\varphi_1(x,y)\in L_2(\Omega)$, $K(x,y,t)\in L_\infty(Q)$, $\chi(x,y)\in L_2(\Omega)$ are given functions.

If in equation (1) function $\upsilon(x,y)$ known function, then (1)-(3) will be conditions to find function $u(x,y,t)$.

If in equation function $\upsilon(x,y)$ unknown, then gives addition conditions (4).

We‘ll consider the generalized solution of problem (1)-(3).

**Definition 1.** *Under the generalized solution of the problem (1)-(3) corresponding to the function $\upsilon(x,y)$ we‘ll understand the function $u(x,y,t)\in W_2^1(Q)$, that satisfies to the integral identity*

$$\int_Q\left(-\frac{\rho}{G}h\frac{\partial u}{\partial t}\cdot\frac{\partial\eta}{\partial t}+h\frac{\partial u}{\partial x}\cdot\frac{\partial\eta}{\partial x}+h\frac{\partial u}{\partial y}\cdot\frac{\partial\eta}{\partial y}\right)dxdydt-$$

$$-\int_\Omega\frac{\rho}{G}h\varphi_1(x,y)\eta(x,y,0)dxdy=\int_Q\upsilon(x,y)f(t)\eta(x,y,t)dxdydt \tag{5}$$

*for any $\forall\eta(x,y,t)\in W_2^1(Q),\eta(x,y,T)=0$, satisfying the conditions*

$$\eta(0,y,t)=0,\ \ \eta(a,y,t)=0,\ \ 0\le y\le b,\ \ 0\le t\le T,$$

$$\eta(x,0,t)=0,\ \ \eta(x,b,t)=0,\ \ 0\le x\le a,\ \ 0\le t\le T.$$

Note, that the conditions (3) and $u(x,y,0)=\varphi_0(x,y)$ are satisfied in the sense almost everywhere.

This problem we reduce to the following optimal control problem: to find the minimum of the functional

$$J(\upsilon)=\frac{1}{2}\int_\Omega\left[\int_0^T K(x,y,t)u(x,y,t;\upsilon)dt-\chi(x,y)\right]^2dxdy, \tag{6}$$

subject to (1)-(3). The function $\upsilon(x,y)$ is called a control. By $u=u(x,y,t;\upsilon)$ we denote the generalized solution of the problem (1)-(3) corresponding to the control $\upsilon(x,y)$. As a class of admissible controls $U_{ad}$ we take convex closed set from $L_2(\Omega)$.

We regularize the problem (1)-(3), (6)by the following way: instead of the functional (6) consider the following functional

$$J_\alpha(\upsilon)=J(\upsilon)+\frac{\alpha}{2}\int_\Omega(\upsilon(x,y)-\omega(x,y))^2dxdy, \tag{7}$$

where $\alpha>0$ is a positive number, $\omega(x,y)\in L_2(\Omega)$ is a given function.

It is required find minimum of functional (7) in the class $U_{ad}$ under conditions (1)-(3).

# 3 Existence of the optimal control

**Theorem 1.** *Under the imposed conditions on the problem data, there exists unique an optimal control in problem (1)-(3), (7).*

*Proof.* Let $\{\upsilon_n(x,y)\} \in U_{ad}$ be a minimizing sequence, i.e.

$$\lim_{n\to\infty} J_\alpha(\upsilon_n) = \inf_{\upsilon\in U_{ad}} J_\alpha(\upsilon). \tag{8}$$

From expression of functional (7) we obtain, that

$$\|\upsilon_n\|_{L_2(\Omega)} \leq C. \tag{9}$$

Taking it into account, for solutions of problem (1)-(3) corresponding to $\upsilon_n(x,y)$, we obtain the estimation (Ladizhenskaya (1973), pp.209-215)

$$\|u_n\|_{W_2^1(Q)} \leq C\left[\|\upsilon_n\|_{L_2(\Omega)} + \|f\|_{L_2(0,T)} + \|\varphi_0\|_{\overset{o}{W}{}_2^1(\Omega)} + \|\varphi_1\|_{L_2(\Omega)}\right]. \tag{10}$$

By virtue of (9) and (10), we can choos from sequences $\{\upsilon_n\}$, $\{u_n\}$ subsequences that (we also denote this by $\{\upsilon_n\}$, $\{u_n\}$ $n\to\infty$.)

$$\upsilon_n \to \upsilon_0 \text{ weakly in } L_2(\Omega), \upsilon_0 \in U_{ad},$$

$$u_n \to u_0, \frac{\partial u_n}{\partial t} \to \frac{\partial u_0}{\partial t}, \frac{\partial u_n}{\partial x} \to \frac{\partial u_0}{\partial x}, \frac{\partial u_n}{\partial y} \to \frac{\partial u_0}{\partial y} \text{ weakly in } L_2(Q).$$

By virtue of compactness theorems, (Ladizhenskaya (1973), p.64) it follows that $u_n \to u_0$ is strong in $L_2(Q)$.

Considering these relations, in the definition of the generalized solution for the problem (1)-(3), by $\upsilon = \upsilon_n$, $u = u_n$, passing to limit as $n\to\infty$ we have

$$\int_Q \left(-\frac{\rho}{G}h\frac{\partial u_0}{\partial t}\cdot\frac{\partial \eta}{\partial t} + h\frac{\partial u_0}{\partial x}\cdot\frac{\partial \eta}{\partial x} + h\frac{\partial u_0}{\partial y}\cdot\frac{\partial \eta}{\partial y}\right)dxdydt-$$

$$-\int_\Omega \frac{\rho}{G}h\varphi_1(x,y)\eta(x,y,0)dxdy = \int_Q \upsilon_0(x,y)f(t)\eta(x,y,t)dxdydt.$$

Hence it follows, that $u_0 = u(\upsilon_0)$, i.e. $u_0(x,y,t)$ is a generalized solution for the problem (1)-(3), corresponding to the control $\upsilon_0(x,y)$. Thanks to functional $J_\alpha(\upsilon)$ is weak semi-continuous below, then

$$\underline{\lim}_{n\to\infty} J_\alpha(\upsilon_n) \geq J_\alpha(\upsilon_0). \tag{11}$$

Then from (8) and (11) follows that

$$J_\alpha(\upsilon_0) = \inf_{\upsilon\in U_{ad}} J_\alpha(\upsilon).$$

It shows, that $\upsilon_0(x,y)$ provides the minimum to functional (7), i.e. is an optimal control. □

The theorem is proved.

# 4 Differentiability of the functional (7), necessary and sufficient condition optimality

Let us introduce the adjoint problem to (1)-(3), (7) for the given control $\upsilon(x,y) \in L_2(\Omega)$:

$$\frac{\rho}{G}h\frac{\partial^2\psi}{\partial t^2} - \frac{\partial}{\partial x}\left(h\frac{\partial\psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(h\frac{\partial\psi}{\partial y}\right) =$$

$$= -K\left(\int_0^T Kudt - \chi\right), (x,y,t) \in Q, \tag{12}$$

$$\psi(x,y,T) = 0, \frac{\partial\psi(x,y,T)}{\partial t} = 0, (x,y) \in \Omega, \tag{13}$$

$$\begin{aligned} &\psi(0,y,t) = 0, \;\; \psi(a,y,t) = 0, \;\; 0 \le y \le b, \;\; 0 \le t \le T, \\ &\psi(x,0,t) = 0, \;\; \psi(x,b,t) = 0, \;\; 0 \le x \le a, \;\; 0 \le t \le T. \end{aligned} \tag{14}$$

From the conditions imposed on the data of the problem (1)-(4) follows that this adjoint problem has unique generalized solution from the space $W_2^1(Q)$ (Ladizhenskaya, 1973).

To derive the necessary conditions for optimality in the considered problem we take two arbitrary admissible controls $\upsilon(x,y)$ and $\upsilon(x,y) + \delta\upsilon(x,y)$. The corresponding solutions of problem (1)-(3) are denoted by $u(x,y,t;\upsilon)$ and $u(x,y,t;\upsilon+\delta\upsilon) \equiv u(x,y,t;\upsilon)+\delta u(x,y,t)$. Then $\delta u(x,y,t) = u(x,y,t;\upsilon+\delta\upsilon) - u(x,y,t;\upsilon)$ is a solution of the boundary value problem

$$\frac{\rho}{G}h\frac{\partial^2(\delta u)}{\partial t^2} - \frac{\partial}{\partial x}\left(h\frac{\partial(\delta u)}{\partial x}\right) - \frac{\partial}{\partial y}\left(h\frac{\partial(\delta u)}{\partial y}\right) = \delta\upsilon(x,y)f(t), \;\; (x,y,t) \in Q, \tag{15}$$

$$\delta u(x,y,0) = 0, \frac{\partial(\delta u(x,y,0))}{\partial t} = 0, \;\; (x,y) \in \Omega, \tag{16}$$

$$\begin{aligned} &\delta u(0,y,t) = 0, \;\; \delta u(a,y,t) = 0, \;\; 0 \le y \le b, \;\; 0 \le t \le T, \\ &\delta u(x,0,t) = 0, \;\; \delta u(x,b,t) = 0, \;\; 0 \le x \le a, \;\; 0 \le t \le T. \end{aligned} \tag{17}$$

Let's show that

$$\|\delta u(x,y,t)\|_{W_2^1(Q)} \le C\,\|\delta\upsilon(x,y)\|_{L_2(\Omega)}. \tag{18}$$

For this purpose, we use Faedo-Galerkin's method. Take the basis $\{\omega_i(x,y)\}_{i=1}^{\infty}$ from $\overset{\circ}{W}{}_2^1(\Omega)$, where the system $\{\omega_i(x,y)\}_{i=1}^{\infty}$ is orthonormal in $L_2(\Omega)$ and the approximate solution for the problem (15)-(17) search in the form

$$\delta u^N(x,y,t) = \sum_{i=1}^{N} c_i^N(t)\omega_i(x,y)$$

from the equalities

$$\int_\Omega \frac{\rho}{G}h\frac{\partial^2(\delta u^N)}{\partial t^2}\omega_j(x,y)dxdy + \int_\Omega \left(h\frac{\partial(\delta u^N)}{\partial x}\right)\frac{\partial}{\partial x}\omega_j(x,y)dxdy +$$

$$+\int_\Omega \left(h\frac{\partial(\delta u^N)}{\partial y}\right)\frac{\partial}{\partial y}\omega_j(x,y)dxdy =$$

$$= \int_\Omega \delta\upsilon\,(x,y)\,f(t)\omega_j(x,y)dxdy, \quad 1 \le j \le N, \tag{19}$$

$$c_i^N(0)=0,\ \frac{d}{dt}c_i^N(t)\Big|_{t=0}=0.$$

Both sides of (19) multiply by $\frac{d}{dt}c_j^N(t)$ and sum over $j$ from 1 to $N$. Then we get

$$\int_\Omega \frac{\rho}{G}h\frac{\partial^2(\delta u^N)}{\partial t^2}\frac{\partial(\delta u^N)}{\partial t}dxdy+\int_\Omega\left(h\frac{\partial(\delta u^N)}{\partial x}\right)\frac{\partial^2(\delta u^N)}{\partial x\partial t}dxdy+$$

$$+\int_\Omega\left(h\frac{\partial(\delta u^N)}{\partial y}\right)\frac{\partial^2(\delta u^N)}{\partial y\partial t}dxdy=\int_\Omega \upsilon\left(x,y\right)f(t)\frac{\partial(\delta u^N)}{\partial t}dxdy.$$

It gives that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\left[\frac{\rho}{G}h\left(\frac{\partial\delta\,u^N}{\partial t}\right)^2+h\left(\frac{\partial(\delta u^N)}{\partial x}\right)^2+h\left(\frac{\partial(\delta u^N)}{\partial y}\right)^2\right]dxdy=$$

$$=\int_\Omega \delta\upsilon\left(x,y\right)f(t)\frac{\partial(\delta u^N)}{\partial t}dxdy.$$

If to integrate this equality over $t$ from 0 to $t$ by the imposed conditions we get

$$\int_\Omega\left[\left(\frac{\partial\delta u^N(x,y,t)}{\partial t}\right)^2+\left(\frac{\partial(\delta u^N)}{\partial x}\right)^2+\left(\frac{\partial(\delta u^N)}{\partial y}\right)^2\right]dxdy\le$$

$$\le C+C\int_0^t\int_\Omega\left[\left(\delta u^N(x,y,s)\right)^2+\left(\frac{\partial\delta u^N(x,y,s)}{\partial t}\right)^2+\left(\frac{\partial\delta u^N(x,y,s)}{\partial x}\right)^2+\right.$$

$$\left.+\left(\frac{\partial\delta u^N(x,y,s)}{\partial y}\right)^2\right]dxdyds+\int_\Omega\left|\delta\upsilon(x,y)\right|^2dxdy,\forall t\in[0,T],$$

where by $C$ we denote the constants not depending on the estimating quantities and admissible controls.

Due to equivalency of the norms in $\overset{\circ}{W}{}_2^1(\Omega)$ we obtain

$$\int_\Omega\left[\left(\delta u^N(x,y,t)\right)^2+\left(\frac{\partial\delta u^N(x,y,t)}{\partial t}\right)^2+\left(\frac{\partial\delta u^N(x,y,t)}{\partial x}\right)^2+\right.$$

$$\left.+\left(\frac{\partial\delta u^N(x,y,t)}{\partial y}\right)^2\right]dxdy\le C+C\int_0^t\int_\Omega\left[\left(\delta u^N(x,y,s)\right)^2+\right.$$

$$\left.+\left(\frac{\partial\delta u^N(x,y,s)}{\partial t}\right)^2+\left(\frac{\partial\delta u^N(x,y,s)}{\partial x}\right)^2+\left(\frac{\partial\delta u^N(x,y,s)}{\partial y}\right)^2\right]dxdys+$$

$$+C\int_\Omega\left|\delta\upsilon(x,y)\right|^2dxdy. \tag{20}$$

Application of the Gronwall's lemma leads to

$$\int_\Omega\left[\left(\delta u^N(x,y,t)\right)^2+\left(\frac{\partial\delta u^N(x,y,t)}{\partial t}\right)^2+\right.$$

$$\left.+\left(\frac{\partial\delta u^N(x,y,t)}{\partial x}\right)^2+\left(\frac{\partial\delta u^N(x,y,t)}{\partial y}\right)^2\right]dxdy\le$$

$$\le C\left\|\delta\upsilon(x,y)\right\|^2_{L_2(\Omega)},\ \forall t\in[0,T]. \tag{21}$$

From this integrating over $t$ we get

$$\left\|\delta u^N(x,y,t)\right\|^2_{W_2^1(Q)} \leq C \left\|\delta \upsilon(x,y)\right\|^2_{L_2(\Omega)}. \tag{22}$$

From (22) we get that from the sequence $\left\{\delta u^N(x,y,t)\right\}$ one can choose a subsequence (which is also denoted by$\left\{\delta u^N(x,y,t)\right\}$) that converges weakly in $W_2^1(Q)$ to some function $\delta u(x,y,t)$ by $N\to\infty$.

Then, by the weak semi-continuity below of the norm in the Hilbert space, (22) implies estimate (18).

Since $u(x,y,t;\upsilon+\delta\upsilon) = u(x,y,t;\upsilon) + \delta u(x,y,t)$ we can calculate the increment of the functional $J_\alpha(\upsilon)$:

$$\begin{gathered}\Delta J_\alpha(\upsilon) = J_\alpha(\upsilon+\delta\upsilon) - J_\alpha(\upsilon) = \\ = \frac{1}{2}\int_\Omega \left[\left(\int_0^T K(u+\delta u)dt - \chi\right)^2 - \left(\int_0^T Kudt - \chi\right)^2\right] dxdy + \\ +\frac{\alpha}{2}\int_\Omega \left[(\upsilon+\delta\upsilon-\omega)^2 - (\upsilon-\omega)^2\right] dxdy = \\ = \int_\Omega \left(\int_0^T Kudt - \chi\right)\left(\int_0^T K\delta udt\right) dxdy + \\ +\alpha\int_\Omega (\upsilon-\omega)\delta\upsilon dxdy + R,\end{gathered} \tag{23}$$

where

$$R = \frac{1}{2}\int_\Omega \left|\int_0^T K\delta u(x,y,t)dt\right|^2 dxdy + \frac{\alpha}{2}\int_\Omega \left|\delta\upsilon(x,y)\right|^2 dxdy$$

is a remained term.

Since $\delta u(x,y,t)$ is a generalized solution of the problem (15)-(17), for arbitrary function $\eta(x,y,t)\in W_2^1(Q)$, $\eta(x,y,T)=0$ is valid

$$\begin{gathered}\int_Q \left(-\frac{\rho}{G}h\frac{\partial(\delta u)}{\partial t}\cdot\frac{\partial\eta}{\partial t} + h\frac{\partial(\delta u)}{\partial x}\cdot\frac{\partial\eta}{\partial x} + h\frac{\partial(\delta u)}{\partial y}\cdot\frac{\partial\eta}{\partial y}\right) dxdydt = \\ = \int_Q \delta\upsilon(x,y)f(t)\eta(x,y,t)dxdydt\end{gathered} \tag{24}$$

and $\delta u(x,y,0)=0$ true.

Similarly, since $\psi(x,y,t)$ is a solution of the problem (12)-(14), for any function $g(x,y,t)\in W_2^1(Q)$, $g(x,y,0)=0$ we have

$$\begin{gathered}\int_Q \left(-\frac{\rho}{G}h\frac{\partial\psi}{\partial t}\cdot\frac{\partial g}{\partial t} + h\frac{\partial\psi}{\partial x}\cdot\frac{\partial g}{\partial x} + h\frac{\partial\psi}{\partial y}\cdot\frac{\partial g}{\partial y}\right) dxdydt = \\ = -\int_Q K\left(\int_0^T Kudt - \chi\right) gdxdydt\end{gathered} \tag{25}$$

and $\psi(x,y,T)=0$.

If in (24) we take $\psi(x,y,t)$ instead of $\eta(x,y,t)$, and in (25) to take $\delta u(x,y,t)$ instead of $g(x,y,t)$ and subtract (25) from (24), then considering (23) we obtain

$$\Delta J_\alpha(\upsilon) = \int_Q \left[-\psi(x,y,t)f(t) + \alpha(\upsilon_0(x,y) - \omega(x,y))\right]\delta\upsilon(x,y)dxdydt + R. \tag{26}$$

Now let us estimate the remained term $R$ taking into account the estimation (18)

$$0 \leq R = \frac{1}{2}\int_\Omega \left|\int_0^T K^2dt\cdot\int_0^T(\delta u)^2dt\right| dxdy + \frac{\alpha}{2}\int_\Omega \left|\delta\upsilon\right|^2 dxdy \leq$$

$$\leq C\int_{\Omega}\int_{0}^{T}(\delta u)^{2}dxdydt + C\left\|\delta v\right\|_{L_{2}(\Omega)}^{2}. \tag{27}$$

Taking into account (18) for here we get

$$R \leq C\left\|\delta v\right\|_{L_{2}(\Omega)}^{2}.$$

Then as follows from (26) the gradient of the functional has a form

$$gradJ_{\alpha}(v) = -\psi(x,y)f(t) + \alpha(v(x,y) - \omega(x,y)).$$

Thus due to known theorem from (Vasiliev (1980), p.28) in order to the $v_0(x,y)$ control was optimal it is necessary fulfillment of the inequality

$$\int_{\Omega}\left(-\int_{0}^{T}\psi(x,y,t)f(t)dt + \alpha\left(v_{0}(x,y) - \omega(x,y)\right)\right)\times$$

$$\times\left(v(x,y) - v_{0}(x,y)\right)dxdy \geq 0 \ \ \forall v \in U_{ad}. \tag{28}$$

Since the functional (7) is strongly convex in $U_{ad}$ and the problem (1)-(3) is linear, condition (28) is also sufficient condition of optimality.

Thus the following theorem is proved.

**Theorem 2.** *Let the imposed on the data of the problem (1)-(3), (7) be fulfilled. Then for the optimality of the control $v_0(x,y)$ in this problem it is necessary and sufficient fulfillment of the inequality (28).*

## 5 Conclusion

In the presented work, considers the inverse problem of determining the right side of the linear equation of vibrations of plate structures. The problem is reduced to the optimal control problem. Existence theorem proved for optimal control. Differentiability of the functional is studied and necessary and sufficient conditions of optimality are derived.

An optimal control problem for an equation of vibrations of plate-like constructions can be applied in various fields. Such problems play an important role in such fields as structural engineering, robotic technologies and modeling of physical processes.

In these applications, optimal control aims to direct properly system behavior, minimize energy consumption and generally to increase performance. Equations of vibrations of plate-like constructions are widely used in solving optimization problems by analytic and computer-based methods in engineering problems like these.

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