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DISCRETE HYPERBOLIC MIXTURE DISTRIBUTION: PROPERTIES AND APPLICATIONS

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Abstract. This article introduces a new three-parameter discrete probability distribution, namely, the discrete hyperbolic mixture (DHM) distribution. It is derived by taking a mixture of two discrete distributions, called the hyperbolic cosine (HC) and hyperbolic sine (HS) distributions, which are also introduced here. Several probability and statistical properties of the proposed distributions are derived. Maximum likelihood estimation is employed to find estimators of the DHM distribution's parameters. Numerical experiments are then conducted to evaluate the performance of the proposed estimator. The results show that the average estimate for each parameter approaches its true value as the sample size increases. The final section of this article presents applications of the DHM distribution to real datasets and performs a comparative study with some existing distributions to demonstrate its potential as an alternative model for count data.

Keywords: hyperbolic cosine, hyperbolic sine, maximum likelihood estimation, moment generating function, mixture distribution.

AMS Subject Classification: 60E05.

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1 Introduction

Throughout the years, developing probability distributions has been one of the major focuses among statisticians, including the distribution for modeling count data. Some notable discrete type distributions such as Poisson, binomial, negative binomial, geometric distributions, lack the flexibility in capturing a count data given its dispersal behavior. This limitation has been the concern of many researchers in recent years as the Poisson distribution is only well-fitted for equidispersed data, the binomial distribution is only suitable for underdispersed data, and the negative binomial and geometric distributions are suitable for overdispersed data.

Lately, there has been an increasing demand for new discrete type distributions due to a variety of randomness in nature. Many researchers have proposed new probability distributions that are extended or transformed from well-known distributions to provide the flexibility in modeling various data or phenomenon. Some researchers have applied survival function or cumulative density function discretization method to the continuous probability distribution for generating a discrete distribution as discussed in the articles of Déniz & Ojeda (2011), Chakraborty & Chakravarty (2012), Noughabi et al. (2013), El-Morshedy et al. (2020), Nekoukhou et al. (2011), Gillariose et al. (2021), and Almetwally et al. (2020). Others have compounded two existing distributions, as proposed by Déniz (2013), Kuş et al. (2019), Bhati & Ahmed (2019), Shafiq et al. (2022), Karakaya (2023), Alrumayh & Khogeer (2023), and Nandi et al. (2024). However, all those mentioned distributions are compatible only for modeling data with particular dispersion behavior. Nevertheless, Khruachalee et al. (2021) introduced the partial-geometric distribution that is the extended version of the geometric distribution which is able to capture both underdispersed and overdispersed data.

In this article, a novel three-parameter distribution is introduced, called the discrete hyperbolic mixture (DHM) distribution. The main objective of this article is to present a new alternative discrete distribution that provides flexibility. The basic idea of constructing DHM distribution is by mixing a couple of discrete probability distributions, referred to as the hyperbolic cosine (HC) and hyperbolic sine (HS) distributions, which are also discussed in this article. Both HC and HS distributions are first exhibited in (Kharazmi & Saadatinik, 2016) and (Kharazmi et al., 2017), respectively. However, the respective articles only mentioned the probability mass function (pmf) of HC and HS without proving that the displayed pmf is a valid pmf of a distribution. Moreover, the domain of its parameter is restricted and there is no explanation for this restriction. Those authors only utilized HC and HS distributions as generators to create a new continuous family of distributions. Therefore, apart from introducing the DHM distribution, this article also provides comprehensive studies of the HC and HS distributions by discovering some important features. These two baseline distributions are eventually the submodels of the DHM distribution and are major key to its flexibility in accommodating different real-life data conditional to its index of dispersion.

2 Proposed Distribution

The formulation of the discrete hyperbolic mixture (DHM) distribution is based on the hyperbolic cosine (HC) and hyperbolic sine (HS) distributions. These two distributions are constructed from the power series expansions of the HC and HS functions, respectively. The selection of these two notable functions is in accordance with the mathematical form produced by the respective series expansions. More specifically, while all differentiable functions can be represented as power series, not all of them can be power series distribution as they may not satisfy some properties of probability distribution.

As preliminary to the DHM distribution, the HC and HS distributions are discussed. The idea of constructing the HC distribution came from the power series expansion of the HC function, $\cosh(\theta)$, as given by Equation (1).

$$\cosh(\theta) = \sum_{i=0}^{\infty} \frac{\theta^{2i}}{(2i)!}, \quad \text{for all } \theta \in \mathbb{R}. \tag{1}$$

If the two hand-side of Equation (1) are multiplied by $\frac{1}{\cosh(\theta)}$, then it leads to the following equation:

$$\sum_{i=0}^{\infty} \frac{1}{\cosh(\theta)} \frac{\theta^{2i}}{(2i)!} = 1, \quad \text{for all } \theta \in \mathbb{R}. \tag{2}$$

Equation (2) shows one of the properties of a probability mass function (pmf). The next step is to ensure that each term in the summation is non-negative. It is obvious that $f(i) = \frac{\theta^{2i}}{(\cosh(\theta))(2i)!} \geq 0$ for all $i \in \{0, 1, 2, \ldots\}$ and $\theta \in \mathbb{R}$, and since $f(i)$ satisfies Equation (2), then $f(i) = \frac{\theta^{2i}}{(\cosh(\theta))(2i)!}$ can be a pmf of the proposed distribution that called the hyperbolic cosine distribution. The value of θ can be negative because the pmf $f(i;\theta) = \frac{\theta^{2i}}{(\cosh(\theta))(2i)!}$ is an even function of θ (i.e. $f(\theta) = f(-\theta)$), and it still satisfies the properties of a pmf. The case of $\theta = 0$ could belong to the consideration where in this case, an event of zero-valued is the only event to occur. However, it is not a useful probability model for most practical applications. Hence in this paper, the domain of θ is restricted to non-zero real numbers.

A non-negative integer random variable X has the hyperbolic cosine distribution with a non-zero parameter θ, denoted by HC(θ), if its pmf is given as follows:

$$p_X(k) = \frac{1}{\cosh(\theta)} \frac{\theta^{2k}}{(2k)!}, \quad k = 0, 1, 2, \ldots \tag{3}$$

The hyperbolic sine (HS) distribution is also inspired by the power series representation. The HS function, $\sinh(\lambda)$, has the following power series expansion

$$\sinh(\lambda) = \sum_{i=0}^{\infty} \frac{\lambda^{2i+1}}{(2i+1)!}, \quad \text{for all } \lambda \in \mathbb{R}. \tag{4}$$

If the two hand-side of Equation (4) are multiplied by $\frac{1}{\sinh(\lambda)}$, then it leads to the following equation:

$$\sum_{i=0}^{\infty} \frac{1}{\sinh(\lambda)} \frac{\lambda^{2i+1}}{(2i+1)!} = 1, \quad \lambda \neq 0. \tag{5}$$

Equation (5) shows one of the properties of a pmf. However, the domain of λ must be restricted to non-zero real numbers to ensure that the function $f(i) = \frac{\lambda^{2i+1}}{(\sinh(\lambda))(2i+1)!}$ is defined and non-negative for all $i \in \{0, 1, 2, \ldots\}$, and hence $f(i)$ can be a pmf of the proposed distribution that called the hyperbolic sine distribution. The prospective pmf $f(i; \lambda) = \frac{\lambda^{2i+1}}{(\sinh(\lambda))(2i+1)!}$ is an even function of λ (i.e. $f(\lambda) = f(-\lambda)$) since $g(\lambda) = \frac{\lambda^{2i+1}}{(2i+1)!}$ and $h(\lambda) = \sinh(\lambda)$ are odd functions of λ. This property implies that the value of λ can be negative as still holding all properties of pmf. As applied to HC distribution, in this paper, the domain of λ is preserved to be a non-zero real number since it does not violate any properties of probability distribution.

A non-negative integer random variable X has the hyperbolic sine distribution with parameter $\lambda \neq 0$, denoted by HS(λ), if its pmf is given as follows:

$$p_X(k) = \frac{1}{\sinh(\lambda)} \frac{\lambda^{2k+1}}{(2k+1)!}, \quad k = 0, 1, 2, \ldots \tag{6}$$

The preceding discussion of the pmfs of HC and HS distributions is beneficial to understand the construction of the proposed distribution, which is presented in the following subsection.

2.1 Discrete Hyperbolic Mixture Distribution

The discrete hyperbolic mixture distribution is created by taking a mixture of HC and HS distributions. The discrete hyperbolic mixture distribution offers greater flexibility than the HC and HS distributions alone and is able to capture a variety of dispersed data.

Let N_1 and N_2 be two random variables following HC(θ) and HS(λ) distributions, respectively. Define a non-negative random variable X as follows

$$X = \begin{cases} N_1 & \text{if } A = 0 \\ N_2 & \text{if } A = 1 \end{cases},$$

where $\Pr(A = 0) = \omega$ and $\Pr(A = 1) = 1 - \omega$. The X defined above is called the discrete 2-point mixture of random variables N_1 and N_2, and follows a distribution with the following pmf

$$\begin{aligned} p_X(k) &= \Pr(X = k) \\ &= \sum_{a=0}^{1} \Pr(X = k | A = a)\Pr(A = a) \\ &= \Pr(X = k|A = 0)\Pr(A = 0) + \Pr(X = k|A = 1)\Pr(A = 1) \\ &= \omega \Pr(N_1 = k) + (1 - \omega) \Pr(N_2 = k) \\ &= \omega \frac{\theta^{2k}}{(\cosh(\theta))(2k)!} + (1 - \omega)\frac{\lambda^{2k+1}}{(\sinh(\lambda))(2k + 1)!}. \end{aligned}$$

Definition 1. *A non-negative integer random variable X has the discrete hyperbolic mixture (DHM) distribution with parameters $\theta \neq 0$, $\lambda \neq 0$, and $\omega \in [0, 1]$, denoted by DHM(θ, λ, ω), if its pmf is given as follows:*

$$p_X(k) = \omega \frac{\theta^{2k}}{(cosh(\theta))(2k)!} + (1 - \omega)\frac{\lambda^{2k+1}}{(sinh(\lambda))(2k + 1)!}, \quad k = 0, 1, 2, \ldots \tag{7}$$

By Equation (7), the HC and HS distributions are special cases of the DHM distribution, when $\omega = 1$ and $\omega = 0$, respectively. The plots of pmf given in Equation (7) are presented by Figure 1 and Figure 2.

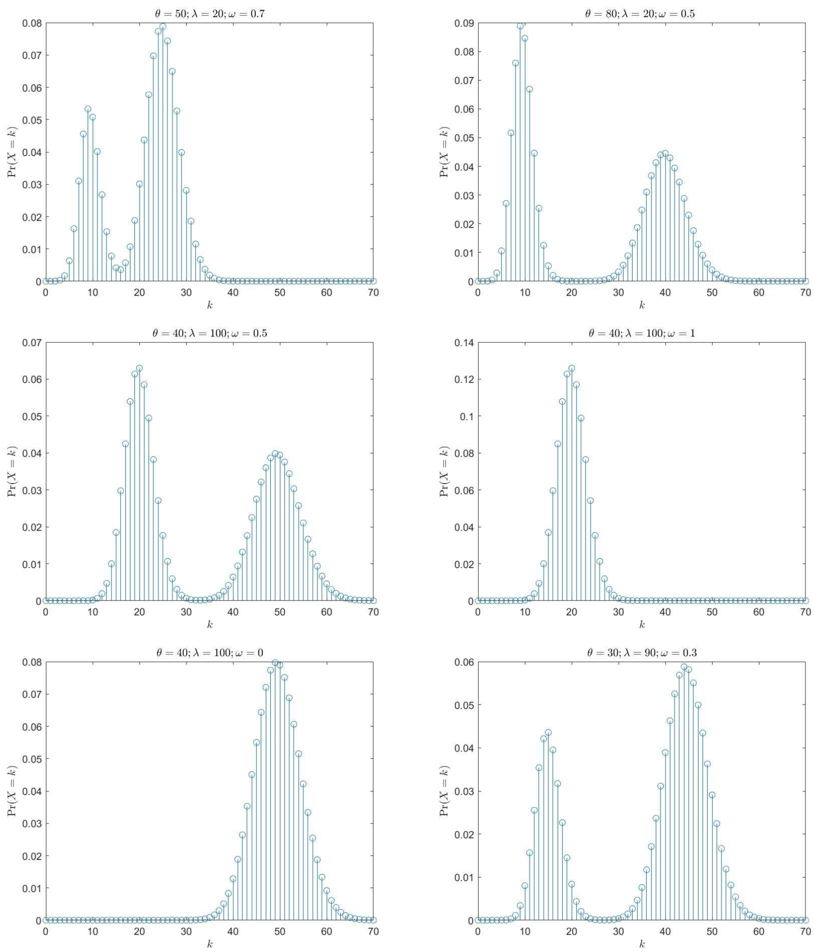

Figure 1: The pmf plots of the discrete hyperbolic mixture distribution in various combinations of $(\theta, \lambda, \omega)$.

From Figure 1, when $\omega = 1$, the high probabilities are concentrated around $\frac{\theta}{2}$, while $\omega = 0$, the high probabilities are concentrated around $\frac{\lambda-1}{2}$. From Figure 2, for the case $\theta < \lambda$, the first peak shifts to the right approaching the second peak as θ approaches λ. This also holds for the case $\theta > \lambda$ as λ approaches θ. The most significant is shown by the top right and bottom right graphs of Figure 2, when distribution has a long tail towards the right side while $\theta = 75$ and $\lambda = 100$, but it has a long tail towards the left side while both values are switched up. In addition, if the difference of θ and λ is relatively large, the DHM distribution forms a bimodal curve. Otherwise, it shows a unimodal curve. Another perspective of DHM distribution's pmf can be seen in the following Figure 3. For the case of $\theta = \lambda$, the pmfs of DHM distribution form the same pattern for different values of ω and almost similar to each other.

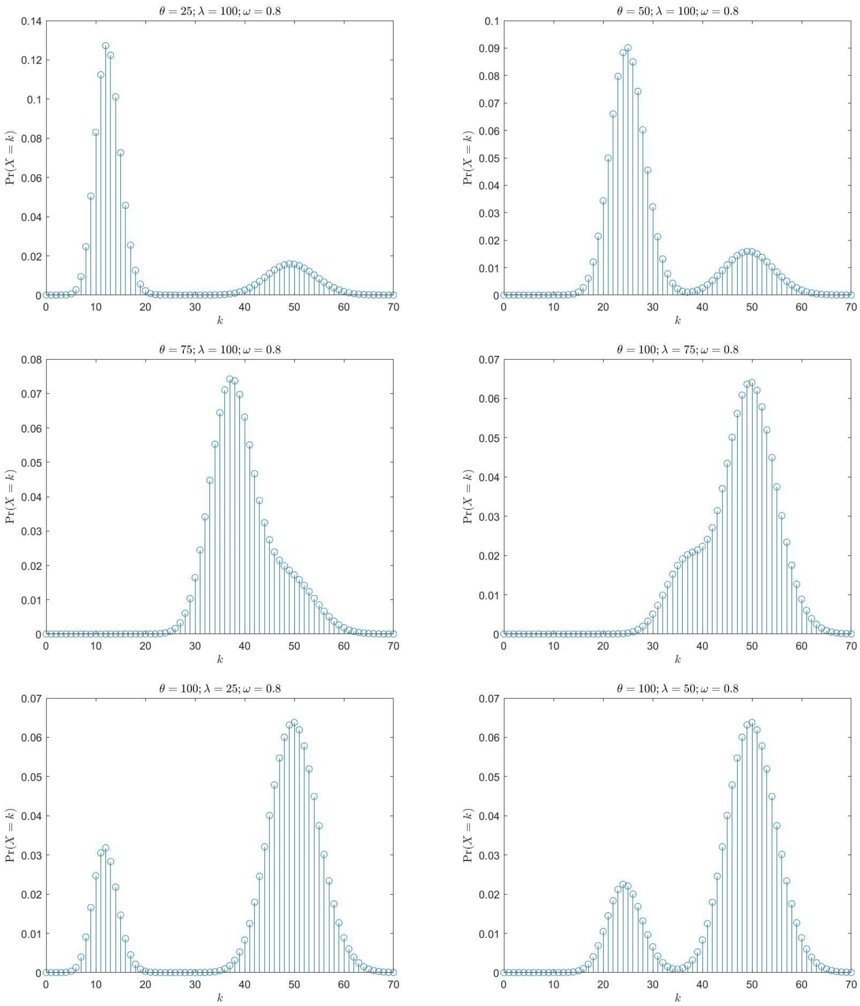

Figure 2: The pmf plots of the discrete hyperbolic mixture distribution with $\theta > \lambda$, $\theta < \lambda$ and a constant value of $\omega = 0.8$.

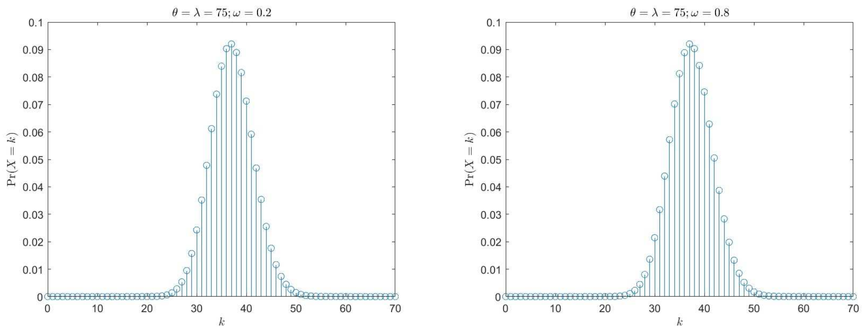

Figure 3: The pmf plots of the discrete hyperbolic mixture distribution with $\theta = \lambda$ and distinct values of ω.

Theorem 1. *Let X be a non-negative integer random variable following the DHM$(\theta, \lambda, \omega)$ with pmf*

$$p_X(k) = \omega \frac{\theta^{2k}}{(cosh(\theta))(2k)!} + (1-\omega)\frac{\lambda^{2k+1}}{(sinh(\lambda))(2k+1)!}, \quad k = 0, 1, 2, \ldots, \tag{8}$$

where $\theta > 0$, $\lambda > 0$ and $\omega \in [0, 1]$. The corresponding cumulative distribution function (cdf) of Equation (8) is

$$F_X(k) = Pr(X \leq k) = \omega F_{N_1}(k) + (1-\omega)F_{N_2}(k), \tag{9}$$

where F_{N_1} and F_{N_2} are cdfs of HC(θ) and HS(λ) distributions respectively, and $F_X(k)$ satisfies the following properties

1. $0 \leq F_X(k) \leq 1$, *for all non-negative integers k.*
2. $F_X(k)$ *is a non-decreasing function.*
3. $F_X(k)$ *is a right-continuous function.*
4. $\lim_{k\to\infty} F_X(k) = 1$ *and* $\lim_{k\to-\infty} F_X(k) = 0$.

Proof. The cdf of X can be obtained as follows:

$$\begin{aligned}
F_X(k) &= \Pr(X \leq k) \\
&= \sum_{x=0}^{k} p_X(x) \\
&= \sum_{x=0}^{k} \left(\omega \frac{\theta^{2x}}{(\cosh(\theta))(2x)!} + (1-\omega)\frac{\lambda^{2x+1}}{(\sinh(\lambda))(2x+1)!} \right) \\
&= \omega \sum_{x=0}^{k} \frac{\theta^{2x}}{(\cosh(\theta))(2x)!} + (1-\omega)\sum_{x=0}^{k} \frac{\lambda^{2x+1}}{(\sinh(\lambda))(2x+1)!} \\
&= \omega F_{N_1}(k) + (1-\omega)F_{N_2}(k).
\end{aligned} \tag{10}$$

The next step is to show that Equation (10) is a valid cdf by checking its properties.
Case 1: $\omega = 1$. In this case, Equation (10) simplifies to

$$F_X(k) = F_{N_1}(k).$$

Since $F_{N_1}(k)$ is a valid cdf of $N_1 \sim$ HC(θ), then $F_X(k) = F_{N_1}(k)$ is a valid cdf as well.
Case 2: $\omega = 0$. In this case, Equation (10) simplifies to

$$F_X(k) = F_{N_2}(k).$$

Since $F_{N_2}(k)$ is a valid cdf of $N_2 \sim$ HS(λ), then $F_X(k) = F_{N_2}(k)$ is also a valid cdf.
Case 3: $\omega \in (0, 1)$. In this case, Equation (10) remains the same.

1. Since $F_{N_1}(k), F_{N_2}(k) \in [0,1]$, then $F_X(k) = \omega F_{N_1}(k) + (1-\omega)F_{N_2}(k) \in [0,1]$ for all non-negative integers k.

2. Let k_1 and k_2 be non-negative integers such that $k_1 < k_2$.

$$F_X(k_1) = \omega F_{N_1}(k_1) + (1-\omega)F_{N_2}(k_1) \le \omega F_{N_1}(k_2) + (1-\omega)F_{N_2}(k_2) = F_X(k_2).$$

Hence, $F_X(k)$ is non-decreasing function.

3. Consider the limit from the right of $F_X(k)$ as $k \to a$, that is $\lim_{k\to a^+} F_X(k)$ or equivalent to $\lim_{\delta\to 0} F_X(k+\delta)$

$$\begin{aligned}\lim_{\delta\to 0} F_X(k+\delta) &= \lim_{\delta\to 0}\left(\omega F_{N_1}(k+\delta) + (1-\omega)F_{N_2}(k+\delta)\right)\\ &= \omega \lim_{\delta\to 0} F_{N_1}(k+\delta) + (1-\omega)\lim_{\delta\to 0} F_{N_2}(k+\delta)\\ &= \omega F_{N_1}(k) + (1-\omega)F_{N_2}(k)\\ &= F_X(k).\end{aligned}$$

Since $\lim_{\delta\to 0} F_X(k+\delta) = F_X(k)$, then $F_X(k)$ is right-continuous function.

4. Consider that

$$\begin{aligned}\lim_{k\to\infty} F_X(k) &= \lim_{k\to\infty}\left(\omega F_{N_1}(k) + (1-\omega)F_{N_2}(k)\right)\\ &= \omega \lim_{k\to\infty} F_{N_1}(k) + (1-\omega)\lim_{k\to\infty} F_{N_2}(k)\\ &= \omega\cdot 1 + (1-\omega)\cdot 1\\ &= 1.\end{aligned}$$

Also,

$$\lim_{k\to-\infty} F_X(k) = \omega \lim_{k\to-\infty} F_{N_1}(k) + (1-\omega)\lim_{k\to-\infty} F_{N_2}(k) = \omega\cdot 0 + (1-\omega)\cdot 0 = 0.$$

Therefore, the finding $F_X(k)$ is a valid cdf of X.

□

2.2 Moment Generating Function of Discrete Hyperbolic Mixture Distribution

Theorem 2. *Let $X \sim DHM(\theta,\lambda,\omega)$, then the moment generating function (mgf) of X is given by:*

$$M_X(t) = \omega\frac{cosh\left(\theta e^{\frac{t}{2}}\right)}{cosh(\theta)} + (1-\omega)\frac{e^{-\frac{t}{2}}sinh\left(\lambda e^{\frac{t}{2}}\right)}{sinh(\lambda)}, \quad for\ all\ t\in\mathbb{R}. \tag{11}$$

Proof. The mgf of the DHM distribution can be obtained as follows:

$$
\begin{aligned}
M_X(t) = E(e^{tk}) &= \sum_{k=0}^{\infty} e^{tk} p_X(k) \\
&= \sum_{k=0}^{\infty} e^{tk} \left(\omega \frac{\theta^{2k}}{(\cosh(\theta))(2k)!} + (1-\omega) \frac{\lambda^{2k+1}}{(\sinh(\lambda))(2k+1)!} \right) \\
&= \omega \sum_{k=0}^{\infty} e^{tk} \frac{\theta^{2k}}{(\cosh(\theta))(2k)!} + (1-\omega) \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^{2k+1}}{(\sinh(\lambda))(2k+1)!} \\
&= \omega \left[\frac{1}{\cosh(\theta)} + \frac{1}{\cosh(\theta)} \sum_{k=1}^{\infty} e^{tk} \frac{\theta^{2k}}{(2k)!} \right] + (1-\omega) \left[\frac{1}{\sinh(\lambda)} \sum_{k=0}^{\infty} \frac{\left(\lambda e^{\frac{t}{2}}\right)^{2k} \lambda}{(2k+1)!} \right] \\
&= \omega \left[\frac{1}{\cosh(\theta)} + \frac{1}{\cosh(\theta)} \left[\cosh\left(\theta e^{\frac{t}{2}}\right) - 1 \right] \right] + (1-\omega) \left[\frac{e^{-\frac{t}{2}}}{\sinh(\lambda)} \sum_{k=0}^{\infty} \frac{\left(\lambda e^{\frac{t}{2}}\right)^{2k+1}}{(2k+1)!} \right] \\
&= \omega \frac{\cosh\left(\theta e^{\frac{t}{2}}\right)}{\cosh(\theta)} + (1-\omega) \frac{e^{-\frac{t}{2}} \sinh\left(\lambda e^{\frac{t}{2}}\right)}{\sinh(\lambda)}.
\end{aligned}
$$

□

By Theorem 2, the mgfs of HC and HS distributions are $\frac{\cosh\left(\theta e^{\frac{t}{2}}\right)}{\cosh(\theta)}$ and $\frac{e^{-\frac{t}{2}}\sinh\left(\lambda e^{\frac{t}{2}}\right)}{\sinh(\lambda)}$, respectively.

2.3 Mean, Variance and Dispersion Index of Discrete Hyperbolic Mixture Distribution

Some probability properties of the DHM distributions, especially the mean, variance, and index of dispersion (IOD) are provided in this section.

Theorem 3. *Let $X \sim DHM(\theta, \lambda, \omega)$, then the mean of X is*

$$
E(X) = \omega \left[\frac{\theta \ sinh(\theta)}{2 \ cosh(\theta)} \right] + (1-\omega) \left[\frac{\lambda \ cosh(\lambda)}{2 \ sinh(\lambda)} - \frac{1}{2} \right]. \tag{12}
$$

Proof.

$$
\begin{aligned}
E(X) &= \sum_{k=0}^{\infty} k \ p_X(k) \\
&= \sum_{k=0}^{\infty} k \left(\omega \frac{\theta^{2k}}{(\cosh(\theta))(2k)!} + (1-\omega) \frac{\lambda^{2k+1}}{(\sinh(\lambda))(2k+1)!} \right) \\
&= \omega \sum_{k=0}^{\infty} k \frac{\theta^{2k}}{(\cosh(\theta))(2k)!} + (1-\omega) \sum_{k=0}^{\infty} k \frac{\lambda^{2k+1}}{(\sinh(\lambda))(2k+1)!} \\
&= \omega \left[\frac{1}{\cosh(\theta)} \frac{\theta}{2} \sum_{k=1}^{\infty} (2k) \frac{\theta^{2k-1}}{(2k)!} \right] + (1-\omega) \left[\frac{1}{\sinh(\lambda)} \sum_{k=1}^{\infty} (k-1) \frac{\lambda^{2k-1}}{(2k-1)!} \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \omega \left[\frac{1}{\cosh(\theta)} \frac{\theta}{2} \sum_{k=1}^{\infty} \frac{\theta^{2k-1}}{(2k-1)!} \right] + (1-\omega) \left[\frac{1}{\sinh(\lambda)} \left(\frac{1}{2} \sum_{k=1}^{\infty} (2k) \frac{\lambda^{2k-1}}{(2k-1)!} - \sum_{k=0}^{\infty} \frac{\lambda^{2k+1}}{(2k+1)!} \right) \right] \\
&= \omega \left[\frac{\theta \sinh(\theta)}{2 \cosh(\theta)} \right] + (1-\omega) \left[\frac{1}{\sinh(\lambda)} \left(\frac{1}{2} \sum_{k=0}^{\infty} (2k+2) \frac{\lambda^{2k+1}}{(2k+1)!} - \sum_{k=0}^{\infty} \frac{\lambda^{2k+1}}{(2k+1)!} \right) \right] \\
&= \omega \left[\frac{\theta \sinh(\theta)}{2 \cosh(\theta)} \right] + (1-\omega) \left[\frac{1}{\sinh(\lambda)} \left(\frac{\lambda}{2} \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} - \frac{1}{2} \sum_{k=0}^{\infty} \frac{\lambda^{2k+1}}{(2k+1)!} \right) \right] \\
&= \omega \left[\frac{\theta \sinh(\theta)}{2 \cosh(\theta)} \right] + (1-\omega) \left[\frac{1}{\sinh(\lambda)} \left(\frac{\lambda}{2} \cosh(\lambda) - \frac{1}{2} \sinh(\lambda) \right) \right] \\
&= \omega \left[\frac{\theta \sinh(\theta)}{2 \cosh(\theta)} \right] + (1-\omega) \left[\frac{\lambda \cosh(\lambda)}{2 \sinh(\lambda)} - \frac{1}{2} \right].
\end{aligned}
$$

□

By Theorem 3, the mean of HC and HS distributions are $\frac{\theta \sinh(\theta)}{2 \cosh(\theta)}$ and $\frac{\lambda \cosh(\lambda)}{2 \sinh(\lambda)} - \frac{1}{2}$, respectively.

Theorem 4. *Let $X \sim DHM(\theta, \lambda, \omega)$, then the variance of X is*

$$
\begin{aligned}
Var(X) = \omega &\left[\frac{\theta^2 + \theta \ tanh(\theta) + \lambda \ coth(\lambda) - \lambda^2 - 1}{4} \right] + \left[\frac{-\lambda \ coth(\lambda) + \lambda^2 + 1}{4} \right] \\
&- \left[\omega \left(\frac{\theta \ tanh(\theta) - \lambda \ coth(\lambda) + 1}{2} \right) + \frac{\lambda \ coth(\lambda) - 1}{2} \right]^2.
\end{aligned} \tag{13}
$$

Proof.

$$
\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 \\
&= \omega \left[\frac{\theta^2}{4} + \frac{\theta \sinh(\theta)}{4 \cosh(\theta)} \right] + (1-\omega) \left[-\frac{\lambda \cosh(\lambda)}{4 \sinh(\lambda)} + \frac{\lambda^2}{4} + \frac{1}{4} \right] \\
&- \left(\omega \left[\frac{\theta \sinh(\theta)}{2 \cosh(\theta)} \right] + (1-\omega) \left[\frac{\lambda \cosh(\lambda)}{2 \sinh(\lambda)} - \frac{1}{2} \right] \right)^2 \\
&= \omega \left[\frac{\theta^2 + \theta \tanh(\theta)}{4} \right] + (1-\omega) \left[\frac{-\lambda \coth(\lambda) + \lambda^2 + 1}{4} \right] \\
&- \left[\omega \frac{\theta \tanh(\theta)}{2} + (1-\omega) \frac{\lambda \coth(\lambda) - 1}{2} \right]^2 \\
&= \omega \left[\frac{\theta^2 + \theta \tanh(\theta) + \lambda \coth(\lambda) - \lambda^2 - 1}{4} \right] + \left[\frac{-\lambda \coth(\lambda) + \lambda^2 + 1}{4} \right] \\
&- \left[\omega \left(\frac{\theta \tanh(\theta) - \lambda \coth(\lambda) + 1}{2} \right) + \frac{\lambda \coth(\lambda) - 1}{2} \right]^2.
\end{aligned}
$$

□

By Theorem 4, the variance of HC and HS distributions are $\frac{\theta^2 + \frac{1}{2}\theta \sinh(2\theta)}{4 \cosh^2(\theta)}$ and $\frac{\lambda^2 + \lambda \sinh(2\lambda)}{4 \sinh^2(\lambda)}$, respectively.

Theorem 5. *The index of dispersion (IOD) of DHM distribution is given as follows*

$$
\begin{aligned}
IOD = \frac{1}{2} &\left[\frac{\omega(\theta^2 + \theta \ tanh(\theta) + \lambda \ coth(\lambda) - \lambda^2 - 1) + (-\lambda \ coth(\lambda) + \lambda^2 + 1)}{\omega(\theta \ tanh(\theta) - \lambda \ coth(\lambda) + 1) + (\lambda \ coth(\lambda) - 1)} \right] \\
&- \frac{1}{2} [\omega(\theta \ tanh(\theta) - \lambda \ coth(\lambda) + 1) + (\lambda \ coth(\lambda) - 1)].
\end{aligned} \tag{14}
$$

Proof.

$$
\begin{aligned}
\mathrm{IOD} = \frac{Var(X)}{E(X)} &= \frac{E(X^2)}{E(X)} - E(X) \\
&= \frac{\omega\left[\frac{\theta^2}{4} + \frac{\theta \sinh(\theta)}{4 \cosh(\theta)}\right] + (1-\omega)\left[-\frac{\lambda \cosh(\lambda)}{4 \sinh(\lambda)} + \frac{\lambda^2}{4} + \frac{1}{4}\right]}{\omega\left[\frac{\theta \sinh(\theta)}{2 \cosh(\theta)}\right] + (1-\omega)\left[\frac{\lambda \cosh(\lambda)}{2 \sinh(\lambda)} - \frac{1}{2}\right]} \\
&- \left(\omega\left[\frac{\theta \sinh(\theta)}{2 \cosh(\theta)}\right] + (1-\omega)\left[\frac{\lambda \cosh(\lambda)}{2 \sinh(\lambda)} - \frac{1}{2}\right]\right) \\
&= \frac{1}{2}\left[\frac{\omega(\theta^2 + \theta \tanh(\theta) + \lambda \coth(\lambda) - \lambda^2 - 1) + (-\lambda \coth(\lambda) + \lambda^2 + 1)}{\omega(\theta \tanh(\theta) - \lambda \coth(\lambda) + 1) + (\lambda \coth(\lambda) - 1)}\right] \\
&- \frac{1}{2}[\omega(\theta \tanh(\theta) - \lambda \coth(\lambda) + 1) + (\lambda \coth(\lambda) - 1)].
\end{aligned}
$$

□

The IOD of the DHM distribution, given in Equation (14), can be varied as it greater than, less than, or equal to one. The mathematical proof of this property is provided case by case as follows.
Case 1: $\omega = 1$. In this case, the IOD in Equation (14) simplifies to

$$\mathrm{IOD} = \frac{\theta}{\sinh(2\theta)} + \frac{1}{2},$$

which is the IOD of HC distribution. Consider the following behavior of $\frac{\theta}{\sinh(2\theta)} + \frac{1}{2}$ as $\theta \to \infty$,

$$\lim_{\theta\to\infty}\left(\frac{\theta}{\sinh(2\theta)} + \frac{1}{2}\right) = \lim_{\theta\to\infty}\left(\frac{1}{2\cosh(2\theta)}\right) + \frac{1}{2} = \frac{1}{2}\lim_{\theta\to\infty}(\mathrm{sech}(2\theta)) + \frac{1}{2} = \frac{1}{2}.$$

Consider $\frac{\theta}{\sinh(2\theta)} + \frac{1}{2}$ as $\theta \to 0$,

$$\lim_{\theta\to 0}\left(\frac{\theta}{\sinh(2\theta)} + \frac{1}{2}\right) = \lim_{\theta\to 0}\left(\frac{1}{\frac{2\sinh(2\theta)}{2\theta}}\right) + \frac{1}{2} = \lim_{\theta\to 0}\frac{1}{2} + \frac{1}{2} = 1.$$

It can be concluded that for this case the IOD of X is bounded above by one and bounded below by a half. This implies that the HC distribution is underdispersion or equidispersion.

Case 2: $\omega = 0$. In this case, the IOD in Equation (14) simplifies to

$$\mathrm{IOD} = \frac{1}{2}\left[\frac{-\lambda \coth(\lambda) + \lambda^2 + 1}{\lambda \coth(\lambda) - 1}\right] - \frac{1}{2}[\lambda \coth(\lambda) - 1] = \frac{\lambda^2 + \lambda \sinh(2\lambda)}{\lambda \sinh(2\lambda) - 2\sinh^2(\lambda)},$$

which is the IOD of HS distribution. Since $\lambda^2 > -2\sinh^2(\lambda)$ for all $\lambda \in \mathbb{R}$, then

$$\lambda \sinh(2\lambda) + \lambda^2 > \lambda \sinh(2\lambda) - 2\sinh^2(\lambda),$$

or equivalent to

$$\frac{\lambda^2 + \lambda \sinh(2\lambda)}{\lambda \sinh(2\lambda) - 2\sinh^2(\lambda)} > 1.$$

Hence, it can be concluded that for this case the IOD of X is bounded below by one. This implies that the HS distribution is overdispersion.

Case 3: $\omega \in (0,1)$. The following inequalities hold for any given values of θ and λ.

$$0 < \frac{1}{2}\left[\frac{\omega(\theta^2 + \theta \tanh(\theta) + \lambda \coth(\lambda) - \lambda^2 - 1) + (-\lambda \coth(\lambda) + \lambda^2 + 1)}{\omega(\theta \tanh(\theta) - \lambda \coth(\lambda) + 1) + (\lambda \coth(\lambda) - 1)}\right] < \infty,$$

and

$$-\infty < -\frac{1}{2}[\omega(\theta \cdot \tanh(\theta) - \lambda \cdot \coth(\lambda) + 1) + (\lambda \cdot \coth(\lambda) - 1)] < 0.$$

However, $\frac{1}{2}\left[\frac{\omega(\theta^2+\theta\cdot\tanh(\theta)+\lambda\cdot\coth(\lambda)-\lambda^2-1)+(-\lambda\cdot\coth(\lambda)+\lambda^2+1)}{\omega(\theta\cdot\tanh(\theta)-\lambda\cdot\coth(\lambda)+1)+(\lambda\cdot\coth(\lambda)-1)}\right] > -\frac{1}{2}[\omega(\theta \cdot \tanh(\theta) - \lambda \cdot \coth(\lambda) + 1) + (\lambda \cdot \coth(\lambda) - 1)]$. This implies:

$$\begin{aligned}0 < \frac{1}{2}&\left[\frac{\omega(\theta^2 + \theta \cdot \tanh(\theta) + \lambda \cdot \coth(\lambda) - \lambda^2 - 1) + (-\lambda \cdot \coth(\lambda) + \lambda^2 + 1)}{\omega(\theta \cdot \tanh(\theta) - \lambda \cdot \coth(\lambda) + 1) + (\lambda \cdot \coth(\lambda) - 1)}\right] \\ &- \frac{1}{2}[\omega(\theta \cdot \tanh(\theta) - \lambda \cdot \coth(\lambda) + 1) + (\lambda \cdot \coth(\lambda) - 1)] < \infty.\end{aligned}$$

Therefore, based on the three cases above, the IOD of the DHM distribution can take any positive real value. The value can be less or more than one depending on the parameter settings. It implies that the DHM distribution can be an appropriate alternative for modeling the overdispersed, underdispersed, and equidispersed count data.

2.4 Probability Generating Function of Discrete Hyperbolic Mixture Distribution

Theorem 6. *Let $X \sim DHM(\theta, \lambda, \omega)$, then the probability generating function (pgf) of X is given by:*

$$G_X(t) = \omega \frac{cosh\left(\theta\sqrt{t}\right)}{cosh(\theta)} + (1-\omega)\frac{sinh(\lambda\sqrt{t})}{\sqrt{t}\ sinh(\lambda)}, \quad for\ all\ t > 0. \tag{15}$$

Proof. The pgf of the DHM distribution can be obtained as follows:

$$\begin{aligned}G_X(t) = E(t^k) &= \sum_{k=0}^{\infty} t^k p_X(k) \\ &= \sum_{k=0}^{\infty} t^k \left(\omega \frac{\theta^{2k}}{(\cosh(\theta))(2k)!} + (1-\omega)\frac{\lambda^{2k+1}}{(\sinh(\lambda))(2k+1)!}\right) \\ &= \omega \sum_{k=0}^{\infty} t^k \frac{\theta^{2k}}{(\cosh(\theta))(2k)!} + (1-\omega)\sum_{k=0}^{\infty} t^k \frac{\lambda^{2k+1}}{(\sinh(\lambda))(2k+1)!} \\ &= \omega\left[\frac{1}{\cosh(\theta)} + \frac{1}{\cosh(\theta)}\sum_{k=1}^{\infty} t^k \frac{\theta^{2k}}{(2k)!}\right] + (1-\omega)\left[\frac{1}{\sqrt{t}\sinh(\lambda)}\sum_{k=0}^{\infty}\frac{\left(\lambda\sqrt{t}\right)^{2k+1}}{(2k+1)!}\right] \\ &= \omega\left[\frac{1}{\cosh(\theta)} + \frac{1}{\cosh(\theta)}\left[\cosh\left(\theta\sqrt{t}\right) - 1\right]\right] + (1-\omega)\frac{\sinh(\lambda\sqrt{t})}{\sqrt{t}\sinh(\lambda)} \\ &= \omega\frac{\cosh\left(\theta\sqrt{t}\right)}{\cosh(\theta)} + (1-\omega)\frac{\sinh(\lambda\sqrt{t})}{\sqrt{t}\sinh(\lambda)}.\end{aligned}$$

□

By Theorem 6, the pgfs of HC and HS distributions are $\frac{\cosh\left(\theta\sqrt{t}\right)}{\cosh(\theta)}$ and $\frac{\sinh(\lambda\sqrt{t})}{\sqrt{t}\sinh(\lambda)}$, respectively.

3 Maximum Likelihood Estimation

In this study, maximum likelihood estimation (MLE) is applied to estimate the parameters of the DHM distribution as the MLE will result in unbiased estimator for big sample size. The estimation is performed case by case.

Case 1: $\omega = 1$. In this case, the DHM distribution reduces to HC distribution. Let $X_1, X_2, \ldots, X_n$ be an independent and identically distributed (i.i.d) random sample of size n from the HC distributed population, HC(θ), and $x_1, x_2, \ldots, x_n$ be the realization values of random variables $X_1, X_2, \ldots, X_n$ respectively. For every $k = 0, 1, 2, \ldots, n_k$ denotes the number of observations where $X = k$. Let n denotes total number of observations (sample size). Hence

$$n = \sum_{k=0}^{\infty} n_k.$$

The likelihood function can be written as

$$\begin{aligned} L(\theta; x_1, x_2, \ldots, x_n) = \prod_{k=0}^{\infty} [\Pr(X = k)]^{n_k} &= [\Pr(X = 0)]^{n_0} \prod_{k=1}^{\infty} [\Pr(X = k)]^{n_k} \\ &= \left[\frac{1}{\cosh(\theta)}\right]^{n_0} \prod_{k=1}^{\infty} \left[\frac{\theta^{2k}}{(\cosh(\theta))(2k)!}\right]^{n_k}. \end{aligned}$$

Taking the logarithm of the likelihood function, we obtain the log-likelihood function:

$$l(\theta; x_1, x_2, \ldots, x_n) = -n_0 \log(\cosh(\theta)) + \sum_{k=1}^{\infty} 2kn_k \log(\theta) - \sum_{k=1}^{\infty} n_k \log(\cosh(\theta)) - \sum_{k=1}^{\infty} n_k \log((2k)!).$$

The MLE of θ can be obtained by solving the following equation:

$$\frac{\partial l}{\partial \theta} = 0 \Longleftrightarrow -n_0 \tanh(\theta) + \frac{1}{\theta} \sum_{k=1}^{\infty} 2kn_k - \tanh(\theta) \sum_{k=1}^{\infty} n_k = 0,$$

or it can be rewritten as follows

$$\frac{2}{\theta} \bar{X} - \tanh(\theta) = 0, \tag{16}$$

where $\bar{X} = \frac{\sum_{k=1}^{\infty} kn_k}{n}$. The solution of nonlinear Equation (16) does not have a closed-form, so a numerical approach is required to find the MLE of θ. This study uses the Newton-Raphson method to find $\hat{\theta}$.

Case 2: $\omega = 0$. In this case, the DHM distribution reduces to the HS distribution. Let $X_1, X_2, \ldots, X_n$ be i.i.d random sample of size n from the HS distributed population, HS(λ). The likelihood function can be written as

$$\begin{aligned} L(\lambda; x_1, x_2, \ldots, x_n) = \prod_{k=0}^{\infty} [\Pr(X = k)]^{n_k} &= [\Pr(X = 0)]^{n_0} \prod_{k=1}^{\infty} [\Pr(X = k)]^{n_k} \\ &= \left[\frac{\lambda}{\sinh(\lambda)}\right]^{n_0} \prod_{k=1}^{\infty} \left[\frac{\lambda^{2k+1}}{(\sinh(\lambda))(2k+1)!}\right]^{n_k}. \end{aligned}$$

Taking the logarithm of the likelihood function, we obtain the log-likelihood function:

$$\begin{aligned} l(\lambda; x_1, x_2, \ldots, x_n) = n_0 \log(\lambda) - n_0 \log(\sinh(\lambda)) + \sum_{k=1}^{\infty} (2k+1) n_k \log(\lambda) \\ - \sum_{k=1}^{\infty} n_k \log(\sinh(\lambda)) - \sum_{k=1}^{\infty} n_k \log((2k+1)!). \end{aligned}$$

The MLE of λ can be obtained by solving the following equation:

$$\frac{\partial l}{\partial \lambda} = 0 \Longleftrightarrow \frac{n_0}{\lambda} - n_0 \coth(\lambda) + \frac{1}{\lambda} \sum_{k=1}^{\infty} (2k+1) n_k - \coth(\lambda) \sum_{k=1}^{\infty} n_k = 0,$$

or it can be rewritten as follows

$$\frac{1}{\lambda}(2\bar{X}+1)-\coth(\lambda)=0, \tag{17}$$

where $\bar{X}=\frac{\sum_{k=1}^{\infty}kn_k}{n}$. The Newton-Raphson method is also applied to find the solution of nonlinear Equation (17) since this equation cannot be solved analytically.

Case 3: $\omega\in(0,1)$. Let $X_1,X_2,\ldots,X_n$ form a random sample of size n from the DHM distributed population, DHM(θ,λ,ω), and $x_1,x_2,\ldots,x_n$ be the realization values of random variables $X_1,X_2,\ldots,X_n$ respectively. The likelihood function can be written as

$$\begin{aligned}L(\theta,\lambda,\omega;x_1,x_2,\ldots,x_n)&=\prod_{k=0}^{\infty}[\Pr(X=k)]^{n_k}\\&=\prod_{k=0}^{\infty}\left[\omega\frac{\theta^{2k}}{(\cosh(\theta))(2k)!}+(1-\omega)\frac{\lambda^{2k+1}}{(\sinh(\lambda))(2k+1)!}\right]^{n_k}.\end{aligned}$$

Taking the logarithm of the likelihood function, we obtain the log-likelihood function:

$$l(\theta,\lambda,\omega;x_1,x_2,\ldots,x_n)=\sum_{k=0}^{\infty}n_k\log\left[g(k;\lambda)+\omega(h(k;\theta)-g(k;\lambda))\right], \tag{18}$$

where $g(k;\lambda)=\frac{\lambda^{2k+1}}{(\sinh(\lambda))(2k+1)!}$ and $h(k;\theta)=\frac{\theta^{2k}}{(\cosh(\theta))(2k)!}$. The value of θ, λ, and ω that maximizes Equation (18) can be obtained by solving the following system of equations:

$$\frac{\partial l}{\partial\omega}=\sum_{k=0}^{\infty}n_k\frac{h(k;\theta)-g(k;\lambda)}{g(k;\lambda)+\omega(h(k;\theta)-g(k;\lambda))}=0, \tag{19}$$

$$\frac{\partial l}{\partial\theta}=\sum_{k=0}^{\infty}\frac{\omega\frac{\partial}{\partial\theta}[h(k;\theta)]}{g(k;\lambda)+\omega(h(k;\theta)-g(k;\lambda))}=0, \tag{20}$$

$$\frac{\partial l}{\partial\lambda}=\sum_{k=0}^{\infty}\frac{(1-\omega)\frac{\partial}{\partial\lambda}[g(k;\lambda)]}{g(k;\lambda)+\omega(h(k;\theta)-g(k;\lambda))}=0. \tag{21}$$

The system of nonlinear Equations (19), (20) and (21) does not have a closed-form solution. Therefore, it requires a numerical approach to find $\hat{\theta}$, $\hat{\lambda}$, and $\hat{\omega}$. As in two previous cases, the Newton-Raphson method is applied as well to support the MLE for this purpose.

4 Data Simulation

This section presents numerical simulations to assess the performance of proposed estimators for some specified values of the parameters of the HC, HS, and DHM distributions, using several sample sizes. Monte Carlo experiment is repeated 10,000 times for each specified value of the parameters of corresponding distribution. The estimated parameters are obtained using the MLE method discussed in Section 3. The performance of the MLE estimator is analyzed using two measures, the average estimate and the mean squared error (MSE), which are calculated as follows:

$$\text{Average estimate}=\frac{\sum_{j=1}^{10,000}\hat{\tau}_j}{10,000};\quad \text{MSE}=\frac{\sum_{j=1}^{10,000}(\hat{\tau}_j-\tau)^2}{10,000},$$

where $\hat{\tau}_j$ is the fitted value for the considered parameter on the j-th experiment and τ is the true value of the considered parameter. The numerical results are presented in Tables 1, 2 and 3.

Table 1: Numerical results of HC distribution parameter estimation for simulated data of size n.

Parameter	Average Estimate				MSE			
	$n=50$	$n=100$	$n=500$	$n=1000$	$n=50$	$n=100$	$n=500$	$n=1000$
$\theta=0.05$	0.0117	0.0170	0.0339	0.0401	0.0037	0.0033	0.0017	0.0009
$\theta=0.5$	0.4900	0.4897	0.4974	0.4987	0.0128	0.0023	0.0004	0.0002
$\theta=5$	4.9988	4.9997	5.0007	5.0001	0.0990	0.0171	0.0033	0.0017
$\theta=50$	50.0117	50.0077	49.9994	49.9958	1.0081	0.1567	0.0996	0.0477

Table 2: Numerical results of HS distribution parameter estimation for simulated data of size n.

Parameter	Average Estimate				MSE			
	$n=50$	$n=100$	$n=500$	$n=1000$	$n=50$	$n=100$	$n=500$	$n=1000$
$\lambda=0.05$	0.0042	0.0151	0.0272	0.0295	0.0035	0.0048	0.0030	0.0021
$\lambda=0.5$	0.4417	0.4740	0.4972	0.4984	0.0517	0.0199	0.0029	0.0016
$\lambda=5$	5.0134	5.0011	4.9987	5.0033	0.0897	0.0503	0.0102	0.0047
$\lambda=50$	50.0801	50.0010	50.0078	50.0042	1.0255	0.5052	0.0937	0.0491

Table 3: Numerical results of DHM distribution parameter estimation for simulated data of size n.

Parameter	Average Estimate				MSE			
	$n=50$	$n=100$	$n=500$	$n=1000$	$n=50$	$n=100$	$n=500$	$n=1000$
$\theta=0.5$	0.4386	0.4651	0.4022	0.4452	0.2073	0.1838	0.1258	0.0935
$\lambda=5$	5.0761	5.0811	5.0104	5.0129	0.2394	0.1298	0.0275	0.0128
$\omega=0.2$	0.2213	0.2206	0.2059	0.2048	0.0107	0.0063	0.0019	0.0011
$\theta=5$	5.0219	5.1121	4.5276	4.5644	1.1642	1.8939	1.0850	1.1607
$\lambda=5$	5.9884	5.4094	4.9858	4.9748	33.2003	15.0432	1.9805	0.9016
$\omega=0.5$	0.7574	0.6782	0.5499	0.4936	0.1208	0.1203	0.0698	0.0637
$\theta=5$	5.1270	5.0841	5.0243	5.0157	0.2788	0.1007	0.0190	0.0094
$\lambda=0.5$	0.6602	0.6035	0.4789	0.4555	0.7892	0.4773	0.1891	0.1284
$\omega=0.8$	0.7686	0.7797	0.7930	0.7963	0.0114	0.0037	0.0007	0.0003

Based on the results presented in the tables, the average estimates of the parameters are generally close to their true values, with the distance never exceeding 0.5. Moreover, the average estimates are close to the true parameters as the sample size increases. It is also supported by its MSE, as the sample size gets larger, the MSE gets smaller. Therefore, it is sufficient to conclude that the finding estimators are appropriate to be used as the point estimation of each parameter.

5 Applications with Real Life Data

This section provides the application of the DHM distribution to two real datasets to illustrate the usefulness of the DHM distribution and to be compared with Poisson, geometric, and partial-geometric distributions. The pmfs of these competing models are as follows:

Poisson:

$$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots, \quad \text{where } \lambda > 0.$$

Geometric:

$$p_X(k) = (1-p)^k p, \quad k = 0, 1, 2, \ldots, \quad \text{where } 0 < p < 1.$$

Partial-Geometric:

$$p_X(k) = \begin{cases} 1 - \frac{\alpha(1-p)}{p}, & k = 0 \\ \alpha(1-p)^k, & k = 1, 2, \ldots \end{cases}, \quad \text{where } 0 < p < 1 \text{ and } 0 \le \alpha \le \frac{p}{1-p}.$$

The first dataset represents the number of sperm in an egg with respect to fertilization in a sea-urchin's egg (Morgan, 1975). The second dataset is the number of claims for 64,548 motor vehicle insurance policyholders of an insurance company in Sweden during 1994-1998, obtained from the R package namely "insuranceData". Both datasets are listed in Table 4.

Table 4: Datasets description.

Dataset 1		Dataset 2	
Number of Sperm	Observed Frequency	Number of Claims	Observed Frequency
0	28	0	63,878
1	44	1	643
2	7	2	27
3	1		
Total	80	Total	64,548
Mean	0.7625	Mean	0.0108
Variance	0.4311	Variance	0.0115
IOD	0.5657	IOD	1.0648

Table 4 indicates that the datasets 1 and 2 exhibit underdispersion and overdispersion, respectively. The results for fitting respective dataset to all four distributions are evaluated by providing adequate measures such as the maximum log-likelihood value, AIC, BIC, and chi-squared goodness-of-fit statistic with its p-values, as presented in Tables 5 and 6, respectively.

Table 5: Fitting results for dataset 1.

Number of Sperm	Observed Frequency	Expected Frequency DHM	Poisson	Geometric	Partial-Geometric
0	28	29.8171	37.3199	45.392	28
1	44	40.2645	28.4564	19.6366	44.3279
2	7	9.0621	10.8490	8.4948	6.5402
3	1	0.8158	2.7575	3.6748	0.9649
Fitted parameters		$\hat{\theta} = 1.6434$ $\hat{\lambda} = 1.4913$ $\hat{\omega} = 0.9997$	$\hat{\lambda} = 0.7625$	$\hat{p} = 0.5674$	$\hat{p} = 0.8525$ $\hat{\alpha} = 3.7556$
chi-squared test statistic		0.9681	13.3035	39.1017	0.0360
p-value		0.6163	0.0013	0.0000	0.8495
Log-likehood		-77.67407	-84.18411	-96.4497	-77.31934
AIC		157.3481	170.3682	194.8994	158.6387
BIC		159.7302	172.7502	197.2814	163.4027

Table 6: Fitting results for dataset 2.

Number of Claims	Observed Frequency	Expected Frequency DHM	Poisson	Geometric	Partial-Geometric
0	63878	63878.0111	63854.6318	63857.3364	63877.8822
1	643	643.4201	689.6302	683.2735	644.0503
2	27	26.1351	3.7240	7.3110	25.0536
Fitted parameters		$\hat{\theta} = 0.7008$ $\hat{\lambda} = 0.0218$ $\hat{\omega} = 0.0506$	$\hat{\lambda} = 0.0108$	$\hat{p} = 0.9893$	$\hat{p} = 0.9611$ $\hat{\alpha} = 0.2565$
chi-squared test statistic		0.0289	148.6428	55.4043	0.1529
p-value		0.8650	0.0000	0.0000	0.6958
Log-likehood		-3840.598	-3871.995	-3857.031	-3753.578
AIC		7687.196	7745.990	7716.062	7511.156
BIC		7714.422	7755.065	7725.137	7529.307

According to Tables 5 and 6, the p-values of the chi-squared goodness-of-fit test for the DHM distribution are greater than the significance level of 0.05 for both datasets. These indicate that both datasets are compatible with the DHM distribution. The DHM distribution provides the closest expected frequencies to the observed frequencies, particularly for the second dataset.

For dataset 1, the DHM and partial-geometric distributions are the most appropriate models. Of these, the DHM distribution is the most suitable, as it yields the smallest AIC and BIC values. For dataset 2, the DHM and partial-geometric distributions are again the most appropriate models. In this case, the partial-geometric distribution is slightly more suitable, as it yields the smallest AIC and BIC values, although the differences between the two distributions are relatively small. Nevertheless, the DHM distribution serves as a viable alternative to the partial-geometric distribution for this dataset.

We have elaborated that dataset 1 describes the fertilization process in marine biology, which naturally produces underdispersed count data, since biological constraints limit the number of sperm involved. The DHM distribution is shown to effectively capture this behavior. Dataset 2 describes claims incurred in an insurance company, which produces overdispersed count data, since a small group of individuals has multiple claims who increases the variance beyond the mean, which is a common cause of overdispersion. The DHM distribution has also been shown to effectively capture overdispersion. Furthermore, the DHM distribution is also potential to capture other real-world scenarios, such as accident counts, disease incidence rates, and industrial failure counts.

6 Conclusion

This research introduces the novel DHM distribution, constructed by considering a mixture of hyperbolic cosine and hyperbolic sine distributions. As confirmed by its IOD, the DHM distribution offers greater flexibility than many existing distributions, as it can generate underdispersed, equidispersed, and overdispersed data. Moreover, the pmf of the DHM distribution can be unimodal or bimodal and can exhibit either left-skewness or right-skewness. This study also derived essential probability proper-ties of the DHM distribution, including its mgf and pgf.

Parameter estimation for the DHM distribution and its sub-models is performed using MLE with a numerical method because the log-likelihood function cannot be obtained analytically. The performance of the MLE is examined by conducting Monte Carlo simulations with various parameter values and sample sizes. The simulation results show that the estimated parameters are very close to the true parameter values, particularly for large sample sizes.

Furthermore, the DHM distribution compares favorably with the Poisson, partial-geometric, and geometric distributions. The compatibility of the DHM distribution with datasets 1 and 2 is confirmed by the p-values of the chi-squared goodness-of-fit test. The DHM distribution also provides the smallest AIC and BIC values for dataset 1. Therefore, the DHM distribution can be a useful alternative to other existing distributions for modeling count data.

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