Advanced Mathematical Models & Applications
Vol.10, No.2, 2025, pp.363-371
https://doi.org/10.62476/amma.102363

Jomard Publishing

EDGE MODERN ROMAN DOMINATION OF GRAPHS

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Abstract. In graph theory, domination is an important concept with many practical applications. Different types of domination have been defined based on the structure of dominating sets. In this study, we propose a new domination parameter by focusing on the edges of a graph, inspired by the concept of Modern Roman domination, which models defense strategies. This new concept, called the Edge Modern Roman domination, assigns labels to the edges of a graph in such a way that each edge without defense is supported by neighboring edges that provide protection. We calculate the minimum sum of these assigned labels, called the Edge Modern Roman domination number, for some standard and known graphs. Furthermore, we investigate the behavior of this parameter under some certain graph operations such as the corona product and the Cartesian product.

Keywords: Edge domination, roman domination, edge roman domination, modern roman domination, edge modern roman domination.

AMS Subject Classification: 68R10, 05C69, 05C70, 05C22, 05C76.

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Received: 27 December 2024; Revised: 18 April 2025; Accepted: 27 April 2025; Published: 4 August 2025.

1 Introduction

Since the 1970s, domination in graph theory has been one of the most researched concepts. This is because of its numerous applications in various fields, including facility location problems, which aim to minimize the distance a person must travel to reach the closest facility.

In simple graph $G = (V, E)$, a set $S \subseteq V$ is a dominating set if every vertex $v \in V$ is an element of S or adjacent to an element of S (Haynes et al., 1998). Alternatively, we can say that $S \subseteq V$ is a dominating set of G if $N[S] = V(G)$. A dominating set S is a minimal dominating set if no proper subset $S' \subset S$ is a dominating set. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G. Similarly, an edge dominating set is a subset S of E if $e \in E{-}S$ is adjacent to an edge in S and the minimum cardinality of S called edge domination number $\gamma'(G)$ (Mitchell et al., 1977). In addition, other examples of domination types that have been studied are co-even domination, edge co-even domination, modern roman domination, double edge-vertex domination, etc. (Hedetniemi et al., 2009; Shalaan et al., 2020; Kılıç et al., 2020; Demirpolat et al., 2025)

In the article published in 1999, Emperor Constantine's plan for protecting the Roman Empire was examined by Ian Stewart (Stewart et al., 1999). With inspiration from this article, many studies have been done on Roman Domination. The Modern Roman Domination is the labeling of each vertex by a function $f : V(G) \to \{0, 1, 2, 3\}$ under certain rules. This definition is a new parameter defined by Salah et al. (2022) considering defense systems.

This article presents the Edge Modern Roman domination (EMRD), a new graph domination model based on the Roman dominating function. The EMRD function's defense strategy is based on the fact that edges labeled 2 or 3 are considered as strong defenders, while edges labeled 0 or 1 require protection. Each unsecured edge (label 0) must be adjacent to both a label-2 and a label-3 edge, ensuring two-legion support in case of an attack.

In this paper, we introduced a new domination parameter called edge modern roman domination and studied this number of some standard graph types such as P_n, C_n, $K_{1,n}$, $K_{m,n}$. In addition, we determined this number of some known graphs such as the binomial tree B_n, the jellyfish graph $J(m,n)$, the diamond snake graph D_n, the coconut tree $CT(m,n)$, the firecracker graph $F(m,n)$ and studied under some graph operations as $P_2 \times P_n$, $P_2 \times C_n$, $C_n \odot \bar{K_m}$ on $\gamma'_{mR}(G)$.

2 Edge Modern Roman Domination Number

In this section, a new domination parameter in graphs called briefly "EMRD" is introduced. The number of this parameter was obtained on the basic graphs.

Definition 1. *An edge Modern Roman dominating function (EMRDF) on a graph $G = (V, E)$ is a labeling function $f : E(G) \to \{0, 1, 2, 3\}$ such that every edge with label 0 is adjacent to two edges one of them is of label 2 and the other is of label 3 and every edge with label 1 is adjacent to an edge with label 2 or 3. The edge Modern Roman domination number $\gamma'_{mR}(G)$ is the minimum weight $f(E) = \sum_{e \in E} f(e)$ over all such functions of G.*

The Edge Modern Roman Domination number is calculated in given Figure 1 below.

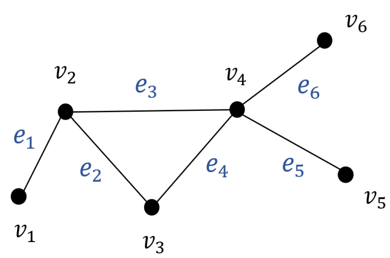

Figure 1: A graph G with six vertices

To minimize the total weight while satisfying the $EMRDF$ rules, the labeling for Figure 1 is assigned as follows: $f(e_1) = 1$, $f(e_2) = 0$, $f(e_3) = 3$, $f(e_4) = 2$, $f(e_5) = 0$ and $f(e_6) = 0$. Therefore, $\gamma'_{mR}(G) = 1 + 0 + 3 + 2 + 0 + 0 = 6$.

Theorem 1. *For $n \geq 3$,*

$$\gamma'_{mR}(P_n) = \begin{cases} 5\frac{n}{4}, & if\ n \equiv 0(mod4) \\ 5\lfloor\frac{n}{4}\rfloor + 1, & if\ n \equiv 1(mod4) \\ 5\lfloor\frac{n}{4}\rfloor + 2, & if\ n \equiv 2(mod4) \\ 5\lfloor\frac{n}{4}\rfloor + 3, & if\ n \equiv 3(mod4) \end{cases}$$

Proof. Let $E(P_n) = \{e_1, ..., e_{n-1}\}$. For $n = 2$, it is clear that the P_2 graph consist of only one edge and is labeled $f(e_1) = 2$ by the $EMRDF$ rules. For $n \geq 3$, two types of labeling can be done. One of them is in the form of $f(e_1) = 1$, $f(e_2) = 2$, $f(e_3) = 1$. Therefore in this labeling form, for every three edges corresponding to n vertices, that is $\frac{n}{4}$, the sum is 4.The other is in the form of $f(e_1) = 2$, $f(e_2) = 0$, $f(e_3) = 3$ and $f(e_4) = 0$. Therefore for every four edges corresponding to n vertices, that is $\frac{n}{5}$, the sum is 5. Since we target to obtain the minimum weight for P_n, the second technique is used.

There are four cases according to the divisibility of n by four.

Case 1. When $n \equiv 0(mod4)$, $\frac{n}{4}$ is obtained since labeling is done in quadruples and each weight of these is 5. Therefore, $\gamma^{'}_{mR}(P_n) = 5\frac{n}{4}$.

Case 2. When $n \equiv 1(mod4)$, then $\frac{n-1}{4}$ is obtained since labeling of edges is done in quadruples and this is expressed as an approximate lower bound by $\lfloor\frac{n}{4}\rfloor$. Each of these labeled quad edges has a total weight of five and the last remaining edge e_{n-1} is labeled $f(e_{n-1}) = 1$ since it has a neighbor $f(e_{n-2}) = 3$. Therefore, $\gamma^{'}_{mR}(P_n) = 5\lfloor\frac{n}{4}\rfloor + 1$.

Case 3. When $n \equiv 2(mod4)$, there are $n-2$ vertices are divisible by four and one edge remains. Similarly, $\frac{n-2}{4}$ is obtained since labeling is done in quadruples and this is expressed as an approximate lower bound by $\lfloor\frac{n}{4}\rfloor$. Each of these labeled quad edges has a total weight of five and the remaining edge e_{n-1} is labeled as $f(e_{n-1}) = 2$ since it has a neighbor $f(e_{n-2}) = 0$. Therefore, $\gamma^{'}_{mR}(P_n) = 5\lfloor\frac{n}{4}\rfloor + 2$.

Case 4. When $n \equiv 3(mod4)$, there are $n-2$ vertices are divisible by four and two edges remain. Comparably, $\frac{n-3}{4}$ is obtained since labeling is done in quadruples and this is expressed as an approximate lower bound by $\lfloor\frac{n}{4}\rfloor$. Each of these labeled quad edges has a total weight of five and the remaining edges e_{n-2} and e_{n-1} are labeled as $f(e_{n-2}) = 2$, $f(e_{n-1}) = 1$. Therefore, $\gamma^{'}_{mR}(P_n) = 5\lfloor\frac{n}{4}\rfloor + 2 + 1 = 5\lfloor\frac{n}{4}\rfloor + 3$.

The proof is complete from all cases.

□

Theorem 2. *For $n \geq 3$,*

$$\gamma^{'}_{mR}(C_n) = \begin{cases} 5\frac{n}{4}, & \text{if } n \equiv 0(mod4) \\ 5\lfloor\frac{n}{4}\rfloor + 2, & \text{if } n \equiv 1(mod4) \\ 5\lfloor\frac{n}{4}\rfloor + 3, & \text{if } n \equiv 2(mod4) \\ 5\lfloor\frac{n}{4}\rfloor + 4, & \text{if } n \equiv 3(mod4) \end{cases}$$

Proof. Let $E(C_n) = \{e_1, ..., e_n\}$ denote the edges of C_n. Similarly Theorem 2, there are two different ways to label these edges and the minimum of them is $e_1 = 2$, $e_2 = 0$, $e_3 = 3$, $e_4 = 0$ and so on. There are four cases that depend on the order of the cycle.

Case 1. When $n \equiv 0(mod4)$, $\frac{n}{4}$ is obtained since labeling is done in quadruples and each weight of these is 5. Therefore, $\gamma^{'}_{mR}(C_n) = 5\frac{n}{4}$.

Case 2. When $n \equiv 1(mod4)$, then $\frac{n-1}{4}$ is obtained since labeling is done in quadruples and this is expressed as an approximate lower bound by $\lfloor\frac{n}{4}\rfloor$. Each of these labeled quad edges has a total weight of five and the last remaining edge e_n is labeled with $f(e_n) = 2$ since it has a neighbor labeled $f(e_{n-1}) = 0$ which means the edge e_{n-1} needs neighbors labeled both 2 and 3 according to the EMRDF rules. Therefore, $\gamma^{'}_{mR}(C_n) = 5\lfloor\frac{n}{4}\rfloor + 2$.

Case 3. When $n \equiv 2(mod4)$, there are $n-2$ edges are divisible by four and two edges remain. Similiarly, $\frac{n-2}{4}$ is obtained since labeling is done in quadruples and this is expressed as an approximate lower bound by $\lfloor\frac{n}{4}\rfloor$. Each of these labeled quad edges has a total weight of five and the remaining edges are labeled as $f(e_n) = 1$ and $f(e_{n-1}) = 2$. Therefore, $\gamma^{'}_{mR}(C_n) = 5\lfloor\frac{n}{4}\rfloor + 2 + 1 = 5\lfloor\frac{n}{4}\rfloor + 3$.

Case 4. When $n \equiv 3(mod4)$, there are $n-3$ edges are divisible by four and three edges remain. $\frac{n-3}{4}$ is obtained since labeling is done in quadruples and this is expressed as an approximate lower bound by $\lfloor\frac{n}{4}\rfloor$. Two types of labeling can be done for the remaining three edges. One of them is $f(e_{n-2}) = 2$, $f(e_{n-1}) = 0$ and $f(e_n) = 3$ and the other is $f(e_{n-2}) = 2$, $f(e_{n-1}) = 1$ and $f(e_n) = 1$ since it has a neighbor labeled $f(e_1) = 2$. Since it is the second labeling that gives the minimum value, that labeling is used for the last three edges. Therefore, $\gamma'_{mR}(C_n) = 5\lfloor\frac{n}{4}\rfloor + 2 + 1 + 1 = 5\lfloor\frac{n}{4}\rfloor + 4$.

From all the above cases, the proof is completed. □

Theorem 3. *For $n \geq 3$, $\gamma'_{mR}(K_{1,n}) = 5$.*

Proof. Let $E(K_{1,n}) = \{e_1, ..., e_n\}$ be the edges of the graph. If $f(e_1) = 2$ and $f(e_2) = 3$, then $f(e_i) = 0$ can be labeled for all remaining edges e_i where $i = 3, ..., n-1$. Hence, $\gamma'_{mR}(K_{1,n}) = 5$. □

Theorem 4. *For $m, n \geq 2$ and $m \leq n$, $\gamma'_{mR}(K_{m,n}) = 5m$.*

Proof. Let V_1 and V_2 are the bipartite sets of order m and n respectively. The edges connected to every $v \in V_1$ is labeled one with 3 and other is labeled with 2 and all the remaining edges adjacent to this vertex are labeled with 0. Under this labeling, each of the edges adjacent to each vertex in V_1 has a total weight of 5. Since V_1 consist of m vertices, $\gamma'_{mR}(K_{m,n}) = 5m$. □

Theorem 5. *For $n \geq 2$ and and $v \in V(G)$, $\gamma'_{mR}(G - v) \leq \gamma'_{mR}(G)$.*

Proof. Let $G' = G - v$ be the graph obtained by removing the vertex v and all edges incident to it from the graph. Therefore, $E(G - v) \subseteq E(G)$. We examine this in two cases.

If all edges labeled 0 in $G - v$ are still adjacent to two edges labeled 2 and 3, then the structure of the graph can still support a same EMRD set and stays stable.

If EMRDF set does not satisfy the domination condition in $G - v$, a new labeling must be done to the new graph. Since the number of edges is reduced, the edges that need to be labeled in the new graph is reduced. Hence, the new EMRD number must be smaller weight than before. Therefore, $\gamma'_{mR}(G - v) \leq \gamma'_{mR}(G)$. □

Theorem 6. *For $n \geq 2$ and $e \in E(G)$, $\gamma'_{mR}(G - e) \leq \gamma'_{mR}(G)$.*

Proof. Deleting an edge of a graph means reducing the number of edges that need to be labeled in the new graph. We examine this in two cases.

If the removal of e does not violate any of the domination conditions (i.e., all edges labeled 0 in $G - e$ are still adjacent to two edges labeled 2 and 3), then the structure of the graph can still support the same edge roman dominating set and stays stable.

If the removal of an edge e change the EMRD conditions (for example, an edge labeled 0 in $G - e$ is no longer adjacent to the edges labeled 2 and 3), a new EMRD number for $G - e$ is constructed by modifying appropriately which must be smaller weight than before. Therefore, $\gamma'_{mR}(G - e) \leq \gamma'_{mR}(G)$.

□

Theorem 7. *For $n \geq 2$ and $e \in E(G)$, $\gamma'_{mR}(G + e) \geq \gamma'_{mR}(G)$.*

Proof. Adding an edge to a graph G to form $G + e$ may break the balance of existing label configurations and require new labeling with newly added edges. It changes the domination relationships among edges and causes the existing EMRD number to no longer satisfy the required conditions in the new graph. Generally, these added edges will be labeled from the EMRD function $f : E(G) \rightarrow \{0, 1, 2, 3\}$ and hence, the edge modern roman domination must increase when they are labeled 1,2,3 or at least stays stable when they are labeled with 0. Therefore, $\gamma'_{mR}(G + e) \geq \gamma'_{mR}(G)$. □

Conjecture 1. *For $n \geq 2$, $2 \leq \gamma'_{mR}(G) \leq 5n$.*

3 EMRD Number of Some Graphs

In this section, we find results on $EMRD$ number of some graphs such as B_n, $J(m,n)$, D_n, $CT(m,n)$, $F(m,n)$ with their proves Corrmen et al. (1990), Sugumaran et al. (2018), Omran et al. (2021), Demirpolat et al. (2021).

Theorem 8. *For $n \geq 1$, $\gamma'_{mR}(B_n) = 2.\gamma'_{mR}(B_{n-1}) + 1$.*

Proof. With root R, the binomial tree of order $n > 0$ is B_n, which has the following definition. If $n = 0$, then $B_n = B_0 = R$, meaning that there is only one vertex R in the binomial tree of order zero. If $n > 0$, then $B_n = R, B_0, B_1, ..., B_{n-1}$, meaning that it consists of the root R and n binomial sub-trees.

• When $n = 0$, then the graph consists a single vertex v. In this case $EMRD$ cannot be calculated.

• When $n = 1$, then it consists a single edge e and it is labeled $f(e) = 2$. Thus, $\gamma'_{mR}(B_1) = 2$.

• When $n = 2$, then the structure of B_2 includes two B_1 connected by the edge e_1 labeled $f(e_2) = 2$ and the both pendant edges that are adjacent to e_2 be labeled with $f(e) = 1$. Therefore, $\gamma'_{mR}(B_2) = 2 + 1 + 1 = 4$.

Similarly, when $n = 3$, then the structure of B_3 includes two B_2 connected by the edge e_3 labeled $f(e_3) = 1$. Therefore, $\gamma'_{mR}(B_3) = 2.\gamma'_{mR}(B_2) + 1 = 2.4 + 1 = 9$. Let's apply this finding to $n \geq 3$ in a recursive formula.

Let $n = 3$, in this case $\gamma'_{mR}(B_n) = 2.\gamma'_{mR}(B_{n-1}) + 1$ and it is true from the above statement. Let $n = k$ and the result is true. That is, we assume that $\gamma'_{mR}(B_k) = 2.\gamma'_{mR}(B_{k-1}) + 1$.

Now, we show that for $i = k+1$. Using the induction hypothesis, it is known that $\gamma'_{mR}(B_k) = 2.\gamma'_{mR}(B_{k-1}) + 1$. Then we get $\gamma'_{mR}(B_{k+1}) = 2.\gamma'_{mR}(B_{k+1-1}) + 1 = 2.\gamma'_{mR}(B_k) + 1$. Hence the formula is true for $i = k + 1$.

The proof is completed.

□

Theorem 9. *For $m, n \geq 2$, then $\gamma'_{mR}(J(m,n)) = 10$.*

Proof. As seen in Figure 2, a $J(m,n)$ is produced by combining x and y with an edge, adding m pendant edges to v and n pendant edges to u, and from a cycle of order four x, y, u, and v. Let the edges of cycle of the graph are labeled as $f(xv) = 3$, $f(vy) = 2$, $f(yu) = 3$ and $f(ux) = 2$. Under this labeling, the remaining edges are labeled with zero. Therefore, $\gamma'_{mR}(G) = 2+3+2+3 = 10$. □

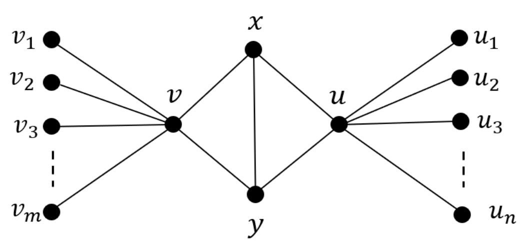

Figure 2: A jellyfish graph $J(m,n)$

Theorem 10. *For $n \geq 1$, $\gamma'_{mR}(D_n) = 5n$.*

Proof. As seen in Figure 3, a D_n consist of a group of n cycles C_4 that are connected so that any two adjacent cycles share a common vertex. There are two different ways to label these vertices. One of them is to label one of the four edges of C_4 with 2 and the others with 1, that is $f(v_1v_2) = 2$, $f(v_1v_3) = 1$, $f(v_2v_4) = 1$, $f(v_3v_4) = 1$ and so on. The other is to label one of the four edges of C_4 with 2, one with 3 and the others with zero, that is $f(v_1v_2) = 2$, $f(v_1v_3) = 0$, $f(v_2v_4) = 0$, $f(v_3v_4) = 3$ and so on. One can conclude that both give the same sum. Therefore, $\gamma^{'}_{mR}(G) = 5n$.

□

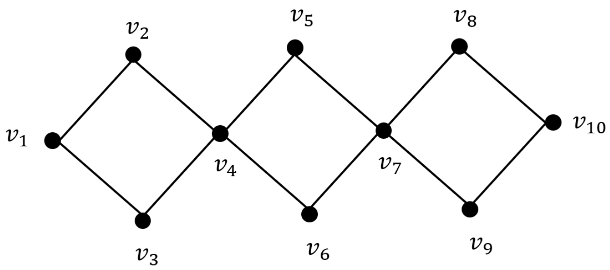

Figure 3: A diamond snake graph D_3

Theorem 11. *For $m, n \geq 2$,*

$$\gamma^{'}_{mR}(CT(m,n)) = \begin{cases} 5\lfloor\dfrac{m-2}{4}\rfloor + 7, & \text{if } n \equiv 0(mod4) \\ 5\lfloor\dfrac{m-2}{4}\rfloor + 8, & \text{if } n \equiv 1(mod4) \\ 5\dfrac{m-2}{4} + 5, & \text{if } n \equiv 2(mod4) \\ 5\lfloor\dfrac{m-2}{4}\rfloor + 6, & \text{if } n \equiv 3(mod4) \end{cases}$$

Proof. As seen in Figure 4, a $CT(m,n)$ is derived from path P_m by adding n new pendant edges at an end vertex of P_m. Let $V = \{v_1, v_2, ..., v_m\}$ denote the vertices of P_m and $U = \{u_1, u_2, ..., u_n\}$ be the vertices of new pendant edges. There are four cases according to the divisibility of m by four.

Case 1. When $n \equiv 0(mod4)$, then let $f(v_1v_2) = 3$ and $f(v_1u_1) = 2$. Hence all n edges and the edge (v_2v_3) are labeled with zero. In this case, the remaining unlabeled graph is the path graph P_{m-2}. From Theorem 3, it is clear that $\gamma^{'}_{mR}(P_{m-2}) = 5\lfloor\frac{m-2}{4}\rfloor + 2$. Therefore, $\gamma^{'}_{mR}(CT(m,n)) = 5\lfloor\frac{m-2}{4}\rfloor + 2 + 3 + 2 = 5\lfloor\frac{m-2}{4}\rfloor + 7$.

Case 2. When $n \equiv 1(mod4)$, then again as mentioned above $f(v_1v_2) = 3$, $f(v_1u_1) = 2$ and all n edges and the edge (v_2v_3) are labeled with zero. From Theorem 3, it is clear that $\gamma^{'}_{mR}(P_{m-2}) = 5\lfloor\frac{m-2}{4}\rfloor + 3$. Therefore, $\gamma^{'}_{mR}(CT(m,n)) = 5\lfloor\frac{m-2}{4}\rfloor + 3 + 3 + 2 = 5\lfloor\frac{m-2}{4}\rfloor + 8$.

Case 3. When $n \equiv 2(mod4)$, then similarly cases mentioned above, it is clear that $\gamma^{'}_{mR}(P_{m-2}) = 5\frac{m-2}{4}$. Therefore, $\gamma^{'}_{mR}(CT(m,n)) = 5\frac{m-2}{4} + 3 + 2 = 5\frac{m-2}{4} + 5$.

Case 4. When $n \equiv 3(mod4)$, then $\gamma^{'}_{mR}(P_{m-2}) = 5\lfloor\frac{m-2}{4}\rfloor + 1$. Therefore, $\gamma^{'}_{mR}(CT(m,n)) = 5\lfloor\frac{m-2}{4}\rfloor + 1 + 3 + 2 = 5\lfloor\frac{m-2}{4}\rfloor + 6$.

From all the above cases, the proof is completed.

□

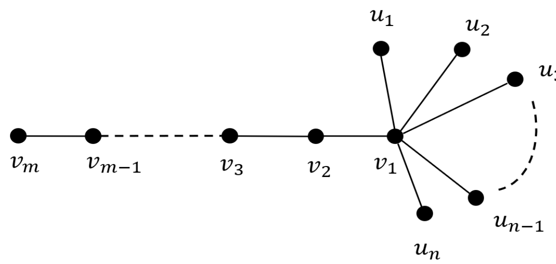

Figure 4: A coconut tree $CT(m, n)$

Theorem 12. *For $m, n \geq 2$, $\gamma'_{mR}(F(m,n)) = 5m$.*

Proof. A $F(m,n)$ is obtained from series of interconnected m copies of n stars by linking one leaf from each. From Theorem 5, it is clear that $\gamma'_{mR}(S_n) = 5$. Since there are m star graph in $F(m,n)$, $\gamma'_{mR}(F(m,n)) = \gamma'_{mR}(S_n)m = 5m$.

□

4 EMRD Number Under Some Graph Operations

In this section, we find results $EMRD$ number under some graph operations such as Cartesian product Klavžar et al. (2008) of $P_2 \times P_n$ and $P_2 \times C_n$, corona Frucht et al. (1970) of $C_n \odot \bar{K_m}$.

Theorem 13. *For $n \geq 2$, then*

$$\gamma'_{mR}(P_2 \times P_n) = \begin{cases} \frac{5n}{2}, & if\ n \equiv 0(mod2) \\ \frac{5n-1}{2}, & if\ n \equiv 1(mod2) \end{cases}$$

Proof. Let $V(P_2) = \{u_1, u_2\}$ and $V(P_n) = \{v_1, v_2, ..., v_n\}$ be the vertices and $E = \{e_1 = (u_1v_1, u_2v_1), e_2 = (u_1v_2, u_2v_2), ..., e_n = (u_1v_n, u_2v_n)\}$ be the edges connecting two paths of G. There are two cases depending on n that be considered.

Case 1. When $n \equiv 0(mod2)$, n edges are labeled with 2 or 3. Thus, $f(e_1) = 2$, $f(e_2) = 3$, $f(e_3) = 2$ and so on. Under this labeling, the sum of the labels of every $\frac{n}{2}$ edges is 5 and all remaining edges are labeled with zero. Therefore, $\gamma'_{mR}(G) = 5\frac{n}{2}$.

Case 2. When $n \equiv 1(mod2)$, then it is labeled $f(e_1) = 2$, $f(e_2) = 3$, ... , $f(e_{n-1}) = 3$ in the same way as Case 1. Under this labeling, the sum of the weights of the labels of every $\frac{n-1}{2}$ edges is 5 and the last edge is $f(e_n) = 2$. Therefore, $\gamma'_{mR}(G) = 5\frac{n-1}{2} + 2 = \frac{5n-1}{2}$.

The proof is completed from all the above cases.

□

Theorem 14. *For $n \geq 3$, then*

$$\gamma'_{mR}(P_2 \times C_n) = \begin{cases} \frac{5n}{2}, & if\ n \equiv 0(mod2) \\ \frac{5n+3}{2}, & if\ n \equiv 1(mod2) \end{cases}$$

Proof. Let $V(P_2) = \{u_1, u_2\}$ and $V(C_n) = \{v_1, v_2, ..., v_n\}$ be the vertices. There are two cases depending on n that be considered.

Case 1. Let $E = \{e_1 = (u_1v_1, u_2v_1), e_2 = (u_1v_2, u_2v_2), ..., e_n = (u_1v_n, u_2v_n)\}$ be the edges connecting two cycles of G. When $n \equiv 0(mod2)$, n edges are labeled with 2 or 3. Therefore, $f(e_1) = 2$, $f(e_2) = 3$, $f(e_3) = 2$ and so on. Under this labeling, the sum of the weights of the labels of every $\frac{n}{2}$ edges is 5 and all remaining edges are labeled with zero. Therefore, $\gamma'_{mR}(G) = 5\frac{n}{2}$.

Case 2. Let $E(C_n) = \{a_1, a_2, ..., a_n\}$ be the edges of the last cycle in addition to Case 1. When $n \equiv 1(mod2)$, then similarly Theorem 9, it is labeled as $f(e_1) = 2$, $f(e_2) = 3$, ... , $f(e_{n-1}) = 3$ in the same way. Under this labeling, the sum of the weights of the labels of every $\frac{n-1}{2}$ edges is 5 and the last edge is $f(e_n) = 2$. Since $f(e_1) = f(e_n) = 2$, then end edge a_n is labeled with 1. Therefore, $\gamma'_{mR}(G) = 5\frac{n-1}{2} + 2 + 1 + 1 = 5\frac{n-1}{2} + 4 = \frac{5n+3}{2}$.

The proof is completed from all the above cases. □

Theorem 15. *For $n \geq 3$ and $m \geq 1$,*

$$\gamma'_{mR}(C_n \odot \bar{K_m}) = \begin{cases} \frac{5n}{2}, & \text{if } n \equiv 0(mod2) \\ \frac{5n-1+2m}{2}, & \text{if } n \equiv 1(mod2) \end{cases}$$

Proof. Let $E(C_n) = \{e_1, e_2, ..., e_n\}$ be the edges of C_n and these edges are labeled with 2 or 3. There are two cases depending on n that be considered.

Case 1. When $n \equiv 0(mod2)$, then it is labeled $f(e_1) = 2$, $f(e_2) = 3$, ... , $f(e_n) = 3$. Under this labeling, the sum of the weights of the labels of every $\frac{n}{2}$ edges is 5 and all remaining edges are labeled with zero. Therefore, $\gamma'_{mR}(G) = 5\frac{n}{2}$.

Case 2. When $n \equiv 1(mod2)$, then it is labeled $f(e_1) = 2$, $f(e_2) = 3$, ... , $f(e_{n-1}) = 3$ in the same way as Case 1. Under this labeling, the sum of the weights of the labels of every $\frac{n-1}{2}$ edges is 5 and the last edge is $f(e_n) = 2$. Therefore, $\gamma'_{mR}(G) = 5\frac{n-1}{2} + 2 + m = \frac{5n-1+2m}{2}$.

The proof is completed from all the above cases. □

5 Conclusion

Domination is one of the most crucial concept in graph theory when analyzing the stability and susceptibility of communication networks that are represented by graphs. Depending on the characteristics and structure of dominating sets, there are several kinds of domination.

In this paper, we introduced a new domination parameter called edge modern roman domination number. We obtain results of this number on certain graph classes such as P_n, C_n, S_n, $K_{1,n}$, $K_{m,n}$ and some graph operations as $P_2 + P_n$, $C_n \odot \bar{K_m}$. Also, some known graphs such as the binomial tree B_n, the jellyfish graph $J(m, n)$, the diamond snake graph D_n, the coconut tree $CT(m, n)$, the firecracker graph $F(m, n)$ are determined.

In future studies, the edge modern roman domination offers much scope for further research. This parameter can be extended under some other graph operation and also on different dominating sets such as total domination, double domination, etc. Moreover, the computational complexity of determining $\gamma'_{mR}(G)$ could be studied in greater depth and it is possible to do research on subjects like complexity analysis and algorithms. Finally, potential real-world applications in areas like communication networks, intrusion-detection networks or defense resource allocation may provide further motivation and practical relevance for this new concept.

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