Advanced Mathematical Models & Applications
Vol.10, No.3, 2025, pp.710-721
https://doi.org/10.62476/amma.103710

EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR NONLINEAR SECOND-ORDER IMPULSIVE DIFFERENTIAL EQUATIONS WITH TWO-POINT BOUNDARY CONDITIONS

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Abstract. In this paper, a system of nonlinear second-order impulsive differential equations with nonlocal boundary conditions is analyzed. The effect of nonlinearity in the boundary conditions is examined in the analysis of nonlinear boundary value problems. The existence of solutions is established using the fixed-point theorems of Schaefer and Krasnoselskii, while their uniqueness is established by means of Banach's contraction mapping principle. Additionally, problems involving two-point boundary conditions are highlighted as a very interesting and important class of problems, with applications in fields such as population dynamics, blood flow models, chemical engineering, and cellular systems. The analysis of such boundary value problems is essential for accurately describing physical systems.

Keywords: Impulsive differential equations, two-point boundary equations, fixed point theorem, impulsive system.

AMS Subject Classification: 47H10, 47H30, 54H25.

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Received: 14 March 2025; Revised: 22 August 2025; Accepted: 25 September 2025; Published: 3 December 2025.

1 Introduction and Problem Statement

The theory of impulsive differential equations is a fundamental branch of differential equations that has a broad physical foundation. Impulsive differential equations frequently appear in the modeling of various physical systems where the state undergoes sudden changes at specific moments. Significant advancements have been made in impulsive theory, particularly in the study of impulsive differential equations with fixed moments (see, for instance, the references therein Benchohra et al. (2006), Samoilenko & Perestyuk(1995). The applications of impulsive differential equations can be found in Liu & Willms (1996), Pasquero (2005). Some classical methods have been used to study impulsive differential equations: variational methods Nieto (2009), Tian & Ge (2010), Zhou & Li (2009), fixed point theorems Bai et al. (2016), Feng & Pang (2009), Hao & Liu (2017), Wang & Feng (2017), Zhang et al. (2010), Zhang & Ge (2009). The study of boundary value problems has become an important area of research over the years due to their wide applications in fields such as fluid mechanics, mathematical physics, and others. In the analysis of nonlinear boundary value problems, our main goal is to investigate the effect of nonlinearity on the solutions of the given problem. The existing literature on this topic covers both theoretical developments and analytical and numerical approaches for solving boundary value problems. Classical boundary conditions fail to account for certain peculiarities of physical, chemical, and other processes occurring within the domain. For this reason, nonlocal conditions have been developed. Nonlocal conditions relate the boundary values of the unknown function to its values at certain interior points of the domain. In the natural sciences, there are certain processes - such as blood flow, chemical engineering, population dynamics, thermoelasticity, and others - whose mathematical models cannot be expressed using classical boundary conditions. In such cases, nonlocal nonlinear boundary value problems are employed. In this direction, the existence and uniqueness of solutions for impulsive and nonlocal boundary value problems have been widely studied (see, for instance, the references therein Ashyralyev & Sharifov (2013), Jiang et al. (2012), Gasimov et al. (2022), Li et al. (2018), Mardanov et al. (2019a,b); (2020); (2014).

In this article, we will investigate the existence and uniqueness of the system of nonlinear impulsive differential equations of the type

$$x''(t) = f(t, x(t)) \text{ for } t \neq t_i, \ i = 1, 2, \dots, p, \ t \in [0, T] \tag{1}$$

subject to two-point boundary conditions

$$Ax'(0) + Bx'(T) = C_1 \tag{2}$$

$$x(0) = C_2 \tag{3}$$

and impulsive

$$x'\left(t_i^+\right) - x'(t_i) = I_i(x(t_i)), \ i = 1, 2, \dots, p \tag{4}$$

$$0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T,$$

where $f : [0, T] \times R^n \to R^n$ and $I_i : R^n \to R^n$ are given functions, $i = 1, 2, \dots, p$; $A, B \in R^{n \times n}$ are given matrices and $\det(A + B) \neq 0$, C_1, $C_2 \in R^n$ are given vectors.

$$\Delta x'(t_i) = x'\left(t_i^+\right) - x'(t_i)$$

$$x'\left(t_i^+\right) = \lim_{h \to 0+} x'(t_i + h), \quad x'\left(t_i^-\right) = \lim_{h \to 0+} x'(t_i - h) = x'(t_i)$$

are the right and left-hand limits of $x(t)$ at $t = t_i$, respectively.

2 Preliminaries

In this part, we provide fundamental definitions and introductory concepts. By $C([0, T], R^n)$, we indicate the Banach space of all vector continuous functions $x(t)$ from $[0, T]$ into R^n with the norm

$$\|x\| = \max\{|x(t)| : t \in [0, T]\},$$

where $|\cdot|$ is the norm in the space R^n.

We examine the linear space

$$PC([0, T], R^n) = \big\{x : [0, T] \to R^n : x'(t) \in C((t_i, t_{i+1}], R^n), \ i = 0, 1, \dots, p,$$
$$x'\left(t_i^-\right) \text{ and } x'\left(t_i^+\right) \text{ exist } i = 1, \dots, p \text{ and } x'\left(t_i^-\right) = x'(t_i)\big\}$$

$PC([0, T], R^n)$ is a Banach space with the norm

$$\|x\|_{PC} = \max\left\{\|x(t)\|_{(t_i, t_{i+1}]}, \, i = 0, 1, \dots, p\right\}.$$

We provide an explanation for the solutions to problems (1) - (4) as follows.

Definition 1. *A function $x \in PC([0,T], R^n)$ is said to be a solution of problems (1)- (4), if*

$$x''(t) = f(t, x(t))$$

for each $t \neq t_i,\ i = 1, 2, \ldots, p,\ t \in [0,T]$, and for each

$$x'\left(t_i^+\right) - x'(t_i) = I_i(x(t_i)),\ \ i = 1, 2, \ldots, p,\ \ 0 < t_1 < t_2 < \cdots < t_p < T$$

and boundary conditions (2), (3) are satisfied.

Lemma 1. *Let us assume that $y \in C([0,T] \times R^n)$ and $a_i \in R^n$, $i = 1, 2, \ldots, p$. Then, the solution of the problem*

$$x''(t) = y(t)\ \ t \neq t_i,\ \ i = 1, 2, \ldots, p,\ \ t \in [0,T] \tag{5}$$

$$x'\left(t_i^+\right) - x'(t_i) = a_i,\ \ i = 1, 2, \ldots, p,\ \ 0 < t_1 < t_2 < \cdots < t_p < T \tag{6}$$

$$Ax'(0) + Bx'(t) = C_1 \tag{7}$$

$$x(0) = C_2 \tag{8}$$

has a unique solution $x \in PC([0,T] \times R^n)$ given by

$$x(t) = C_2 + (A+B)^{-1} tC_1 + t\int_0^T K(t,\tau)\, y(\tau)\, d\tau -$$

$$-\int_0^t \tau y(\tau)\, d\tau + t\sum_{0<t_k<T} K(t_i, t_k)\, a_k$$

for $t \in (t_i, t_{i+}1]$, $i = 0, 1, \ldots, p$, where

$$K(t,\tau) = \begin{cases} (A+B)^{-1} A, & 0 \leq \tau \leq t, \\ -(A+B)^{-1} B, & t < \tau \leq T. \end{cases}$$

Proof. Let's assume that $x(t)$ is a solution of the boundary value problem (5)-(8), integrating equation (5) for $t \in (0, t_{j+1}]$, we obtain

$$\int_0^t y(s)\ ds = \int_0^t x''(s)\ ds =$$

$$= \left[x'(t_1) - x'\left(0^+\right)\right] + \left[x'(t_2) - x'\left(t_1^+\right)\right] + \ldots + \left[x'(t) - x'\left(t_j^+\right)\right] =$$

$$= -x'(0) - \left[x'\left(t_1^+\right) - x'(t_1)\right] - \left[x'\left(t_2^+\right) - x'(t_2)\right]$$

$$-\left[x'\left(t_j^+\right) - x'(t_j)\right] + x'(t)\,. \tag{9}$$

Applying this equation together with condition (6), it can be written as

$$x'(t) = x'(0) + \int_0^t y(\tau)\, d\tau + \sum_{0<t_j<t} a_j. \tag{10}$$

Using formula (9) and condition (7), we obtain

$$Ax'(0)+B\left(x'(0)+\int\limits_0^T y(\tau)\,d\tau+\sum_{0<t_j<T}a_j\right)=C_1,$$

$$(A+B)\,x'(0)=C_1-B\int\limits_0^T y(\tau)\,d\tau-B\sum_{0<t_j<T}a_j,$$

$$x'(0)=(A+B)^{-1}C_1-(A+B)^{-1}B\int\limits_0^T y(\tau)\,d\tau-(A+B)^{-1}B\sum_{0<t_k<T}a_k. \tag{11}$$

From formulas (9) and (10), it follows

$$x'(t)=(A+B)^{-1}C_1-(A+B)^{-1}B\int\limits_0^T y(\tau)\,d\tau-$$

$$-(A+B)^{-1}B\sum_{0<t_k<T}a_k+\int\limits_0^t y(\tau)\,d\tau+\sum_{0<t_j<t}a_j.$$

Then, we get

$$x'(t)=(A+B)^{-1}C_1+\int\limits_0^T K(t,\tau)\,y(\tau)\,d\tau+\sum_{0<t_k<T}K(t_i,t_k)\,a_k.$$

Here

$$x(t)=x(0)+$$

$$+\int\limits_0^t\left((A+B)^{-1}C_1+\int\limits_0^T K(s,\tau)\,y(\tau)\,d\tau+\sum_{0<t_k<T}K(t_i,t_k)\,a_k\right)ds=$$

$$=x(0)+(A+B)^{-1}C_1t+\int\limits_0^t(t-\tau)\,(A+B)^{-1}Ay(\tau)\,d\tau-$$

$$-\int\limits_0^t\int\limits_s^T(A+B)^{-1}By(\tau)\,d\tau ds+t\sum_{0<t_k<T}K(t_i,t_k)\,a_k.$$

On the other hand if we consider the following equality

$$\int\limits_0^t\int\limits_s^T(A+B)^{-1}By(\tau)\;d\tau\,ds=\int\limits_0^t\tau\,(A+B)^{-1}By(\tau)\;d\tau+t\int\limits_t^T(A+B)^{-1}By(\tau)\;d\tau,$$

we obtain

$$x(t)=x(0)+$$

$$+(A+B)^{-1}tC_1+\int\limits_0^t(t-\tau)\,(A+B)^{-1}Ay(\tau)\,d\tau-$$

$$-\int_0^t \tau (A+B)^{-1} By(\tau)\,d\tau - t\int_t^T (A+B)^{-1} By(\tau)\,d\tau + t\sum_{0<t_k<T} K(t_i,t_k)\,\alpha_k.$$

Using condition (3)

$$x(t) = C_2 +$$

$$+(A+B)^{-1} tC_1 + \int_0^t (t-\tau)(A+B)^{-1} Ay(\tau)\,d\tau - \int_0^t \tau (A+B)^{-1} By(\tau)\,d\tau -$$

$$-t\int_t^T (A+B)^{-1} By(\tau)\,d\tau + t\sum_{0<t_k<T} K(t_i,t_k)\,a_k.$$

From here

$$x(t) = C_2 + (A+B)^{-1} tC_1 + t\int_0^T K(t,\tau)\,y(\tau)\,d\tau -$$

$$-\int_0^t \tau y(\tau)\,d\tau + t\sum_{0<t_k<t} K(t,t_k)\,a_k \tag{12}$$

for

$$t\in(t_i,t_{i+1}],\, i=0,1,\ldots,p.$$

Lemma 1 has been proven.

□

Lemma 2. *Assume that $f\in C([0,T]\times R^n, R^n)$ and $I_k(x)\in C(R^n)$. Then the function $x(t)\in PC([0,T],R^n)$ is a solution of the boundary-value problem (1)-(4) if and only if $x(t)$ is a solution of the impulsive integral equation*

$$x(t) = C_2 + (A+B)^{-1} tC_1 + t\int_0^T K(t,\tau)\,f(\tau,x(\tau))\,d\tau -$$

$$-\int_0^t \tau f(\tau,x(\tau))\,d\tau + t\sum_{0<t_k<T} K(t_i,t_k)\,I_i(x(t_k))$$

for

$$t\in(t_i,t_{i+1}],\, i=0,1,\ldots,p.$$

Proof. Assume that the boundary value problems (1)-(4) have a solution $x(t)$. It is shown that $x(t)$ satisfies the impulsive integral equations (11) in a manner similar to Lemma 1. It is demonstrated by direct proof that the solution to the impulsive integral equation (11) satisfies both the nonlocal boundary condition (3) and equation (1). Confirmation of the satisfaction of condition (2) is also simple.

Lemma 2 has been proven. □

3 Existence of solution

According to Lemma 1, we convert the boundary value problem (1)-(4) into a fixed point problem that is similar to it as follows:

$$x = \phi x. \tag{13}$$

Let's introduce the $\phi : R^n \to R^n$ operator as follows

$$(\phi x)(t) = C_2 + (A+B)^{-1} tc_1 + t\int_0^T K(t,\tau) f(\tau, x(\tau))\, d\tau -$$

$$-\int_0^t \tau f(\tau, x(\tau))\, d\tau + t\sum_{0<t_k<T} K(t_i, t_k) I_i(x(t_k)). \tag{14}$$

Lemma 3. ***(Krasnoselskii fixed point theorem) Zhou & Li (2009)*** *Let Y be a nonempty, convex, closed bounded subset of a Banach space 4 . Let ϕ_1, ϕ_2 be the operators such that*

(i)$\phi_1 y_1 + \phi_2 y_2 \in Y$ whenever $y_1, y_2 \in Y$;

(ii) ϕ_1 is compact and continuous;

(iii) ϕ_2 is a contraction mapping. Then there exists $y_3 \in Y$ such that $y_3 = \phi_1 y_3 + \phi_2 y_3$.

Theorem 1. *Let $f : [0,T]\times R^n \to R^n$ be a continuous function such that the following conditions hold:*

(H1) $|f(t,x) - f(t,y)| \le l\,|x-y|, \quad \forall\, t \in [0,T], \quad l>0, \quad x,y \in R^n$;

(H2) $|I_i(x) - I_i(y)| \le l_i\,|x-y|, \quad l_i > 0, \quad i=1,2,\dots,p, \quad x,y \in R^n$

(H3) There exists a function $\mu \in C([0,T], R^n)$ and a constant L such that $\max\limits_{i=1,\dots,p} |I_i(x)| \le L$, $\|\mu\| = \max\limits_{[0,T]} |\mu(t)|$,

$$|f(t,x)| \le \|\mu\|, \quad \forall\ (t,x) \in [0,T]\times R^n.$$

If

$$TS\sum_{i=1}^{p} l_i + lST^2 < 1. \tag{15}$$

Then there exists at least one solution for problems (1)-(4).

Proof. Let's consider the closed sphere $B_r = \{x \in PC([0,T], R^n) : \|x\| \le r\}$. Here,

$$r \ge |C_2| + \|\mu\|\left[\frac{T^2}{2} + \left\|(A+B)^{-1} TC_1\right\| + ST^2\right] + TSpL.$$

The operators ϕ_1 and ϕ_2 defined on the sphere B_r will now be defined.

$$(\phi_1 x)(t) = -\int_0^t \tau f(\tau, x(\tau))\, d\tau,$$

$$(\phi_2 x)(t) = C_2 + (A+B)^{-1} tC_1 + t\int_0^T K(t,\tau) f(\tau, x(\tau))\, d\tau +$$

$$+t\sum_{0<t_k<T}K\left(t_i,t_k\right)I_i\left(x\left(t_k\right)\right).$$

Show that $\phi=\phi_1+\phi_2$. For $x,y\in B_r$, then

$$\left\|\phi_1x+\phi_2y\right\|=\left|C_2\right|+\max_{[0,T]}\left|-\int\limits_0^t\tau f\left(\tau,x\left(\tau\right)\right)d\tau+\left(A+B\right)^{-1}tC_1+\right.$$

$$\left.+t\int\limits_0^T K\left(t,\tau\right)f\left(\tau,y\left(\tau\right)\right)d\tau+t\sum_{0<t_k<T}K\left(t_i,t_k\right)I_i\left(y\left(t_k\right)\right)\right|\leq$$

$$\leq\left|C_2\right|+\left\|\mu\right\|\left[\frac{T^2}{2}+\left|\left(A+B\right)^{-1}TC_1\right|+ST^2\right]+TS\rho L_2\leq r.$$

Therefore $\phi_1x+\phi_2y\in B_r$, which verifies the condition (i) in Lemma 3. Using the assumptions (H1) and (H2), we obtain

$$\left\|\phi_1x-\phi_2y\right\|\leq$$

$$\leq\max_{[0,T]}\left\{\left|t\int\limits_0^T K\left(t,\tau\right)f\left(\tau,x\left(\tau\right)\right)d\tau+t\sum_{0<t_k<T}K\left(t_i,t_k\right)I_i\left(x\left(t_k\right)\right)-\right.\right.$$

$$\left.\left.-t\int\limits_0^T K\left(t,\tau\right)f\left(\tau,y\left(\tau\right)\right)d\tau-t\sum_{0<t_k<T}K\left(t_i,t_k\right)I_i\left(y\left(t_k\right)\right)\right|\right\}\leq$$

$$\leq\max_{[0,T]}\left\{t\int\limits_0^T\left|K\left(t,\tau\right)\right|\left|f\left(\tau,x\left(\tau\right)\right)-f\left(\tau,y\left(\tau\right)\right)\right|d\tau+\right.$$

$$\left.+t\sum_{0<t_k<T}\left|K\left(t_i,t_k\right)\right|\left|I_i\left(x\left(t_k\right)\right)-I_i\left(y\left(t_k\right)\right)\right|\right\}\leq$$

$$\leq ST\left(lT+\sum_{i=1}^p l_i\right)\left\|x-y\right\|_{PC}.$$

We obtained

$$\left\|\phi_2x-\phi_2y\right\|\leq ST\left(lT+\sum_{i=1}^p l_i\right)\left\|x-y\right\|_{PC}$$

which indicates that ϕ_2 is a contraction, in view of condition (14).

Later, we show that ϕ_1 is continuous and compact. Observe that the continuity of f indicates that the operator ϕ_1 is continuous. Additionally, ϕ_1 is uniformly bounded on $B_{\bar{r}}$ as

$$\left\|\phi_1x\right\|_{PC}\leq\left\|\mu\right\|\frac{T^2}{2},$$

let $t_1,t_2\in\left[0,T\right],\ t_1<t_2$. Then

$$\left\|\phi_1x\left(t_2\right)-\phi_1x\left(t_1\right)\right\|=\left|-\int\limits_0^{t_2}\tau f\left(\tau,x\left(\tau\right)\right)d\tau+\right.$$

$$\left.+\int\limits_0^{t_1}\tau f\left(\tau,x\left(\tau\right)\right)d\tau\right|\leq\left|\int\limits_{t_1}^{t_2}\tau f\left(\tau,x\left(\tau\right)\right)d\tau\right|\to 0,\quad t_2\to t_1.$$

This suggests that ϕ_1 is relatively compact on B_r. Finally, due to Lemma 2, the boundary value issue (1) has at least one solution on $[0,T]$. □

In this subsection, we discuss the existence of solutions for problems (1)-(4). Our first existence result is based on the following Schaefer fixed point theorem Smart (1980).

Lemma 4. *Assume that X is a Banach space. Assume that the collection*

$$Y = \{x \in D \,|\, x = \lambda F x,\, 0 < \lambda < 1\}$$

is bounded and that $F : D \to D$ is a completely continuous operator. Then F has a fixed point in D.

Theorem 2. *Let $f : [0,T] \times R^n \to R^n$ be a continuous function and satisfy the condition (H3). Then problem (1)-(4) has at least one solution on $[0,T]$.*

Proof. In the first step, we show that the operator ϕ defined by (8) is completely continuous. Note that the continuity of ϕ follows from the continuity of f. For a positive constant w, let $B_w = \{x \in PC\left([0,T], R^n\right) : \|x\| \le w\}$ be a bounded set in P. Then, for $t \in [0,T]$, it will be shown that the operator ϕ maps bounded sets into bounded sets of P. For $x \in B_w$, $t \in [0,T]$, we have

$$\|(\phi x)\| \le \max_{[0,T]} \left| C_2 + (A+B)^{-1} t C_1 + t \int_0^T K(t,\tau) f(\tau, x(\tau))\, d\tau - \right.$$

$$\left. - \int_0^t \tau f(\tau, x(\tau))\, d\tau + t \sum_{0<t_k<T} K(t_i, t_k)\, I_i(x(t_k)) \right| \le$$

$$\le |C_2| + \left|(A+B)^{-1} T C_1\right| + \|\mu\| \left(S T^2 + \frac{T^2}{2} \right) + T S p L.$$

Next, we indicate that the operator ϕ maps bounded sets into equicontinuous sets of P. For $0 < t_1 < t_2 < T$ and $x \in B_w$ we have.

$$(\phi x)(t_2) - (\phi x)(t_1) = (A+B)^{-1} C_1 (t_2 - t_1) - \int_{t_1}^{t_2} \tau f(\tau, x(\tau))\, d\tau +$$

$$+ t_2 \int_0^{t_2} (A+B)^{-1} A f(\tau, x(\tau))\, d\tau - t_2 \int_{t_2}^{T} (A+B)^{-1} B f(\tau, x(\tau))\, d\tau -$$

$$- t_1 \int_0^{t_1} (A+B)^{-1} A f(\tau, x(\tau))\, d\tau + t_1 \int_{t_1}^{T} (A+B)^{-1} B f(\tau, x(\tau))\, d\tau +$$

$$+ (t_2 - t_1) \sum_{0<t_k<T} K(t_i, t_k)\, l_i(x(t_k)) =$$

$$= (A+B)^{-1} C_1 (t_2 - t_1) - \int_{t_1}^{t_2} \tau f(\tau, x(\tau))\, d\tau +$$

$$+ (t_2 - t_1) \int_0^{t_1} (A+B)^{-1} A f(\tau, x(\tau))\, d\tau + t_2 \int_{t_1}^{t_2} (A+B)^{-1} A f(\tau, x(\tau))\, d\tau +$$

$$+ (t_2 - t_1) \int_{t_2}^{T} (A+B)^{-1} B f(\tau, x(\tau))\, d\tau + t_1 \int_{t_2}^{t_1} (A+B)^{-1} B f(\tau, x(\tau))\, d\tau +$$

$$+\left(t_2-t_1\right)\sum_{0<t_k<T}K\left(t_i,t_k\right)I_i\left(x\left(t_k\right)\right).$$

From here

$$\left\|\phi x\left(t_2\right)-\phi x\left(t_1\right)\right\|\leq$$

$$\leq\left|t_2-t_1\right|\left[\left\|(A+B)^{-1}C_1\right\|+\int\limits_0^t\left\|(A+B)^{-1}Af\left(\tau,x\left(\tau\right)\right)\right\|\right]d\tau+$$

$$+\int\limits_{t_2}^T\left\|(A+B)^{-1}Bf\left(\tau,x\left(\tau\right)\right)\right\|d\tau+\sum_{0<t_k<T}\left\|K\left(t_i,t_k\right)I_i\left(x\left(t_k\right)\right)\right\|+$$

$$+\int\limits_{t_1}^{t_2}\tau\left\|f\left(\tau,x\left(\tau\right)\right)\right\|d\tau+\left|t_2\right|\int\limits_{t_1}^{t_2}\left\|(A+B)^{-1}Af\left(\tau,x\left(\tau\right)\right)\right\|d\tau+$$

$$+\left|t_1\right|\int\limits_{t_1}^{t_2}\left\|(A+B)^{-1}Bf\left(\tau,x\left(\tau\right)\right)\right\|d\tau\to 0\quad\text{as}\quad\left(t_2-t_1\right)\to 0,$$

Independent of $x\in B_w$. Thus, by the Arzela-Ascoli theorem, the operator $\phi: B_w\to B_w$ is completely continuous.

Finally, we analyze the set

$$Y=\left\{x\in PC\left([0,T],R^n\right):\left|x=\lambda\phi x,\quad 0<\lambda<1\right.\right\}$$

and indicate that Y is bounded. Let $x=\lambda\phi x$, for some $0<\lambda<1$. Then, for any $t\in(t_i,t_{i+1}]$, $i=0,1,\ldots,p$, we can get

$$x\left(t\right)=\lambda\left[C_2+(A+B)^{-1}tC_1+t\int\limits_0^T K\left(t,\tau\right)f\left(\tau,x\left(\tau\right)\right)d\tau-\right.$$

$$\left.-\int\limits_0^t\tau f\left(\tau,x\left(\tau\right)\right)d\tau+t\sum_{0<t_k<T}K\left(t_i,t_k\right)I_i\left(x\left(t_k\right)\right)\right]$$

From here, we obtain

$$\|x\|\leq\left|C_2\right|+\left|(A+B)^{-1}TC_1\right|+\|\mu\|\left(ST^2+\frac{T^2}{2}\right)+TSpL.$$

This means that the set Y is bounded. We obtain that the function ϕ has a fixed point, which is a solution to problem (1)-(4), as a consequence of the Schaefer fixed point theorem.

Theorem 2 is proven. □

4 Uniqueness of solution

Here we initiate the uniqueness of solutions for problem (1) by means of Banach's contraction mapping principle (Smart, 1980)

Theorem 3. *(Banach fixed point theorem). Suppose that $f:[0,T]\times R^n\times R^n$ is a continuous function satisfying the conditions (H1) and (H2). Then there exists a unique solution for problem (1)-(4) on $[0,T]$ if*

$$TS\sum_{i=1}^p l_i+lST^2+\frac{lT^2}{2}<1.$$

Proof. Let us fix $\max\limits_{[0,T]} |f(t,0)| = M,\ I_i(0) = a_i$

$$w \geq \frac{|C_2| + \left|(A+B)^{-1} TC_1\right| + M\left(ST^2 + \frac{T^2}{2}\right) + TS\sum\limits_{i=1}^{p} a_i}{1 - \left(TS\sum\limits_{i=1}^{p} l_i + lST^2 + \frac{lT^2}{2}\right)}$$

and indicate that $\phi B_w \subset B_w$, the where $B_w = \{x \in PC : \|x\| \leq w\}$. For any $x \in B_w$, $t \in [0,T]$. We find that

$$|f(t,x(t))| = |f(t,x(t)) - f(t,0) + f(t,0)| \leq$$

$$\leq |f(t,x(t)) - f(t,0)| + |f(t,0)| \leq l\,|x(t)| + M \leq lw + M,$$

and

$$|I_i(x)| = |I_i(x) - I_i(0) + I_i(0)| \leq l_i\,|x| + a_i$$

Then, we obtain

$$\|(\phi x)\| \leq$$

$$\leq \max_{[0,T]} \left| C_2 + (A+B)^{-1} tC_1 + t\int_0^T K(t,\tau) f(\tau,x(\tau))\,d\tau - \right.$$

$$\left. - \int_0^t \tau f(\tau,x(\tau))\,d\tau + t\sum_{0<t_k<T} K(t_i,t_k)\, I_i(x(t_k)) \right| \leq$$

$$\leq |C_2| + \left|(A+B)^{-1} TC_1\right| + (lw+M)\left(ST^2 + \frac{T^2}{2}\right) + TS\left(\sum_{i=1}^{p}(l_i w + a_i)\right) \leq w.$$

This indicates that $\phi B_w \subset B_w$. Then we show that the operator ϕ is a contraction. For $x, y \in P$, we obtain

$$\|\phi x - \phi y\| \leq$$

$$\leq \max_{t\in[0,T]} \left| t\int_0^T \left(K(t,\tau) f(\tau,x(\tau)) - f(\tau,y(\tau))\right) d\tau + \right.$$

$$+ \int_0^t \tau\left(f(\tau,x(\tau)) - f(\tau,x(t))\right) d\tau +$$

$$\left. +t\sum_{0<t_k<t} K(t_i,t_k)\left(I_i(x(t_k)) - l_i(y(t_k))\right) \right| \leq$$

$$\leq TSl\,\|x-y\| + \frac{T^2}{2} l\,\|x-y\| + Sp\sum_{i=1}^{p} l_i\,\|x-y\| =$$

$$= \left(TSl + \frac{T^2}{2} l + Sp\sum_{i=1}^{p} l_i\right) < 1.$$

Which, due to the presumption (H2), indicate that the operator ϕ is a contraction. Thus, by Banach's contraction mapping principle, we identify that operator ϕ has a fixed point, this corresponds to a single solution for problem (1)-(4) defined on $[0,T]$.

□

5 Conclusion

The article investigates the existence and uniqueness of the solution to a boundary value problem involving a second-order differential equation with two-point boundary conditions. It should be noted that the scheme presented in the paper can also be applied to more general boundary value problems. For example, the following nonlocal boundary value problem for equation (1) can also be considered.

$$\begin{cases} x''(t) = f(t, x(t)), \quad t \in [0, T] \\ Ax'(0) + Bx'(t) = \alpha \\ \int_0^T m(t)\, x(t)\, dt = \beta \end{cases}$$

Here, $A, B \in R^{n\times n}$ are given matrices and , $\det(A+B) \neq 0$; $\alpha, \beta \in R^n$ are given vectors, $m(t) \in R^{n\times n}$ and $\det \int_0^T m(t) \neq 0$.

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