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FIXED POINT THEOREMS IN STRONGLY SEQUENTIAL S-METRIC SPACES AND THEIR APPLICATION TO THE SOLUTION OF A SECOND-ORDER BOUNDARY VALUE DIFFERENTIAL EQUATION

Behnoosh Ekhveh[1], **Seyed Jalaleddin Hosseini Ghoncheh**[2,*], **Vahid Parvaneh**[3,*]

[1]Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran
[2]Department of Mathematics, Takestan Branch, Islamic Azad University, Takestan, Iran
[3]Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran

Abstract. Many mathematicians are often interested in establishing the existence and uniqueness of solutions, as well as finding exact or approximate solutions, for various classes of integral and differential equations, including their fractional variants with initial or boundary conditions. Fixed point theorems serve as useful and powerful tools for addressing such problems. In this paper, the notion of a strongly sequential S-metric space is first introduced as a new generalization of S_b-metric spaces and S^{JS}-metric spaces. Several fixed point results for $\overline{S}$-JS-Hardy-Rogers-type contractions and other types of contractions are obtained. To confirm the validity of these results, an illustrative application is provided, demonstrating the existence and uniqueness of a solution for a second-order boundary value differential equation, which arises in many engineering applications.

Keywords: fixed point, S-metric space, sequential S-metric space, strongly sequential S-metric space, pointwise contractive self-mappings, differential equation.

AMS Subject Classification: 47H10, 54H25.

[*]**Corresponding authors:** Seyed Jalaleddin Hosseini Ghoncheh, Department of Mathematics, Takestan Branch, Islamic Azad University, Takestan, Iran, Tel.: +989125824896, e-mail: *sjhghoncheh@gmail.com*
Vahid Parvaneh, Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran, Tel.: +989186875070, e-mail: *zam.dalahoo@gmail.com*

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1 Introduction

Since boundary value differential equations of the second order describe a wide range of phenomena pertaining to physical systems that vary with respect to time and space, they are crucial in many branches of science and engineering. In fact, when a system is controlled by a process that depends on two variables (such as position and time) and when the behavior of the system is affected by particular conditions at the boundaries, second-order boundary value problems occur. These equations are essential to both theoretical and applied sciences since they apply to a wide range of systems, including mechanical system vibrations, electrical circuits, etc. These are a few of their main uses as follows:

a. Mechanical System Vibrations: Second-order differential equations are frequently used to represent the motion of a spring-mass-damper system. Boundary conditions, such as the system's fixed or free ends, might affect how the system behaves. For more details we refer the reader to Singiresu (1995).

b. Circuits and Systems in Electrical Engineering: Second-order linear differential equations are frequently used in electrical circuits to describe the behavior of an RLC (Resistor, Inductor, Capacitor) circuit, particularly its voltage and current. The particular behavior of the circuit over time is determined in part by boundary conditions, such as initial voltages or currents. For more details on RLC circuits we refer the reader to Nilsson et al. (2010).

Fixed-point theorems are essential techniques for proving the existence and uniqueness of solutions to a variety of mathematical problems, including integral and differential equations and their fractional variants. Metric spaces have been generalized and expressed in various ways by numerous authors in recent years. Double-controlled metric spaces (Abdeljawad et al., 2018), rectangular metric spaces (Branciari (2000)), b-metric-like spaces (Hussain et al., 2014), extended b-metric spaces (Kamran et al., 2017), JS-metric spaces (Jleli & Samet, 2015), controlled graphical metric-type spaces (Younis & Bahuguna, 2023), graphical extended b-metric spaces (Mudasir et al., 2022), and so forth are examples of the various types of extended metric spaces.

Sedghi et al. (2012) introduced the idea of S-metric spaces in 2012. There are many generalizations of the S-metric concept. The notion of S_b-metric space, which is a generalization of S-metric spaces was then presented by Souayah & Mlaiki (2016).

Qaralleh et al. (2023) talked about an extended S-metric space of type (α, β). They did this by using two functions instead of a constant in the second condition of the definition of an S-metric space. In Adewale & Iluno (2022), the notion of rectangular S-metric space, which extends rectangular metric spaces, has been introduced by Adewale and Iluno.

A generalization of S-metric spaces known as S^{JS}-metric spaces was recently presented by Beg et al. (2021); nevertheless, it does not meet the symmetry property and triangle inequality. Note that JS-metric spaces are a class of abstract metric spaces that were introduced in 2015. They are larger than many known classes of extended metric spaces. Accordingly, JS-metric spaces include any standard metric space, b-metric space and dislocated metric space (see Jleli & Samet (2015)). In Karami et al. (2021), the notion of sequential S_p-metric spaces has been introduced as a generalization of usual S-metric spaces, S_b-metric spaces, S^{JS} metric spaces, and specially of S_p-metric spaces Mustafa et al. (2019).

This study aims to provide the structure of strongly sequential S-metric spaces, a novel form of generalized metric spaces. Additionally, we establish some fixed point theorems for mappings that are JS-contractive in a strongly sequential S-metric space.

Before starting the main discussion, we recall some necessary definitions and results.

Definition 1. *(Bakhtin, 1989). Let $s \geq 1$ and L be a nonempty set. The function $\mathcal{B} : L \times L \to [0, +\infty)$ is said to be a b-metric on L if,*

1. $\mathcal{B}(\jmath, \iota) = 0 \Leftrightarrow \jmath = \iota \ \forall \jmath, \iota, \in L$,
2. $\mathcal{B}(\jmath, \iota) = \mathcal{B}(\iota, \jmath) \ \forall \jmath, \iota \in L$
3. $\mathcal{B}(\jmath, \iota) \leq s\Big(\mathcal{B}(\jmath, \ell) + \mathcal{B}(\ell, \iota)\Big) \ \forall \jmath, \iota, \ell \in L.$

The pair $(L, \mathcal{B})$ is called a b-metric space.

Definition 2. *(Sedghi et al., 2012). Let L be a nonempty set. An S-metric on L is a function $S : L \times L \times L \to [0, \infty)$ provided that,*

(1) $S(\jmath, \iota, \ell) = 0 \Leftrightarrow \jmath = \iota = \ell \ \forall \jmath, \iota, \ell \in L$,

(2) $S(\jmath, \iota, \ell) \leq S(\jmath, \jmath, a) + S(\iota, \iota, a) + S(\ell, \ell, a), \ \forall \jmath, \iota, \ell, a \in L.$

The pair (L, S) is called an S-metric space.

Definition 3. *(Souayah & Mlaiki, 2016). Let L be a nonempty set and $b \geq 1$ be a given real number. Suppose that a mapping $S_b : L \times L \times L \to [0, \infty)$ satisfies,*

(1) $S_b(\jmath, \iota, \ell) = 0 \Leftrightarrow \jmath = \iota = \ell$ *for each* $\jmath, \iota, \ell \in L$,

(2) $S_b(\jmath, \iota, \ell) \leq b(S_b(\jmath, \jmath, a) + S_b(\iota, \iota, a) + S_b(\ell, \ell, a))$, $\forall \jmath, \iota, \ell, a \in L$.

Then S_b is called an S_b-metric and the pair (L, S_b) is called an S_b-metric space.

Let L be a non-empty set and $\rho_g : L \times L \to [0, \infty]$ be a mapping. For any $\kappa \in L$, let us define the set

$$C(\rho_g, L, \kappa) = \{\{\kappa_n\} \subset L : \lim_{n\to\infty} \rho_g(\kappa_n, \kappa) = 0\}. \tag{1}$$

A generalised metric space was introduced by Jleli & Samet (2015) as an interesting extension of the concept of metric space.

Definition 4. *(Jleli & Samet, 2015). Consider a mapping $\rho_g : L \times L \to [0, \infty]$ that meets the following:*

1. $\rho_g(\jmath, \iota) = 0$ *suggests that* $\jmath = \iota$,
2. $\rho_g(\jmath, \iota) = \rho_g(\iota, \jmath)$ *for each* $\jmath, \iota \in L$,
3. *for some* $p > 0$, $\rho_g(\jmath, \iota) \leq p \limsup_{n\to\infty} \rho_g(\jmath_n, \iota)$, *if* $(\jmath, \iota) \in L \times L$ *and* $\{\jmath_n\} \in C(\rho_g, L, \jmath)$.

A JS-metric space is defined as the pair (L, ρ_g).

Remark 1. *(Beg et al., 2021). Rectangular metric spaces might not be JS-metric spaces; for further details, see Example 2.2. of Beg et al. (2021). It should be noted that any metric space, b-metric space, and Hitzler-Seda (dislocated) metric space are special examples of JS-metric spaces.*

Definition 5. *(Gajić & Ralević, 2018). Let $\mathcal{D} : L \times L \to [0, \infty]$ be a mapping that meets the following:*

1. $\mathcal{D}(\jmath, \iota) = 0$ *suggests that* $\jmath = \iota$,
2. $\mathcal{D}(\jmath, \iota) = \mathcal{D}(\iota, \jmath)$ *for all points* $\jmath, \iota \in L$,
3. *there is a* $p > 0$ *so that for every* $\jmath, \iota \in L$, $\{\jmath_n\} \in C(\mathcal{D}, L, \jmath)$ *and* $\{\iota_n\} \in C(\mathcal{D}, L, \iota)$,

$$\mathcal{D}(\jmath, \iota) \leq p \limsup_{n\to\infty} \mathcal{D}(\jmath_n, \iota_n).$$

$(L, \mathcal{D})$ is referred to as a strong JS-metric space thereafter.

Let L be a non-empty set and $J : L \times L \times L \to [0, \infty]$ be a function. For any $\jmath \in L$, let us define the set

$$\underline{\mathcal{C}}(J, L, \jmath) = \{\{\jmath_n\} \subset L : \lim_{n\to\infty} J(\jmath, \jmath, \jmath_n) = 0\}. \tag{2}$$

S^{JS}-metric spaces are a generalization of S-metric spaces, as proposed by Beg et al. (2021).

Definition 6. *(Beg et al., 2021). Consider a mapping $J : L \times L \times L \to [0, \infty]$ that meets the following:*

1. $J(\jmath, \iota, \ell) = 0$ *suggests that* $\jmath = \iota = \ell$;
2. *for any* $(\jmath, \iota, \ell) \in L \times L \times L$ *and* $\{\jmath_n\} \in \underline{\mathcal{C}}(J, L, \jmath)$, *there exists some* $K > 0$ *such that*

$$J(\jmath, \iota, \ell) \leq K \limsup_{n\to\infty} \big(J(\iota, \iota, \jmath_n) + J(\ell, \ell, \jmath_n)\big).$$

The pair (L, J) is called an S^{JS}-metric space.

In the framework of Branciari (rectangular) metric spaces, (Jleli & Samet (2014)) introduced a new generalisation of the Banach fixed point theorem.

Definition 7. *Jleli & Samet (2014). The set of all functions $\wp : (0, \infty) \to (1, \infty)$ meeting the following statements is denoted as Φ.*

($\wp$1) $\wp$ is increasing;

($\wp$2) for any sequence $\{\sigma_\nu\} \subseteq (0, \infty)$

$$\lim_{\nu \to \infty} \wp(\sigma_\nu) = 1 \Leftrightarrow \lim_{\nu \to \infty} \sigma_\nu = 0;$$

($\wp$3) there exists $\alpha \in (0, 1)$ and $\tau \in (0, \infty]$ such that

$$\lim_{u \to 0^+} \frac{\wp(u) - 1}{u^\alpha} = \tau.$$

Theorem 1. *Jleli & Samet (2014). Let (L, d) be a metric space. Let a mapping $\Delta : L \to L$ be given.*

If $\wp \in \Phi$ and $\mu \in (0, 1)$ exist such that,

$$d(\Delta \jmath, \Delta \iota) \neq 0 \Rightarrow \wp(d(\Delta \jmath, \Delta \iota)) \leq (\wp(d(\jmath, \iota))^\mu, \tag{3}$$

for every $\jmath, \iota \in L$, then Δ possesses a unique fixed point.

2 Strongly sequential S-metric spaces

We present the notion of strongly sequential S-metric spaces in this section. We first define,

$$\eth(\overline{S}, L, \jmath) := \left\{ \{\jmath_n\} \subset L : \lim_{n \to \infty} \overline{S}(\jmath, \jmath, \jmath_n) = 0 \right\},$$

where $\overline{S} : L \times L \times L \to [0, +\infty)$ is a given mapping.

Definition 8. *Let L be a non-empty set. A mapping $\overline{S} : L \times L \times L \to [0, +\infty)$ is said to be a strongly sequential S-metric (SSSM) if $\forall \jmath, \iota, \ell \in L$:*

(a) $\overline{S}(\jmath, \iota, \ell) = 0$ implies $\jmath = \iota = \ell$;

(b) there is a $C > 0$ such that $\forall \jmath, \iota, \ell \in L$, $\{\jmath_n\} \in \eth(\overline{S}, L, \jmath)$, $\{\iota_n\} \in \eth(\overline{S}, L, \iota)$ and $\{\ell_n\} \in \eth(\overline{S}, L, \ell)$,

$$\overline{S}(\jmath, \iota, \ell) \leq C \Big(\limsup_{n \to \infty} \overline{S}(\jmath_n, \jmath_n, \iota_n) + \limsup_{n \to \infty} \overline{S}(\jmath_n, \jmath_n, \ell_n) \Big).$$

The combination $(L, \overline{S})$ is an SSSM space, and $\overline{S}$ is referred to as a SSSM.

Furthermore, we refer to it as a symmetric SSSM space $\Leftrightarrow \overline{S}(\jmath, \jmath, \iota) = \overline{S}(\iota, \iota, \jmath)$, $\forall \jmath, \iota \in L$.

Remark 2. *Every symmetry S^{JS}-metric space is a SSSM space since sequences $\{\iota_n\} = \iota$ and $\{\ell_n\} = \ell$, ($n \in \mathbb{N}$), $\overline{S}$-converge to ι and ℓ, respectively.*

It is noteworthy that there are many classes of SSSM spaces. SSSM spaces include, for instance, every S-metric space, every S_b-metric space, every dislocated S_b-metric space, and every symmetry S^{JS}-metric space.

Definition 9. *Let $(L, \overline{S})$ be a SSSM space and $\{\jmath_n\}$ be a sequence in $(L, \overline{S})$ and $a \in L$.*

(i) If $\{\jmath_n\} \in \eth(\overline{S}, L, \jmath)$, then $\{\jmath_n\}$ is convergent and converges to $\jmath$.

(ii) If $\lim_{n,m \to \infty} \overline{S}(\jmath_n, \jmath_n, \jmath_m) = 0$, then $\{\jmath_n\}$ is considered as a Cauchy sequence.

(iii) If each Cauchy sequence in $(L, \overline{S})$ is a convergent sequence, then $(L, \overline{S})$ is considered as a complete SSSM space.

Example 1. *Let $L=\mathbb{R}$ and $\overline{S}(\jmath,\iota,\ell)=|\jmath-\iota|+|\jmath-\ell|$, $\forall \jmath,\iota,\ell\in L$. Obviously, $\overline{S}(\jmath,\iota,\ell)=0$ implies that $|\jmath-\iota|+|\jmath-\ell|=0$ which gives us $\jmath=\iota=\ell$. Subsequently, the Definition 8's first requirement is met. Furthermore, $\overline{S}$ symmetry requirement is satisfied, since we have*

$$\overline{S}(\jmath,\jmath,\iota)=|\jmath-\iota|=\overline{S}(\iota,\iota,\jmath).$$

Let $\jmath,\iota,\ell\in L$ and $\{\jmath_n\},\{\iota_n\}$ and $\{\ell_n\}$ are convergent sequences in L such that

$$\lim_{n\to\infty}\overline{S}(\jmath,\jmath,\jmath_n)=0,\ \lim_{n\to\infty}\overline{S}(\iota,\iota,\iota_n)=0 \text{ and } \lim_{n\to\infty}\overline{S}(\ell,\ell,\ell_n)=0,$$

respectively. So,

$$\limsup_{n\to\infty}\jmath_n=\jmath,\ \limsup_{n\to\infty}\iota_n=\iota \text{ and } \limsup_{n\to\infty}\ell_n=\ell.$$

Then we have

$$\begin{aligned}\overline{S}(\jmath,\iota,\ell)&=|\jmath-\iota|+|\jmath-\ell|\\&\leq|\jmath-\jmath_n+\jmath_n-\iota_n+\iota_n-\iota|+|\jmath-\jmath_n+\jmath_n-\ell_n+\ell_n-\ell|.\end{aligned}$$

So,

$$\overline{S}(\jmath,\iota,\ell)=|\jmath-\iota|+|\jmath-\ell|\leq\limsup_{n\to\infty}\overline{S}(\jmath_n,\jmath_n,\iota_n)+\limsup_{n\to\infty}\overline{S}(\jmath_n,\jmath_n,\ell_n).$$

After then, the presumptions of Definition 8 are all met. As a result, $\overline{S}$ is a symmetric SSSM spacewith $C=1$.

Obviously, the expected triangle inequality in an S-metric space is not true in this example. Also, this space cannot be an S_b metric space and an S_b-metric-like space.

Example 2. *Let $L=[0,\infty)$ and $\overline{S}(\jmath,\iota,\ell)=\sqrt{|\jmath-\iota|^3}+\sqrt[3]{|\jmath-\ell|^2}$, $\forall \jmath,\iota,\ell\in L$. Evidently, $\overline{S}(\jmath,\iota,\ell)=0$ implies that $\sqrt{|\jmath-\iota|}+\sqrt[3]{|\jmath-\ell|}=0$ which gives us $\jmath=\iota=\ell$. Subsequently, the Definition 8's first requirement is met.*

Let $\jmath,\iota,\ell\in L$ and $\{\jmath_n\},\{\iota_n\}$ and $\{\ell_n\}$ are convergent sequences in L such that

$$\lim_{n\to\infty}\overline{S}(\jmath,\jmath,\jmath_n)=0,\ \lim_{n\to\infty}\overline{S}(\iota,\iota,\iota_n)=0 \text{ and } \lim_{n\to\infty}\overline{S}(\ell,\ell,\ell_n)=0,$$

respectively. So,

$$\limsup_{n\to\infty}\jmath_n=\jmath,\ \limsup_{n\to\infty}\iota_n=\iota \text{ and } \limsup_{n\to\infty}\ell_n=\ell.$$

Now, we have

$$\begin{aligned}\overline{S}(\jmath,\iota,\ell)&=\sqrt{|\jmath-\iota|^3}+\sqrt[3]{|\jmath-\ell|^2}\\&\leq\sqrt{|\jmath-\jmath_n+\jmath_n-\iota_n+\iota_n-\iota|^3}+\sqrt[3]{|\jmath-\jmath_n+\jmath_n-\ell_n+\ell_n-\ell|^2}\\&\leq\sqrt{4|\jmath-\jmath_n|^3+16|\jmath_n-\iota_n|^3+16|\iota_n-\iota|^3}+\sqrt[3]{2|\jmath-\jmath_n|^2+4|\jmath_n-\ell_n|^2+4|\ell_n-\ell|^2}\\&\leq\sqrt{4|\jmath-\jmath_n|^3}+\sqrt{16|\jmath_n-\iota_n|^3}+\sqrt{16|\iota_n-\iota|^3}+\sqrt[3]{2|\jmath-\jmath_n|^2}+\sqrt[3]{4|\jmath_n-\ell_n|^2}+\sqrt[3]{4|\ell_n-\ell|^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\overline{S}(\jmath,\iota,\ell)&=\sqrt{|\jmath-\iota|^3}+\sqrt[3]{|\jmath-\ell|^2}\\&\leq4\limsup_{n\to\infty}(\overline{S}(\jmath_n,\jmath_n,\iota_n)+\overline{S}(\jmath_n,\jmath_n,\ell_n)).\end{aligned}$$

After then, Definition 8's presumptions are all met. As a result, although $\overline{S}$ is not a usual S-metric, it is a SSSM with $C=4$.

We now provide some instances of S-metric spaces that are strongly sequential.

Proposition 1. *If (L, S) be an S-metric space, then S is also an SSSM on L.*

Proof. Let (L, S) be an S-metric space. We only demonstrate that S meets the condition (b) of Definition 8.

Let $\jmath, \iota, \ell \in L$, $\{\jmath_n\} \to \jmath$, $\{\iota_n\} \to \iota$ and $\{\ell_n\} \to \ell$. Using triangle inequality

$$\begin{aligned} S(\jmath, \iota, \ell) &\leq S(\jmath, \jmath, \jmath_n) + S(\iota, \iota, \jmath_n) + S(\ell, \ell, \jmath_n) \\ &\leq S(\jmath, \jmath, \jmath_n) + \big(2S(\iota, \iota, \iota_n) + S(\jmath_n, \jmath_n, \iota_n)\big) + \big(2S(\ell, \ell, \ell_n) + S(\jmath_n, \jmath_n, \ell_n)\big), \end{aligned}$$

for all natural number n. So, we have

$$S(\jmath, \iota, \ell) \leq \limsup_{n\to\infty} S(\jmath_n, \jmath_n, \iota_n) + \limsup_{n\to\infty} S(\jmath_n, \jmath_n, \ell_n).$$

So, condition (b) of Definition 8 is satisfied with $C = 1$.

□

Proposition 2. *If (L, S) be an S_b-metric space, then S_b is also an SSSM on L.*

Proof. Let (L, S) be an S_b-metric space with parameter D. We just show that S_b satisfies the condition (b) of Definition 8.

Let $\jmath, \iota, \ell \in L$, $\{\jmath_n\} \to \jmath$, $\{\iota_n\} \to \iota$ and $\{\ell_n\} \to \ell$. By using triangle inequality

$$\begin{aligned} S(\jmath, \iota, \ell) &\leq D\big(S(\jmath, \jmath, \jmath_n) + S(\iota, \iota, \jmath_n) + S(\ell, \ell, \jmath_n)\big) \\ &\leq D\Big(S(\jmath, \jmath, \jmath_n) + D\big(2S(\iota, \iota, \iota_n) + S(\jmath_n, \jmath_n, \iota_n)\big) + D\big(2S(\ell, \ell, \ell_n) + S(\jmath_n, \jmath_n, \ell_n)\big)\Big), \end{aligned}$$

$\forall$ natural number n. Thus, one has

$$S(\jmath, \iota, \ell) \leq D^2\big(\limsup_{n\to\infty} S(\jmath_n, \jmath_n, \iota_n) + \limsup_{n\to\infty} S(\jmath_n, \jmath_n, \ell_n)\big).$$

So, condition (b) of Definition 8 is satisfied with $D^2 = C$. □

Proposition 3. *If $(L, \mathcal{D})$ be a strong JS-metric space with parameter p, then there exists an $\overline{S}$ which is an SSSM on L.*

Proof. We define $\overline{S}(\jmath, \iota, \ell) = \mathcal{D}(\jmath, \iota) + \mathcal{D}(\jmath, \ell)$. Let $\jmath, \iota, \ell \in L$, $\{\jmath_n\} \to \jmath$, $\{\iota_n\} \to \iota$ and $\{\ell_n\} \to \ell$. By Definition 5, we know that

$$\mathcal{D}(\jmath, \iota) \leq p \limsup_{n\to\infty} \mathcal{D}(\jmath_n, \iota_n) \tag{4}$$

and

$$\mathcal{D}(\jmath, \ell) \leq p \limsup_{n\to\infty} \mathcal{D}(\jmath_n, \ell_n). \tag{5}$$

From (4) and (5), we have

$$\begin{aligned} \mathcal{D}(\jmath, \iota) + \mathcal{D}(\jmath, \ell) &\leq p\Big(\limsup_{n\to\infty} \mathcal{D}(\jmath_n, \iota_n) + \limsup_{n\to\infty} \mathcal{D}(\jmath_n, \ell_n)\Big) \\ &\leq p\Big(\limsup_{n\to\infty} \overline{S}(\jmath_n, \jmath_n, \iota_n) + \limsup_{n\to\infty} \overline{S}(\jmath_n, \jmath_n, \ell_n)\Big). \end{aligned}$$

So, condition (b) of Definition 8 is satisfied with $p = C$ and the proof is finished. □

Remark 3. *It should be noted that (L, σ) is a strong JS-metric on L if it is a b-metric-like space. In light of Proposition 3, it follows that $\overline{S}(\jmath, \iota, \ell) = \sigma(\jmath, \iota) + \sigma(\jmath, \ell)$ is likewise an SSSM on L.*

The following examples are based on (Hussain et al. (2014)) (Example 11) and Proposition 3.

Example 3. *Let* $L = \mathbb{R}$ *and* $\overline{S} : L \times L \times L \to [0, +\infty)$ *be defined by*

$$\begin{aligned}
\overline{S}(\jmath, \iota, \ell) &= (\jmath^2 + \iota^2)^p + (\jmath^2 + \ell^2)^p, \\
\overline{S}(\jmath, \iota, \ell) &= (|\jmath| + |\iota| + a)^p + (|\jmath| + |\ell| + a)^p, \\
\overline{S}(\jmath, \iota, \ell) &= (|\jmath - b| + |\iota - b|)^p + (|\jmath - b| + |\ell - b|)^p.
\end{aligned}$$

Then they are SSSM on L *with* $C = 2^{2(p-1)}$, *where* $p > 1$, $a \geq 0$ *and* $b \in L$.

Proposition 4. *Let* $(L, \overline{S})$ *be an SSSM space and* $\{\jmath_n\}$ *be a sequence in* L *with* $\lim_{n\to\infty} \overline{S}(\jmath_n, \jmath_n, \jmath_n) = 0$ *so that* $\{\jmath_n\} \in \eth(\overline{S}, L, \jmath)$. *Then* $\overline{S}(\jmath, \jmath, \jmath) = 0$.

Proof. From the condition (b) of Definition 8 we have,

$$\overline{S}(\jmath, \jmath, \jmath) \leq C \limsup_{n\to\infty} 2\ \overline{S}(\jmath_n, \jmath_n, \jmath_n),$$

which implies that $\overline{S}(\jmath, \jmath, \jmath) = 0$. □

Remark 4. *Let* $(L, \overline{S})$ *be an SSSM space and* $\{\jmath_n\}$ *be a Cauchy sequence in* L *so that* $\{\jmath_n\} \in \eth(\overline{S}, L, \jmath)$. *Then* $\overline{S}(\jmath, \jmath, \jmath) = 0$.

Proposition 5. *Any sequence* $\{\jmath_n\}$ *satisfying* $\lim_{n\to\infty} \overline{S}(\jmath_n, \jmath_n, \jmath_n) = 0$ *in an SSSM space* $(L, \overline{S})$ *can converge to at most one point.*

Proof. Let $\{\jmath_n\} \in \eth(\overline{S}, L, \jmath) \cup \eth(\overline{S}, L, \iota)$ where $\jmath, \iota \in L$. Then we have

$$\overline{S}(\jmath, \jmath, \iota) \leq C \limsup_{n\to\infty} 2\overline{S}(\jmath_n, \jmath_n, \jmath_n),$$

which implies that $\overline{S}(\jmath, \jmath, \iota) = 0$, that is, $\jmath = \iota$. □

Remark 5. *In the complete SSSM space* $(L, \overline{S})$, *a Cauchy sequence can converge to a maximum of one point.*

Proposition 6. *In an SSSM space* $(L, \overline{S})$, *if a self-mapping* Δ *is continuous at* $\jmath \in L$, *then* $\{\Delta\jmath_n\} \in \eth(\overline{S}, L, \Delta\jmath)$ *for any sequence* $\{\jmath_n\} \in \eth(\overline{S}, L, \jmath)$.

Proof. Since Δ is continuous at $\jmath$, then for any $\epsilon > 0$ there is $\delta_\epsilon > 0$ so that $\overline{S}(\ell, \ell, \jmath) < \delta_\epsilon$ implies $\overline{S}(\Delta\ell, \Delta\ell, \Delta\jmath) < \epsilon$. As $\{\jmath_n\}$ converges to $\jmath$, so for $\delta_\epsilon > 0$ there is $N \in \mathbb{N}$ so that $\overline{S}(\jmath, \jmath, \jmath_n) < \delta_\epsilon$, $\forall n \geq N$. Therefore, for any $n \geq N$, $\overline{S}(\Delta\jmath, \Delta\jmath, \Delta\jmath_n) < \epsilon$ and thus $\Delta\jmath_n \to \Delta\jmath$ as $n \to \infty$. □

Now, we define the topology on SSSM spaces.

Let $(L, \overline{S})$ be an SSSM space. Define,

$$\mathcal{B}(\jmath, \eta) := \left\{\xi \in L : \overline{S}(\jmath, \jmath, \xi) < \overline{S}(\jmath, \jmath, \jmath) + \eta\right\}$$

and

$$\mathcal{B}[\jmath, \eta] := \left\{\xi \in L : \overline{S}(\jmath, \jmath, \xi) \leq \overline{S}(\jmath, \jmath, \jmath) + \eta\right\}$$

$\forall\ \jmath \in L$ and $\eta > 0$.

Remark 6. *It can be easily checked that the collection*

$$\tau_{\overline{S}} := \{\emptyset\} \cup \{\mathcal{V}(\neq \emptyset) \subset L : \text{ for any } \jmath \in \mathcal{V}, \text{ there is } \eta > 0 \text{ so that } \mathcal{B}(\jmath, \eta) \subset \mathcal{V}\}$$

forms a topology on L.

3 Main results

3.1 Some fixed point theorems in an SSSM space

We further limit the requirements on the control function $\wp$ in this section. The set of functions $\wp : [0, \infty) \to [1, \infty)$ is indicated for this by the symbol Θ where

($\wp$1) $\wp$ is continuous and increasing,

($\wp$2) for any sequence $\{\sigma_\nu\} \subseteq (0, \infty)$

$$\lim_{\nu\to\infty} \wp(\sigma_\nu) = 1 \Leftrightarrow \lim_{\nu\to\infty} \sigma_\nu = 0.$$

Definition 10. *Let $(L, \overline{S})$ be a complete SSSM space with parameter $C \geq 1$. A mapping $\Delta : L \to L$ is called an $\overline{S}$-JS-Hardy-Rogers type contraction if*

$$\wp(\overline{S}(\Delta\kappa, \Delta\jmath, \Delta\iota)) \leq (\wp(\overline{S}(\kappa, \jmath, \iota)))^\alpha (\wp(\overline{S}(\kappa, \kappa, \Delta\kappa)))^\beta (\wp(\overline{S}(\jmath, \jmath, \Delta\jmath)))^\gamma (\wp(\frac{\overline{S}(\iota, \iota, \Delta\iota)}{C}))^\varrho$$
$$(\wp(\frac{\overline{S}(\kappa, \kappa, \Delta\jmath) + \overline{S}(\jmath, \jmath, \Delta\iota) + \overline{S}(\iota, \iota, \Delta\kappa)}{3}))^\omega \tag{6}$$

$\forall\ \kappa, \jmath, \iota \in L$ and for some $\alpha, \beta, \gamma, \varrho, \omega \geq 0$ with $\alpha + \beta + \gamma + \varrho + \omega < 1$ where $\wp \in \Theta$.

Theorem 2. *Let $(L, \overline{S})$ be a complete SSSM space and $\Delta : L \to L$ be a mapping such that:*

(i) Δ is an $\overline{S}$-JS-Hardy-Rogers type contraction,

(ii) there is $\jmath_0 \in L$ so that

$$\delta(\overline{S}, \Delta, \jmath_0) := \sup\left\{\overline{S}\left(\Delta^i\jmath_0, \Delta^i\jmath_0, \Delta^j\jmath_0\right) : i, j = 1, 2, \cdots\right\} < \infty.$$

This suggests that L contains at least one fixed point for Δ. Furthermore, if $(L, \overline{S})$ is symmetry and for all fixed point $\jmath$ of Δ in L one has $\overline{S}(\jmath, \jmath, \jmath) = 0$, then the fixed point is unique.

Proof. We define $\delta(\overline{S}, \Delta^{p+1}, \jmath_0) := \sup\left\{\overline{S}\left(\Delta^{p+i}\jmath_0, \Delta^{p+i}\jmath_0, \Delta^{p+j}\jmath_0\right) : i, j = 1, 2, \cdots\right\}$, for all $p \geq 1$. Clearly, $\delta(\overline{S}, \Delta^{p+1}, \jmath_0) \leq \delta(\overline{S}, \Delta, \jmath_0) < \infty$ for all $p \geq 1$.

Then $\forall p \geq 1$ and $\forall i, j = 1, 2, \cdots,$

$$\begin{aligned}
&\wp(\overline{S}(\Delta^{p+i}\jmath_0, \Delta^{p+i}\jmath_0, \Delta^{p+j}\jmath_0))\\
&\leq (\wp(\overline{S}(\Delta^{p-1+i}\jmath_0, \Delta^{p-1+i}\jmath_0, \Delta^{p-1+j}\jmath_0)))^\alpha (\wp(\overline{S}(\Delta^{p-1+i}\jmath_0, \Delta^{p-1+i}\jmath_0, \Delta^{p+i}\jmath_0)))^\beta\\
&(\wp(\overline{S}(\Delta^{p-1+i}\jmath_0, \Delta^{p-1+i}\jmath_0, \Delta^{p+i}\jmath_0)))^\gamma (\wp(\frac{\overline{S}(\Delta^{p-1+j}\jmath_0, \Delta^{p-1+j}\jmath_0, \Delta^{p+j}\jmath_0)}{C}))^\varrho\\
&(\wp(\frac{\overline{S}(\Delta^{p-1+i}\jmath_0, \Delta^{p-1+i}\jmath_0, \Delta^{p+i}\jmath_0) + \overline{S}(\Delta^{p-1+i}\jmath_0, \Delta^{p-1+i}\jmath_0, \Delta^{p+j}\jmath_0) + \overline{S}(\Delta^{p-1+j}\jmath_0, \Delta^{p-1+j}\jmath_0, \Delta^{p+i}\jmath_0)}{3}))^\omega\\
&\leq (\wp(\delta(\overline{S}, \Delta^p, \jmath_0)))^\eta,
\end{aligned}$$

where $\eta = \alpha + \beta + \gamma + \varrho + \omega$. This implies that, $\forall\ p \geq 1$,

$$\begin{aligned}
\wp\left(\delta(\overline{S}, \Delta^{p+1}, \jmath_0)\right) &= \wp\left(\sup_{i,j\geq 1} \overline{S}(\Delta^{p+i}\jmath_0, \Delta^{p+i}\jmath_0, \Delta^{p+j}\jmath_0)\right)\\
&\leq \left(\wp(\delta(\overline{S}, \Delta^p, \jmath_0))\right)^\eta\\
&\leq \left(\wp(\delta(\overline{S}, \Delta^{p-1}, \jmath_0))\right)^{\eta^2}\\
&\vdots\\
&\leq \left(\wp(\delta(\overline{S}, \Delta, \jmath_0))\right)^{\eta^p}.
\end{aligned}$$

Let $\jmath_i = \Delta \jmath_{i-1} = \Delta^i \jmath_0$, $\forall i \in \mathbb{N}$. For all $m > n \geq 1$ we have,

$$\begin{aligned}
\wp\left(\overline{S}(\jmath_n, \jmath_n, \jmath_m)\right) &= \wp\left(\overline{S}(\Delta^n \jmath_0, \Delta^n \jmath_0, \Delta^m \jmath_0)\right) \\
&= \wp\left(\overline{S}(\Delta^{n-1+1} \jmath_0, \Delta^{n-1+1} \jmath_0, \Delta^{n-1+(m-n+1)} \jmath_0)\right) \\
&\leq \wp\left(\delta(\overline{S}, \Delta^n, \jmath_0)\right) \\
&\leq \wp\left(\delta(\overline{S}, \Delta, \jmath_0)\right)^{\eta^{n-1}} \longrightarrow 1 \quad (\text{as } n \to \infty).
\end{aligned}$$

Therefore, $\{\jmath_n\}$ is a Cauchy sequence in L. From the completeness of L, $\{\jmath_n\}$ is convergent. Let $\lim_{n\to\infty} \jmath_n = \jmath \in L$.

On the other hand, using condition (b) of Definition 8 and the continuity assumption of Δ, we have

$$\overline{S}(\jmath, \jmath, \Delta\jmath) \leq C\Big(\limsup_{n\to\infty} \overline{S}(\jmath_n, \jmath_n, \jmath_{n+1})\Big) = 0.$$

Hence, we have $\Delta\jmath = \jmath$, *i.e.*, $\jmath \in L$ is a fixed point of Δ.

In the case that we have not the continuity assumption of Δ we should use the contractive condition 6 and Definition 8 as follows:

$$\begin{aligned}
\wp\left(\overline{S}(\jmath_{n+1}, \jmath_{n+1}, \Delta\jmath)\right) &= \wp\left(\overline{S}(\Delta\jmath_n, \Delta\jmath_n, \Delta\jmath)\right) \\
&\leq (\wp(\overline{S}(\jmath_n, \jmath_n, \jmath)))^{\alpha} (\wp(\overline{S}(\jmath_n, \jmath_n, \Delta\jmath_n)))^{\beta+\gamma} (\wp(\frac{\overline{S}(\jmath, \jmath, \Delta\jmath)}{C}))^{\varrho} \\
&\quad (\wp(\frac{2\overline{S}(\jmath_n, \jmath_n, \Delta\jmath) + \overline{S}(\jmath, \jmath, \Delta\jmath_n)}{3}))^{\omega}.
\end{aligned}$$

Taking the limsup and using Definition 8 in the above we obtain that:

$$\begin{aligned}
&\limsup_{n\to\infty} \wp\left(\overline{S}(\jmath_{n+1}, \jmath_{n+1}, \Delta\jmath)\right) \\
&= \limsup_{n\to\infty} \wp\left(\overline{S}(\Delta\jmath_n, \Delta\jmath_n, \Delta\jmath)\right) \\
&\leq \limsup_{n\to\infty} \Big[(\wp(\frac{\overline{S}(\jmath, \jmath, \Delta\jmath)}{C}))^{\varrho} (\wp(\frac{2\overline{S}(\jmath_n, \jmath_n, \Delta\jmath) + \overline{S}(\jmath, \jmath, \Delta\jmath_n)}{3}))^{\omega}\Big].
\end{aligned}$$

Consequently, we obtain that

$$\begin{aligned}
&\wp\left(\limsup_{n\to\infty} \overline{S}(\jmath_{n+1}, \jmath_{n+1}, \Delta\jmath)\right) \\
&\leq \Big[(\wp(\frac{\overline{S}(\jmath, \jmath, \Delta\jmath)}{C}))^{\varrho} (\wp(\frac{\limsup_{n\to\infty} 2\overline{S}(\jmath_n, \jmath_n, \Delta\jmath) + \limsup_{n\to\infty} \overline{S}(\jmath, \jmath, \Delta\jmath_n)}{3}))^{\omega}\Big]
\end{aligned}$$

which further implies that

$$\wp\left(\limsup_{n\to\infty} \overline{S}(\jmath_{n+1}, \jmath_{n+1}, \Delta\jmath)\right) \leq (\wp(\frac{\overline{S}(\jmath, \jmath, \Delta\jmath)}{C}))^{\frac{\varrho}{1-\omega}}.$$

On the other hand,

$$\overline{S}(\jmath, \jmath, \Delta\jmath) \leq C\Big(\limsup_{n\to\infty} \overline{S}(\jmath_n, \jmath_n, \Delta\jmath)\Big).$$

Therefore,

$$1 \leq \wp(\frac{\overline{S}(\jmath, \jmath, \Delta\jmath)}{C}) \leq \wp\Big((\limsup_{n\to\infty} \overline{S}(\jmath_n, \jmath_n, \Delta\jmath))\Big) \leq (\wp(\frac{\overline{S}(\jmath, \jmath, \Delta\jmath)}{C}))^{\frac{\varrho}{1-\omega}}$$

which is impossible unless that $\frac{\overline{S}(\jmath,\jmath,\Delta\jmath)}{C} = 0$. Hence $\jmath = \Delta\jmath$.

Now, if $\jmath$ and ι be two fixed points of Δ in L with $\overline{S}(\jmath,\jmath,\jmath) = 0$ and $\overline{S}(\iota,\iota,\iota) = 0$, then

$$\begin{aligned}\wp(\overline{S}(\jmath,\jmath,\iota)) &= \wp(\overline{S}(\Delta\jmath,\Delta\jmath,\Delta\iota))\\ &\leq (\wp(\overline{S}(\jmath,\jmath,\iota)))^{\alpha}(\wp(\overline{S}(\jmath,\jmath,\Delta\jmath)))^{\beta+\gamma}(\wp(\frac{\overline{S}(\iota,\iota,\Delta\iota)}{C}))^{\varrho}\\ &\quad(\wp(\frac{\overline{S}(\jmath,\jmath,\Delta\jmath)+\overline{S}(\jmath,\jmath,\Delta\iota)+\overline{S}(\iota,\iota,\Delta\jmath)}{3}))^{\omega}\\ &\leq \wp(\overline{S}(\jmath,\jmath,\iota)))^{\alpha+\omega},\end{aligned}$$

where $\alpha+\omega<1$. This implies that $\overline{S}(\jmath,\jmath,\iota) = 0$. Hence, $\jmath = \iota$. □

When we use $\omega = 0$ in Theorem 2, we have a contraction of the Reich type, which leads to the following outcome.

Corollary 1. *Let $(L,\overline{S})$ be a complete SSSM space and $\Delta : L \to L$ be a mapping such that:*
(i) Δ is an $\overline{S}$-JS-Reich type contraction, that is,

$$\wp(\overline{S}(\Delta\kappa,\Delta\jmath,\Delta\iota)) \leq (\wp(\overline{S}(\kappa,\jmath,\iota)))^{\alpha}(\wp(\overline{S}(\kappa,\kappa,\Delta\kappa)))^{\beta}(\wp(\overline{S}(\jmath,\jmath,\Delta\jmath)))^{\gamma}(\wp(\frac{\overline{S}(\iota,\iota,\Delta\iota)}{C}))^{\varrho},$$

$\forall\ \kappa,\jmath,\iota \in L$ and for some $\alpha,\beta,\gamma,\varrho \geq 0$ with $\alpha+\beta+\gamma+\varrho<1$ where $\wp\in\Theta$.
(ii) there is $\jmath_0 \in L$ so that

$$\delta(\overline{S},\Delta,\jmath_0) := \sup\left\{\overline{S}\left(\Delta^i\jmath_0,\Delta^i\jmath_0,\Delta^j\jmath_0\right) : i,j = 1,2,\cdots\right\} < \infty.$$

In L, then Δ has at least one fixed point. Furthermore, if $(L,\overline{S})$ is symmetry and for all fixed point $\jmath$ of Δ in L one has $\overline{S}(\jmath,\jmath,\jmath) = 0$, then the fixed point is unique.

Remark 7. *Assuming $\alpha = 0$ in Corollary 1, the Kannan fixed point theorem is expanded.*

Considering $\beta = \gamma = \varrho = \omega = 0$ as an exceptional instance of Theorem 2, we arrive at the subsequent outcome.

Corollary 2. *Let $(L,\overline{S})$ be a complete SSSM space and $\Delta : L \to L$ be a mapping such that:*
(i) Δ is an $\overline{S}$-JS-contraction, that is,

$$\wp(\overline{S}(\Delta\kappa,\Delta\jmath,\Delta\iota)) \leq (\wp(\overline{S}(\kappa,\jmath,\iota)))^{\alpha}, \forall\kappa,\jmath,\iota \in L,$$

and for some $\alpha \in (0,1)$,
(ii) there is $\jmath_0 \in L$ so that

$$\delta(\overline{S},\Delta,\jmath_0) := \sup\left\{\overline{S}\left(\Delta^i\jmath_0,\Delta^i\jmath_0,\Delta^j\jmath_0\right) : i,j = 1,2,\cdots\right\} < \infty.$$

In L, then Δ has at least one fixed point. Furthermore, if $(L,\overline{S})$ is symmetry and for all fixed point $\jmath$ of Δ in L one has $\overline{S}(\jmath,\jmath,\jmath) = 0$, then the fixed point is unique.

Utilizing $\alpha = \beta = \gamma = \varrho = 0$ in Theorem 2, we may derive a generalization of Chatterjea fixed point theorem.

Corollary 3. *Let $(L,\overline{S})$ be a complete SSSM space and $\Delta : L \to L$ be a mapping such that:*
(i) Δ is an $\overline{S}$-JS-Chatterjea type contraction, that is,

$$\wp(\overline{S}(\Delta\kappa,\Delta\jmath,\Delta\iota)) \leq (\wp(\frac{\overline{S}(\kappa,\kappa,\Delta\jmath)+\overline{S}(\jmath,\jmath,\Delta\iota)+\overline{S}(\iota,\iota,\Delta\kappa)}{3}))^{\omega},$$

$\forall\ \kappa,\jmath,\iota \in L$ and for some $\omega \in (0,1)$,
(ii) there is $\jmath_0 \in L$ so that

$$\delta(\overline{S},\Delta,\jmath_0) := \sup\left\{\overline{S}\left(\Delta^i\jmath_0,\Delta^i\jmath_0,\Delta^j\jmath_0\right) : i,j = 1,2,\cdots\right\} < \infty.$$

Then Δ has at least one fixed point in L. In addition, if $(L,\overline{S})$ is symmetry and for all fixed point $\jmath$ of Δ in L one has $\overline{S}(\jmath,\jmath,\jmath) = 0$, then the fixed point is unique.

Setting $\wp(t) = e^t$ in Theorem 2, we have

Corollary 4. *Let $(L, \overline{S})$ be a complete SSSM space and $\Delta : L \to L$ be a mapping such that:*
(i) Δ is an $\overline{S}$-Hardy-Rogers type contraction, that is,

$$\begin{aligned}\overline{S}(\Delta\kappa, \Delta\jmath, \Delta\iota) \leq &\alpha\, \overline{S}(\kappa, \jmath, \iota) + \beta\, \overline{S}(\kappa, \kappa, \Delta\kappa) + \gamma\, \overline{S}(\jmath, \jmath, \Delta\jmath) + \varrho\frac{\overline{S}(\iota, \iota, \Delta\iota)}{C} \\ &+ \omega(\frac{\overline{S}(\kappa, \kappa, \Delta\jmath) + \overline{S}(\jmath, \jmath, \Delta\iota) + \overline{S}(\iota, \iota, \Delta\kappa)}{3}), \forall\kappa, \jmath, \iota \in L,\end{aligned}$$

and for some $\alpha, \beta, \gamma, \varrho, \omega \geq 0$ with $\alpha + \beta + \gamma + \varrho + \omega < 1$,
(ii) there is $\jmath_0 \in L$ so that

$$\delta(\overline{S}, \Delta, \jmath_0) := \sup\left\{\overline{S}\left(\Delta^i\jmath_0, \Delta^i\jmath_0, \Delta^j\jmath_0\right) : i, j = 1, 2, \cdots\right\} < \infty.$$

In L, there is at least one fixed point for Δ. Furthermore, if $(L, \overline{S})$ is symmetry and for all fixed point $\jmath$ of Δ in L one has $\overline{S}(\jmath, \jmath, \jmath) = 0$, then the fixed point is unique.

Setting $\wp(t) = e^{\sqrt{t}}$ in Theorem 2, we have

Corollary 5. *Let $(L, \overline{S})$ be a complete SSSM space and $\Delta : L \to L$ be a mapping such that:*
(i)

$$\begin{aligned}\sqrt{\overline{S}(\Delta\kappa, \Delta\jmath, \Delta\iota)} \leq &\alpha\sqrt{\overline{S}(\kappa, \jmath, \iota)} + \beta\sqrt{\overline{S}(\kappa, \kappa, \Delta\kappa)} + \gamma\sqrt{\overline{S}(\jmath, \jmath, \Delta\jmath)} + \varrho\sqrt{\frac{\overline{S}(\iota, \iota, \Delta\iota)}{C}} \\ &+ \omega\sqrt{\frac{\overline{S}(\kappa, \kappa, \Delta\jmath) + \overline{S}(\jmath, \jmath, \Delta\iota) + \overline{S}(\iota, \iota, \Delta\kappa)}{3}}, \quad \forall\kappa, \jmath, \iota \in L,\end{aligned}$$

and for some $\alpha, \beta, \gamma, \varrho, \omega \geq 0$ with $\alpha + \beta + \gamma + \varrho + \omega < 1$.
(ii) there is $\jmath_0 \in L$ so that

$$\delta(\overline{S}, \Delta, \jmath_0) := \sup\left\{\overline{S}\left(\Delta^i\jmath_0, \Delta^i\jmath_0, \Delta^j\jmath_0\right) : i, j = 1, 2, \cdots\right\} < \infty.$$

In L, then Δ has at least one fixed point. Furthermore, if $(L, \overline{S})$ is symmetry and for all fixed point $\jmath$ of Δ in L one has $\overline{S}(\jmath, \jmath, \jmath) = 0$, then the fixed point is unique.

Example 4. *Consider $L = [0, \infty)$ and $\overline{S}(\jmath, \iota, \ell) = (|\jmath - \iota| + |\iota - \ell|)^2$, $\forall\jmath, \iota, \ell \in L$. Then $\overline{S}$ forms an SSSM on L. Let $a, b, c \in L$ and the sequences $\{\jmath_n\}$, $\{\iota_n\}$ and $\{\ell_n\}$ be such that*

$$\jmath_n = \frac{na + 1}{n},\ \iota_n = \frac{n^2b + 1}{n^2} \text{ and } \ell_n = \frac{n^3c + 1}{n^3}.$$

It is easy to see that

$$\lim_{n\to\infty} \overline{S}(a, a, \frac{na + 1}{n}) = 0, \lim_{n\to\infty} \overline{S}(b, b, \frac{n^2b + 1}{n^2}) = 0 \text{ and } \lim_{n\to\infty} \overline{S}(c, c, \frac{n^3c + 1}{n^3}) = 0.$$

Therefore, for every $a, b, c \in L$ there exists sequences

$$\jmath_n = \frac{na + 1}{n},\ \iota_n = \frac{n^2b + 1}{n^2} \text{ and } \ell_n = \frac{n^3c + 1}{n^3}$$

such that $\eth(\overline{S}, L, a) \neq \emptyset$, $\eth(\overline{S}, L, b) \neq \emptyset$ and $\eth(\overline{S}, L, c) \neq \emptyset$, respectively.

Define $\Delta : L \to L$ *by* $\Delta \jmath = \frac{1}{4}(\jmath - \tan^{-1}\jmath)\ \forall\ \jmath \in L$. *Then* Δ *satisfies all the conditions of Corollary 2 for* $\alpha = \frac{1}{2}$, $\wp(t) = \cosh t$ *and clearly,* Δ *has a unique fixed point* $0 \in L$, *because*

$$
\begin{aligned}
&\wp(\overline{S}(\Delta\kappa, \Delta\jmath, \Delta\iota)) \\
&= \cosh \frac{1}{16}(|\kappa - \tan^{-1}\kappa - \jmath + \tan^{-1}\jmath| + |\jmath - \tan^{-1}\jmath - \iota + \tan^{-1}\iota|)^2 \\
&\leq \cosh \frac{1}{16}(|\kappa - \jmath| + |\tan^{-1}\kappa - \tan^{-1}\jmath| + |\jmath - \iota| + |\tan^{-1}\jmath - \tan^{-1}\iota|)^2 \\
&\leq \cosh \frac{1}{16}(2|\kappa - \jmath| + 2|\jmath - \iota|)^2 \\
&= \cosh \frac{1}{4}(|\kappa - \jmath| + |\jmath - \iota|)^2 \\
&\leq \cosh \frac{1}{2}(|\kappa - \jmath| + |\jmath - \iota|)^2 \\
&\leq \left(\cosh[|\kappa - \jmath| + |\jmath - \iota|]^2\right)^{\frac{1}{2}} \\
&= (\wp(\overline{S}(\kappa, \jmath, \iota)))^{\alpha}.
\end{aligned}
$$

Keep in mind that $\cosh(\frac{1}{2}\jmath) \leq \sqrt{\cosh(\jmath)}$ *for any real number* $\jmath$. *We know that this function has only one fixed point in 0.*

3.2 Fixed point results for semigroups of mappings

Sehgal studied continuous self-mappings that satisfy an iterative contraction at every point in a complete metric space in Sehgal (1969). Sehgal demonstrated that there exists a unique fixed point for these mappings, and that the sequence of iterations of any point converges to the fixed point. The Sehgal result was extended in Guseman (1970) to mappings with an iterative contraction at each point in a proper subset of the space, even if they are not necessarily continuous.

Browder studied continuous self-mappings on a metric space that fulfil a functional inequality in Browder (1968). For these kinds of mappings, Browder provided enough conditions to ensure that the consecutive approximations of any point converge to a single fixed point. Browder's finding is extended in Sehgal & Thomas (1971) to a commutative semigroup of mappings as well as to single-valued mappings that fulfil a weaker version of the functional inequality that Browder applied, even if they are not necessarily continuous.

By a semigroup we mean a non-empty set F equipped with a binary relation "." that has the associate property. It is not necessary for "." to enjoy an identity member and an inverse member for each $f \in F$ in this structure.

Definition 11. Sehgal & Thomas (1971) *Let* L' *be a bounded subset of* L, *where* (L, d) *is a complete metric space. Assume that* F *is a commutative semigroup of self-mappings on* L' *that aren't always continuous. If, for every* $\jmath \in L'$ *there exists an* $f_\jmath \in F$ *such that*

$$d(f_\jmath(\iota), f_\jmath(\jmath)) \leq \phi(d(\iota, \jmath)),$$

for all $\iota \in L'$, *consequently,* F *is called a pointwise contractive semigroup on* L'. *Here, the real valued function* ϕ *is defined on the nonnegative real numbers.*

Theorem 3. Sehgal & Thomas (1971) *Assume that* L *has a closed subset* L'. *Let* F *be a semigroup of self-mappings on* L' *which is commutative. Let* F *be pointwise contractive on* L' *for some nondecreasing and continuous from the right mapping* $\phi : [0, \infty) \to [0, \infty)$, *such that* $\phi(r) < r$ *for all* $r > 0$. *If, for some* $\jmath_0 \in L'$,

$$\sup\{d(f(\jmath_0), \jmath_0) : f \in F\} < \infty,$$

then there exists a unique $\xi \in L'$ *such that* $f(\xi) = \xi$ *for each* $f \in F$. *Moreover, there exists a sequence* $\{g_n\} \subset F$ *with* $g_n(\jmath) \to \xi$ *for each* $\jmath \in L'$.

The following construction is inspired from article Gajić & Ralević (2018). In this part, we assume that SSSM space $(L,\overline{S})$ is symmetry. Let $(L,\overline{S})$ be a complete SSSM space and $B \subseteq L$. Let Q be a semigroup of self-mappings on B which is commutative. The semigroup Q is JS-pointwise contractive on B if for each $\jmath \in B$, there is a $\mathcal{G}_\jmath \in Q$ so that for some $\alpha \in (0,1)$,

$$\wp(\overline{S}(\mathcal{G}_\jmath(\kappa),\mathcal{G}_\jmath(\iota),\mathcal{G}_\jmath(\jmath))) \leq (\wp(\overline{S}(\kappa,\iota,\jmath)))^\alpha, \tag{7}$$

$\forall\ \kappa,\iota \in B$, where $\wp \in \Theta$.

Theorem 4. *Let Q be a commutative semigroup of self-mappings on B which is JS-pointwise contractive on B, where B is a closed nonempty subset of complete SSSM space $(L,\overline{S})$. If for some $\jmath_0 \in B$*

$$\sup\{\overline{S}(\mathcal{G}(\jmath_0),\mathcal{G}(\jmath_0),\jmath_0) : \mathcal{G} \in Q\} < \infty,$$

then, there exists a unique $u \in B$ such that $\mathcal{G}(u) = u$ for each $\mathcal{G} \in Q$. Moreover, there is a sequence $\{g_n\} \subset Q$ with $\{g_n(\jmath)\} \xrightarrow{\overline{S}} u$ for each $\jmath \in B$.

Proof. Let $\jmath_0 \in B$ so that $\overline{S}_0 = \sup\{\overline{S}(\mathcal{G}(\jmath_0),\mathcal{G}(\jmath_0),\jmath_0) : \mathcal{G} \in Q\} < \infty$. Let $\mathcal{G}_0 = \mathcal{G}_{\jmath_0}$, that is, $\mathcal{G}_0$ satisfies 7 for $\jmath_0$. Inductively, we define $\mathcal{G}_n = \mathcal{G}_{\jmath_n}$, where $\jmath_{n+1} = \mathcal{G}_n(\jmath_n)$ in which $\mathcal{G}_n$ satisfies 7 for $\jmath_n$. Then, for a fixed $k \geq 0$,

$$\sup_{n\geq k} \overline{S}(\jmath_{n+1},\jmath_{n+1},\jmath_{k+1}) = \sup_{n\geq k} \overline{S}(\mathcal{G}_n \circ \mathcal{G}_{n-1} \circ \cdots \circ \mathcal{G}_{k+1} \circ \mathcal{G}_k(\jmath_k), \mathcal{G}_n \circ \mathcal{G}_{n-1} \circ \cdots \circ \mathcal{G}_{k+1} \circ \mathcal{G}_k(\jmath_k), \mathcal{G}_k(\jmath_k)).$$

Let $\mathcal{H}_n = \mathcal{G}_n \circ \mathcal{G}_{n-1} \circ \cdots \circ \mathcal{G}_{k+1}$. It follows that

$$\begin{aligned}
\wp(\overline{S}(\jmath_{n+1},\jmath_{n+1},\jmath_{k+1})) &= \wp(\overline{S}(\mathcal{H}_n(\mathcal{G}_k(\jmath_k)),\mathcal{H}_n(\mathcal{G}_k(\jmath_k)),\mathcal{G}_k(\jmath_k)) \\
&= \wp(\overline{S}(\mathcal{G}_k(\mathcal{H}_n(\jmath_k)),\mathcal{G}_k(\mathcal{H}_n(\jmath_k)),\mathcal{G}_k(\jmath_k)) \\
&\leq \Big(\wp(\overline{S}(\mathcal{H}_n(\jmath_k),\mathcal{H}_n(\jmath_k),\jmath_k))\Big)^\alpha \\
&= \Big(\wp(\overline{S}(\mathcal{G}_{k-1}(\mathcal{H}_n(\jmath_{k-1})),\mathcal{G}_{k-1}(\mathcal{H}_n(\jmath_{k-1})),\mathcal{G}_{k-1}(\jmath_{k-1})))\Big)^\alpha \\
&\leq \Big(\wp(\overline{S}(\mathcal{H}_n(\jmath_{k-1}),\mathcal{H}_n(\jmath_{k-1}),\jmath_{k-1})\Big)^{\alpha^2} \\
&\vdots \\
&\leq \Big(\wp(\overline{S}(\mathcal{H}_n(\jmath_0),\mathcal{H}_n(\jmath_0),\jmath_0))\Big)^{\alpha^{(k+1)}} \\
&\leq \Big(\wp(\sup_{\mathcal{G}\in Q} \overline{S}(\mathcal{G}(\jmath_0),\mathcal{G}(\jmath_0),\jmath_0))\Big)^{\alpha^{(k+1)}} \\
&\leq \Big(\wp(\overline{S}_0)\Big)^{\alpha^{(k+1)}} \to 1 \quad (as\ k \to \infty).
\end{aligned}$$

So, by $(\wp_2)$ we have

$$\lim_{n\to\infty} \overline{S}(\jmath_{n+1},\jmath_{n+1},\jmath_{k+1}) = 0.$$

Thus, the sequence $\{\jmath_n\}$ is $\overline{S}$-Cauchy. Let $\{\jmath_n\} \xrightarrow{\overline{S}} u \in B$. By hypotheses, there is a $\mathcal{G}_u \in Q$ so that

$$\wp(\overline{S}(\mathcal{G}_u(\jmath_n),\mathcal{G}_u(\jmath_n),\mathcal{G}_u(u)) \leq (\wp(\overline{S}(\jmath_n,\jmath_n,u)))^\alpha \to 1 \quad (as\ n \to \infty).$$

So, from $(\wp_2)$ we have

$$\overline{S}(\mathcal{G}_u(\jmath_n),\mathcal{G}_u(\jmath_n),\mathcal{G}_u(u)) \to 0 \quad (as\ n \to \infty),$$

so that $\mathcal{G}_u(\jmath_n) \xrightarrow{\overline{S}} \mathcal{G}_u(u)$.

On the other side,

$$\begin{aligned}\wp(\overline{S}(\mathcal{G}_u(\jmath_k),\mathcal{G}_u(\jmath_k),\jmath_k)) &= \wp(\overline{S}(\mathcal{G}_u(\mathcal{G}_{k-1}(\jmath_{k-1})),\mathcal{G}_u(\mathcal{G}_{k-1}(\jmath_{k-1})),\mathcal{G}_{k-1}(\jmath_{k-1})))\\
&= \wp(\overline{S}(\mathcal{G}_{k-1}(\mathcal{G}_u(\jmath_{k-1})),\mathcal{G}_{k-1}(\mathcal{G}_u(\jmath_{k-1})),\mathcal{G}_{k-1}(\jmath_{k-1})))\\
&\le \Big(\wp(\overline{S}(\mathcal{G}_u(\jmath_{k-1}),\mathcal{G}_u(\jmath_{k-1}),\jmath_{k-1}))\Big)^{\alpha}\\
&= \Big(\wp(\overline{S}(\mathcal{G}_{k-2}(\mathcal{G}_u(\jmath_{k-2})),\mathcal{G}_{k-2}(\mathcal{G}_u(\jmath_{k-2})),\mathcal{G}_{k-2}(\jmath_{k-2})))\Big)^{\alpha}\\
&\le \Big(\wp(\overline{S}(\mathcal{G}_u(\jmath_{k-2}),\mathcal{G}_u(\jmath_{k-2}),\jmath_{k-2}))\Big)^{\alpha^2}\\
&\ \ \vdots\\
&\le \Big(\wp(\overline{S}(\mathcal{G}_u(\jmath_0),\mathcal{G}_u(\jmath_0),\jmath_0))\Big)^{\alpha^k}\\
&\le \Big(\wp(\sup_{\mathcal{G}\in Q}\overline{S}(\mathcal{G}(\jmath_0),\mathcal{G}(\jmath_0),\jmath_0))\Big)^{\alpha^k}\\
&= \Big(\wp(\overline{S}_0)\Big)^{\alpha^k}\to 1 \quad (as\ k\to\infty).\end{aligned}$$

Again by $(\wp_2)$ we have $\overline{S}(\mathcal{G}_u(\jmath_k),\mathcal{G}_u(\jmath_k),\jmath_k)\to 0$, so that

$$\overline{S}(\mathcal{G}_u(u),\mathcal{G}_u(u),u)\le C(\limsup_{k\to\infty}\overline{S}(\mathcal{G}_u(\jmath_k),\mathcal{G}_u(\jmath_k),\jmath_k))\to 0 \quad (as\ k\to\infty),$$

as a result $\mathcal{G}_u(u)=u$.

One may demonstrate that u is a unique fixed point of $\mathcal{G}_u$ on B by using (7). Specifically, if we allow u and u' to be two fixed points of $\mathcal{G}_u$ such that $u\ne u'$, then we obtain

$$\begin{aligned}\wp(\overline{S}(u',u',u)) &= \wp(\overline{S}(\mathcal{G}_u(u'),\mathcal{G}_u(u'),\mathcal{G}_u(u)))\\ &\le (\wp(\overline{S}(u',u',u)))^{\alpha},\end{aligned}$$

which is impossible unless that $u=u'$. Moreover, since Q is commutative, for any $\mathcal{G}\in Q$,

$$\mathcal{G}(u)=\mathcal{G}(\mathcal{G}_u(u))=\mathcal{G}_u(\mathcal{G}(u))$$

and therefore $\mathcal{G}(u)=u$, $\forall\ \mathcal{G}\in Q$.

Since Q is a commutative semigroup of self-mappings on B which is JS-pointwise contractive on B, so for fixed point u there exists $\mathcal{G}_u$ in Q such that for any $\jmath\in B$,

$$\begin{aligned}&\wp(\overline{S}(\mathcal{G}_u^n(\jmath),\mathcal{G}_u^n(\jmath),\mathcal{G}_u(u)))\\ &\le (\wp(\overline{S}(\jmath,\jmath,u)))^{\alpha^n}\to 1 \quad (as\ n\to\infty).\end{aligned}$$

Therefore, for each nonnegative integer n, we can set $g_n=\mathcal{G}_u^n$. Then obviously

$$\wp(\overline{S}(g_n(\jmath),g_n(\jmath),u))\le(\wp(\overline{S}(\jmath,\jmath,u)))^{\alpha^n}\to 1 \quad (as\ n\to\infty).$$

By $(\wp_2)$ we have $\overline{S}(u,u,g_n(\jmath))=\overline{S}(g_n(\jmath),g_n(\jmath),u)\to 0$, so $g_n(\jmath)\xrightarrow{\overline{S}} u$. □

Corollary 6. *Let $\mathcal{G}$ be a self-mapping on B, where B is a closed nonempty subset of the complete SSSM space $(L,\overline{S})$. If $\mathcal{G}$ fulfils the requirements:*

1. for all $\jmath,\iota\in B$

$$\sup\{\overline{S}(\mathcal{G}^k(\jmath),\mathcal{G}^k(\jmath),\iota):k\in\mathbb{N}\}<\infty;$$

2. for each $\jmath\in B$, there exists an integer $n(\jmath)\ge 1$ so that for some $\alpha\in(0,1)$

$$\wp(\overline{S}(\mathcal{G}^{n(\jmath)}(\kappa),\mathcal{G}^{n(\jmath)}(\ell),\mathcal{G}^{n(\jmath)}(\jmath)))\le(\wp(\overline{S}(\kappa,\ell,\jmath)))^{\alpha}, \tag{8}$$

$\forall\ \kappa,\ell\in B$, where $\wp\in\Theta$.

Then there is a unique $u\in B$ such that $\mathcal{G}(u)=u$ and $\{\mathcal{G}^k(\jmath)\}\xrightarrow{\overline{S}} u$, for each $\jmath\in B$.

Proof. Pointwise contractive mappings on B form a commutative semigroup which is the family $Q = \{\mathcal{G}^k : k \in \mathbb{N}\}$. According to Theorem 4, for every $\jmath \in B$, there is $\{g_k\} \subset Q$ such that $g_k(\jmath) \xrightarrow{\overline{S}} u$ where u is the unique fixed point of any element of Q. Consequently, there is only one fixed point u of $\mathcal{G}$. We now demonstrate that, for each $\jmath \in B$, $\{\mathcal{G}^k(\jmath)\} \xrightarrow{\overline{S}} u$.

Since Q is a commutative semigroup of self-mappings on B which is JS-pointwise contractive on B, so for fixed point u there exists $\mathcal{G}^{n(u)}$ in Q such that for any $\jmath \in B$,

$$\wp(\overline{S}(\mathcal{G}^{n(u)}(\jmath), \mathcal{G}^{n(u)}(\jmath), \mathcal{G}^{n(u)}(u))) \leq (\wp(\overline{S}(\jmath, \jmath, u)))^{\alpha}.$$

For k sufficiently large we have $k = \mathbf{r}.n(u) + s$, with $\mathbf{r} > 0$ and $0 \leq s < n(u)$ and therefore for any $\jmath \in B$,

$$\begin{aligned}
\wp(\overline{S}(\mathcal{G}^k(\jmath), \mathcal{G}^k(\jmath), u)) &= \wp(\overline{S}(\mathcal{G}^{\mathbf{r}.n(u)+s}(\jmath), \mathcal{G}^{\mathbf{r}.n(u)+s}(\jmath), \mathcal{G}^{n(u)}(u))) \\
&\leq (\wp(\overline{S}(\mathcal{G}^{(\mathbf{r\text{-}1}).n(u)+s}(\jmath), \mathcal{G}^{(\mathbf{r\text{-}1}).n(u)+s}(\jmath), u)))^{\alpha} \\
&\ \vdots \\
&\leq (\wp(\overline{S}(\mathcal{G}^{n(u)+s}(\jmath), \mathcal{G}^{n(u)+s}(\jmath), u)))^{\alpha^{\mathbf{r\text{-}1}}} \\
&\leq (\wp(\overline{S}(\mathcal{G}^{s}(\jmath), \mathcal{G}^{s}(\jmath), u)))^{\alpha^{\mathbf{r}}} \\
&\leq (\wp(\overline{S}_0))^{\alpha^{\mathbf{r}}} \to 1 \quad (as\ \mathbf{r} \to \infty),
\end{aligned}$$

where $\overline{S}_0 = \sup\{\overline{S}(\mathcal{G}^s(\jmath), \mathcal{G}^s(\jmath), u) : 0 \leq s \leq n(u) - 1\}$. So, by $(\wp_2)$ it follows that $\{\mathcal{G}^k(\jmath)\} \xrightarrow{\overline{S}} u$, for each $\jmath \in B$. $\square$

A generalization of the Guseman fixed point finding Guseman (1970) in an SSSM space is the next corollary.

Corollary 7. *Suppose that $\mathcal{G}$ is a self-mapping on B, where B is a closed subset of the complete SSSM space $(L, \overline{S})$. Let $\mathcal{G}$ be a mapping with contractive iterates at any point $\jmath \in B$ (for each $\jmath \in B$, there exists an integer $n(\jmath) \geq 1$ so that for some $\alpha \in (0, 1)$*

$$\overline{S}(\mathcal{G}^{n(\jmath)}(\kappa), \mathcal{G}^{n(\jmath)}(\ell), \mathcal{G}^{n(\jmath)}(\jmath)) \leq \alpha \cdot \overline{S}(\kappa, \ell, \jmath), \tag{9}$$

$\forall\ \kappa, \ell \in B$) and for all $\jmath, \iota \in B$

$$\sup\{\overline{S}(\mathcal{G}^k(\jmath), \mathcal{G}^k(\jmath), \iota) : k \in \mathbb{N}\} < \infty.$$

A unique $u \in B$ then exists such that $\mathcal{G}(u) = u$ and $\{\mathcal{G}^k(\jmath)\} \xrightarrow{\overline{S}} u$ for each $\jmath \in B$.

Proof. The proof is obtained by using $\wp(t) = e^t$ and applying Corollary 6. $\square$

4 Application

This section addresses whether the following boundary value problem (see, also Parvaneh et al. (2019) in which this problem has been discussed in a modular b-metric space) has a solution:

$$\begin{cases} y''(t) = f(t, y(t)),\ t \in [0, 1] \\ y(0) = y(1) = 0 \end{cases} \tag{10}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

The following Fredholm integral equation can be obtained by transforming the equation mentioned above:

$$y(t) = -\int_0^1 K(t, \kappa) f(\kappa, y(\kappa)) d\kappa \tag{11}$$

where the kernel is given by

$$K(t,\kappa)=\begin{cases}\kappa(1-t), & if\ \kappa\in[0,t],\\ t(1-\kappa), & if\ \kappa\in[t,1].\end{cases} \tag{12}$$

See Rahman (2007) for details.

Assume that $L=C(I,\mathbb{R})$, the set of continuous real functions defined on $I=[0,1]$. We now give an existence theorem for a solution of (11) in L.

For $\jmath\in L$, define

$$||\jmath||=\sup_{\kappa\in I}|\jmath(\kappa)|.$$

Here, $(L,||.||)$ is a Banach space.

We prove that L is a complete SSSM space if we choose:

$$\mathcal{S}(\jmath,\iota,\ell)=\sup_{\kappa\in I}(|\jmath(\kappa)-\iota(\kappa)|)+\sup_{\kappa\in I}(|\iota(\kappa)-\ell(\kappa)|)=||\jmath-\iota||+||\iota-\ell||,\forall\jmath,\iota,\ell\in L.$$

We only demonstrate that $\mathcal{S}$ meets the condition (b) of Definition 8. Note that this mapping is not a usual S-metric.

Let $\{\jmath_n\}$, $\{\iota_n\}$ and $\{\ell_n\}$ are convergent sequences such that $\lim_{n\to\infty}\mathcal{S}(\jmath,\jmath,\jmath_n)=0$, $\lim_{n\to\infty}\mathcal{S}(\iota,\iota,\iota_n)=0$ and $\lim_{n\to\infty}\mathcal{S}(\ell,\ell,\ell_n)=0$ which implies that $\lim_{n\to\infty}|\jmath_n(\kappa)-\jmath(\kappa)|=0$, $\lim_{n\to\infty}|\iota_n(\kappa)-\iota(\kappa)|=0$ and $\lim_{n\to\infty}|\ell_n(\kappa)-\ell(\kappa)|=0$, $\forall\ \kappa\in I$. We have

$$\begin{aligned}&\mathcal{S}(\jmath,\iota,\ell)\\ &=\sup_{\kappa\in I}(|\jmath(\kappa)-\iota(\kappa)|)+\sup_{\kappa\in I}(|\iota(\kappa)-\ell(\kappa)|)\\ &=\sup_{\kappa\in I}(|\jmath(\kappa)-\jmath_n(\kappa)+\jmath_n(\kappa)-\iota_n(\kappa)+\iota_n(\kappa)-\iota(\kappa)|)+\sup_{\kappa\in I}(|\iota(\kappa)-\iota_n(\kappa)+\iota_n(\kappa)-\jmath_n(\kappa)+\jmath_n(\kappa)-\ell_n(\kappa)+\ell_n(\kappa)-\ell(\kappa)|)\\ &\le\sup_{\kappa\in I}(|\jmath(\kappa)-\jmath_n(\kappa)|)+2\sup_{\kappa\in I}(|\jmath_n(\kappa)-\iota_n(\kappa)|)+2\sup_{\kappa\in I}(|\iota_n(\kappa)-\iota(\kappa)|)+\sup_{\kappa\in I}(|\jmath_n(\kappa)-\ell_n(\kappa)|)+\sup_{\kappa\in I}(|\ell_n(\kappa)-\ell(\kappa)|),\end{aligned}$$

which implies that

$$\begin{aligned}&\mathcal{S}(\jmath,\iota,\ell)\\ &=\sup_{\kappa\in I}(|\jmath(\kappa)-\iota(\kappa)|)+\sup_{\kappa\in I}(|\iota(\kappa)-\ell(\kappa)|)\\ &\le\limsup_{n\to\infty}[\sup_{\kappa\in I}(2|\jmath_n(\kappa)-\iota_n(\kappa)|)+\sup_{\kappa\in I}(|\jmath_n(\kappa)-\ell_n(\kappa)|)\\ &\le 2\limsup_{n\to\infty}\sup_{\kappa\in I}(|\jmath_n(\kappa)-\iota_n(\kappa)|)+2\limsup_{n\to\infty}\sup_{\kappa\in I}(|\jmath_n(\kappa)-\ell_n(\kappa)|)\\ &\le 2(\limsup_{n\to\infty}\mathcal{S}(\jmath_n,\jmath_n,\iota_n)+\limsup_{n\to\infty}\mathcal{S}(\jmath_n,\jmath_n,\ell_n)).\end{aligned}$$

So, $L=C(I,\mathbb{R})$ is an SSSM space.

Define $\Delta:L\to L$ by

$$\Delta y(t)=-\int_0^1 K(t,\kappa)f(\kappa,y(\kappa))d\kappa, y\in L, t\in I.$$

A function $v\in L$ is, obviously, a solution of (11) $\Leftrightarrow$ it is a fixed point of Δ.

Take into account the following presumptions:

(1) For all $y,\overline{y}\in L$ and $\forall\kappa\in I$,

$$|f(\kappa,y(\kappa))-f(\kappa,\overline{y}(\kappa))|\le\Big(|y(\kappa)-\overline{y}(\kappa)|+\frac{|y(\kappa)-\Delta y(\kappa)|}{2C}+\frac{|\overline{y}(\kappa)-\Delta\overline{y}(\kappa)|}{2C}+(\frac{|y(\kappa)-\Delta\overline{y}(\kappa)|}{3})\Big).$$

(2) there exists $\jmath_0\in L$ such that

$$\sup\{\sup_{\kappa\in I}(|\Delta^i\jmath_0(\kappa)-\Delta^j\jmath_0(\kappa)|),\ i,j=1,2,...\}<\infty.$$

Theorem 5. *Let us assume that all (1)–(2) above assumptions are true. A solution for Equation* (11) *is thus found in L.*

Proof. The proof that Δ is a $\mathcal{S}$-JS-contraction is the last step in demonstrating the satisfaction of all the presumptions of Theorem 2. We have

$$\begin{aligned}
&|\Delta y(t) - \Delta\overline{y}(t)| \\
&\leq \sup_{t\in I}\Big|\int_0^1 K(t,\kappa)f(\kappa,y(\kappa))ds - \int_0^1 K(t,\kappa)f(\kappa,\overline{y}(\kappa))ds\Big| \\
&\leq \sup_{t\in I}\int_0^1 |K(t,\kappa)||f(\kappa,y(\kappa)) - f(\kappa,\overline{y}(\kappa))|ds \\
&\leq \sup_{t\in I}\int_0^1 |K(t,\kappa)|\big(|y(\kappa)-\overline{y}(\kappa)| + \frac{|y(\kappa)-\Delta y(\kappa)|}{2C} + \frac{|\overline{y}(\kappa)-\Delta\overline{y}(\kappa)|}{2C} + (\frac{|y(\kappa)-\Delta\overline{y}(\kappa)|}{3})\big)d\kappa \\
&\leq \sup_{t\in I}\int_0^1 |K(t,\kappa)|ds\Big(||y-\overline{y}|| + \frac{||y-\Delta y||}{2C} + \frac{||\overline{y}-\Delta\overline{y}||}{2C} + (\frac{||y-\Delta\overline{y}||}{3})\Big).
\end{aligned}$$

On the other hand,

$$\int_0^1 |K(t,\kappa)|d\kappa = \int_0^t |\kappa(1-t)|d\kappa + \int_t^1 |t(1-\kappa)|d\kappa = \frac{t(1-t)}{2}.$$

Hence,

$$\sup_{t\in I}\int_0^1 |K(t,\kappa)|d\kappa = \sup_{t\in I}\frac{t(1-t)}{2} = 1/8,$$

which implies that

$$||\Delta y(t) - \Delta\overline{y}(t)|| \leq 1/8\Big(||y-\overline{y}|| + \frac{||y-\Delta y||}{2C} + \frac{||\overline{y}-\Delta\overline{y}||}{2C} + (\frac{||y-\Delta\overline{y}||}{3})\Big).$$

Similarly,

$$||\Delta\overline{y}(t) - \Delta\overline{\overline{y}}(t)|| \leq 1/8\Big(||\overline{y}-\overline{\overline{y}}|| + \frac{||\overline{y}-\Delta\overline{y}||}{2C} + \frac{||\overline{\overline{y}}-\Delta\overline{\overline{y}}||}{2C} + (\frac{||\overline{y}-\Delta\overline{\overline{y}}||}{3})\Big).$$

So, we obtain that

$$\begin{aligned}
\mathcal{S}(\Delta y,\Delta\overline{y},\Delta\overline{\overline{y}}) \leq 1/8\Big(&||y-\overline{y}|| + ||\overline{y}-\overline{\overline{y}}|| + ||y-\Delta y|| + ||\overline{y}-\Delta\overline{y}|| + \frac{||\overline{\overline{y}}-\Delta\overline{\overline{y}}||}{C} \\
&+ (\frac{||y-\Delta\overline{y}|| + ||\overline{y}-\Delta\overline{\overline{y}}|| + ||\overline{\overline{y}}-\Delta y||}{3})\Big).
\end{aligned}$$

From the above inequality, we get

$$e^{\mathcal{S}(\Delta y,\Delta\overline{y},\Delta\overline{\overline{y}})} \leq e^{\frac{1}{8}\Big(||y-\overline{y}||+||\overline{y}-\overline{\overline{y}}||+||y-\Delta y||+||\overline{y}-\Delta\overline{y}||+\frac{||\overline{\overline{y}}-\Delta\overline{\overline{y}}||}{C}+(\frac{||y-\Delta\overline{y}||+||\overline{y}-\Delta\overline{\overline{y}}||+||\overline{\overline{y}}-\Delta y||}{3})\Big)}.$$

This is equivalent to

$$e^{\mathcal{S}(\Delta y,\Delta\overline{y},\Delta\overline{\overline{y}})} \leq (e^{\mathcal{S}(y,\overline{y},\overline{\overline{y}})})^{\frac{1}{8}}(e^{\mathcal{S}(y,y,\Delta y)})^{\frac{1}{8}}(e^{\mathcal{S}(\overline{y},\overline{y},\Delta\overline{y})})^{\frac{1}{8}}(e^{\frac{\mathcal{S}(\overline{\overline{y}},\overline{\overline{y}},\Delta\overline{\overline{y}})}{C}})^{\frac{1}{8}}(e^{\frac{\mathcal{S}(y,y,\Delta\overline{y})+\mathcal{S}(\overline{y},\overline{y},\Delta\overline{\overline{y}})+\mathcal{S}(\overline{\overline{y}},\overline{\overline{y}},\Delta y)}{3}})^{\frac{1}{8}}.$$

Taking $\wp(s) = e^s$ and $\alpha = \beta = \gamma = \varrho = \omega = \dfrac{1}{8}$, we get

$$\begin{aligned}
\wp(\mathcal{S}(\Delta y,\Delta\overline{y},\Delta\overline{\overline{y}})) \leq &(\wp(\mathcal{S}(y,\overline{y},\overline{\overline{y}})))^{\alpha}(\wp(\mathcal{S}(y,y,\Delta y)))^{\beta}(\wp(\mathcal{S}(\overline{y},\overline{y},\Delta\overline{y})))^{\gamma}(\wp(\frac{\mathcal{S}(\overline{\overline{y}},\overline{\overline{y}},\Delta\overline{\overline{y}})}{C}))^{\varrho} \\
&(\wp(\frac{\mathcal{S}(y,y,\Delta\overline{y})+\mathcal{S}(\overline{y},\overline{y},\Delta\overline{\overline{y}})+\mathcal{S}(\overline{\overline{y}},\overline{\overline{y}},\Delta y)}{3}))^{\omega}.
\end{aligned}$$

As a result, it is clear that the mapping Δ estimates each and every one of the Theorem 2 requirements. Because Δ has a unique fixed point, the Fredholm integral equation (11) has a unique solution.

□

5 Conclusions

In the setting of SSSM spaces, we obtained a few fixed point findings for generalised JS-contractions. Finally, we converted a second-order boundary value differential equation to correspondence Fredholm integral equation and proved the existence and uniqueness of its solution. Our findings complement and broaden the pertinent findings in Beg et al. (2021); Gajić & Ralević (2018) and Jleli & Samet (2014).

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