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KHARRAT-TOMA PERTURBATION METHOD FOR SOLVING LINEAR AND NONLINEAR KLEIN-GORDON EQUATIONS

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Abstract. In this paper, we combine the Kharrat-Toma transform and the homotopy perturbation method for solving linear and nonlinear Klein-Gordon equations. Since the Klein-Gordon is two-variable equation, we first generalize the Kharrat-Toma transform to two-dimensional state. Then, we solve the linear Klein-Gordon equation only using the Kharrat-Toma transform and inverse Kharrat-Toma transform. Moreover, to solve the nonlinear Klein-Gordon equation, we combine the Kharrat-Toma transform with the homotopy perturbation method and obtain an approximate solution for it. Finally, we give some examples for both cases and compute the error of the approximate solutions.

Keywords: Kharrat-Toma transform, Homotopy perturbation method, Klein-Gordon equation.
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1 Introduction

The Klein-Gordon equation plays an important role in mathematical physics. This equation appears in quantum field theory, wave phenomena, plasma physics, nonlinear optics and describe the behavior of the inflaton field (Ahmad et al., 2023, 2024). In general, the Klein-Gordon equation has the following form

$$u_{tt}(x,t) - u_{xx}(x,t) + \alpha u(x,t) + f(u(x,t)) = g(x,t), \tag{1}$$

with initial conditions

$$u(x,0) = h_1(x), \tag{2}$$
$$u_t(x,0) = h_2(x), \tag{3}$$

where $u(x,t)$ is unknown function, the nonlinear function f and analytic function g are known and $\alpha \in \mathbb{R}$. Furthermore, $h_1(x)$ and $h_2(x)$ are known functions.

In recent years, the Klein-Gordon equation has attracted the attention of researchers and different methods have been proposed to solve it (Jafari et al., 2009; Meddahi et al., 2022). Some of these methods are Adomian decomposition method (Deeba & Khuri, 1996; Kaya & Sayed, 2004; Sayed, 2003), perturbation method (Alluhaibi, 2015), homotropy perturbation transform method (Chowdhury & Hashim, 2009) and perturbation iteration transform method (Cheniguel & Ayadi, 2011).

An integral transform is to convert a complex problem into a simpler one. The idea behind an integral transform is for the transform function f to F. Then, the function f can be obtained using the inverse transform F. Also, the integral transform play an important role in finding solutions to initial value problems and boundary value problems. In these cases, the differential equations and integral equations can be transformed into solvable equations or systems of algebraic equations. Note that the appropriate choice of integral transformation is very important. The most important integral transformations are Mohand transform, Sawi transform, Elzaki transform, Kamal transform, Laplace transform, Kharrat-Toma transform and Sumudu transform (Debnath & Bhatta, 2016).

In Jafari (2021) authors introduced a general integral transform in the class of Laplace transform, and has proved that new transforms in the class of Laplace transform which are introduced during last few decades are a special case of this general transform. General integral transform is used for solving higher order initial value problems, integral equations and fractional order integral equation.

There are various numerical and analytical methods for solving ordinari and partial differential equations. In Salati et al. (2023) authors obtained numerical solutions of Bagely-Torvik and a class of fractional oscillation equations by using a numerical method based on Hosoya and Clique polynomials. The authors in Jafari & Manjarekar (2022) presented a modified version of the new general integral transform and find out its relationship with Laplace type and some other integral transformations.

Our aim in this paper is to extend the Kharrat-Toma transform for two variable functions and apply Kharrat-Toma perturbation method for solving nonlinear Klein-Gordon equations.

This paper is organized as follows. In Section 2, we introduce the Kharrat-Toma transform for one-variable and then generalize it to the two-variable case. The homotopy perturbation method for solving the nonlinear partial differential equation is presented in Section 3. In Section 4, we solve the linear and nonlinear Klein-Gordon equation by the Kharrat-Toma transform and the homotopy perturbation method. Finally, some conclusions are given in Section 5.

2 Kharrat-Toma transform

In this section, we briefly review the Kharrat-Toma transform for one-variable functions (Kharrat & Toma , 2020) and then generalize it for two-variable functions.

2.1 One-variable Kharrat-Toma transform

Definition 1. *(Kharrat & Toma (2020)) The function $u(t)$ is said to have exponential order on every finite interval in $[0,\infty)$ if there exist a positive number M satisfying*

$$|u(t)| \leq Me^{\alpha t}, \qquad \alpha > 0, \qquad \forall t \geq 0.$$

Definition 2. *(Kharrat & Toma , 2020) Let the function $u(t)$ have exponential order in $[0,\infty)$. The Kharrat-Toma integral transform is*

$$B\Big[u(t)\Big] = s^3 \int_0^\infty u(t)e^{-\frac{t}{s^2}}\,dt = U(s), \qquad t \geq 0. \tag{4}$$

Furthermore, the Kharrat-Toma inverse integral transform is given by

$$u(t) = B^{-1}\Big[U(s)\Big] = B^{-1}\left[s^3 \int_0^\infty u(t)e^{-\frac{t}{s^2}}\,dt\right]. \tag{5}$$

The Kharrat-Toma transform for some functions are given in Table 1.

Theorem 1. *(Kharrat & Toma , 2020) (Sufficient condition for existence Kharrat-Toma transform) The Kharrat-Toma transform $U(s)$ exists if it has exponential order and $\int_0^b |u(t)|dt$ exists for any $b > 0$.*

Table 1: The Kharrat-Toma transform for some elementary functions.

$u(t)$	$U(s)$
1	s^5
t	s^7
t^n	$n!s^{2n+5}$
e^{kt}	$\dfrac{s^5}{1-ks^2}$
$\sin(kt)$	$\dfrac{ks^7}{1+k^2s^4}$
$\cos(kt)$	$\dfrac{s^5}{1+k^2s^4}$
$\sinh(kt)$	$\dfrac{ks^7}{1-k^2s^4}$
$\cosh(kt)$	$\dfrac{s^5}{1-k^2s^4}$

Theorem 2. *(Kharrat & Toma , 2020) (Linearity property) Let $B[u_1(t)] = U_1(s)$, $B[u_2(t)] = U_2(s)$, ..., $B[u_n(t)] = U_n(s)$ and $c_1, c_2, ..., c_n$ be constants. Then*

$$B\left[\sum_{k=1}^{n} c_k u_k(t)\right] = \sum_{k=1}^{n} c_k B[u_k(t)].$$

Theorem 3. *(Kharrat & Toma , 2020) Let $B[u(t)] = U(s)$ and $n \geq 1$. Then*

$$B\Big[u^{(n)}(t)\Big] = \frac{1}{s^{2n}}U(s) - \sum_{k=0}^{n-1} s^{-2n+2k+5}u^{(k)}(0).$$

Theorem 4. *(Kharrat & Toma , 2020) (Convolution Theorem) Let $B[u_1(t)] = U_1(s)$ and $B[u_2(t)] = U_2(s)$. Then*

$$B[u_1(t) * u_2(t)] = \frac{1}{s^3}U_1(s)U_2(s),$$

in which

$$u_1(t) * u_2(t) = \int_0^t u_1(x)u_2(t-x)dx.$$

2.2 Two-variable Kharrat-Toma transform

Similar to (4), we introduce the Kharrat-Toma transform for bivariate functions. In two-variable Kharrat-Toma transform integration is respect to variable t and variable x is constant.

Definition 3. *Let function $u(x,t)$ have exponential order for $t \in [0, \infty)$. Two-variable Kharrat-Toma integral transform is*

$$B\Big[u(x,t)\Big] = s^3 \int_0^\infty u(x,t)e^{-\frac{t}{s^2}}dt = U(x,s). \tag{6}$$

Furthermore, the two-variable Kharrat-Toma inverse integral transform is given by

$$u(x,t) = B^{-1}\Big[U(x,s)\Big] = B^{-1}\left[s^3 \int_0^\infty u(x,t)e^{-\frac{t}{s^2}}dt\right]. \tag{7}$$

Now, we give some properties of two-variable Kharrat-Toma transform.

Theorem 5. *Let* $B[u(x,t)] = U(x,s)$. *Then*

(*i*) $B\left[\dfrac{\partial u(x,t)}{\partial t}\right] = \dfrac{1}{s^2}U(x,s) - s^3 u(x,0)$.

(*ii*) $B\left[\dfrac{\partial^2 u(x,t)}{\partial t^2}\right] = \dfrac{1}{s^4}U(x,s) - su(x,0) - s^3\dfrac{\partial u}{\partial t}(x,0)$.

Proof. (*i*) By (6), we have

$$B\left[\frac{\partial u(x,t)}{\partial t}\right] = s^3\int_0^\infty \frac{\partial u(x,t)}{\partial t}e^{-\frac{t}{s^2}}dt = \lim_{b\to\infty}\left(s^3\int_0^b \frac{\partial u(x,t)}{\partial t}e^{-\frac{t}{s^2}}dt\right).$$

Let

$$z = e^{-\frac{t}{s^2}} \longrightarrow dz = -\frac{1}{s^2}e^{-\frac{t}{s^2}}dt$$
$$dw = \frac{\partial u(x,t)}{\partial t}dt \longrightarrow w = u(x,t).$$

Hence

$$\begin{aligned} B\left[\frac{\partial u(x,t)}{\partial t}\right] &= \lim_{b\to\infty}\left(s^3\left[u(x,t)e^{-\frac{t}{s^2}}\Big|_0^b + \frac{1}{s^2}\int_0^b u(x,t)e^{-\frac{t}{s^2}}dt\right]\right) \\ &= s^3\lim_{b\to\infty}\left(u(x,b)e^{-\frac{b}{s^2}} - u(x,0) + \frac{1}{s^2}\int_0^b u(x,t)e^{-\frac{t}{s^2}}dt\right) \\ &= \frac{1}{s^2}U(x,s) - s^3u(x,0). \end{aligned}$$

(*ii*) Let $g(x,t) = \dfrac{\partial u(x,t)}{\partial t}$. Then, (*i*) gives us

$$\begin{aligned} B\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right] &= B\left[\frac{\partial g(x,t)}{\partial t}\right] \\ &= \frac{1}{s^2}B[g(x,t)] - s^3 g(x,0) \\ &= \frac{1}{s^2}B\left[\frac{\partial u(x,t)}{\partial t}\right] - s^3\frac{\partial u}{\partial t}(x,0) \\ &= \frac{1}{s^2}\Big(\frac{1}{s^2}U(x,s) - s^3u(x,0)\Big) - s^3\frac{\partial u}{\partial t}(x,0) \\ &= \frac{1}{s^4}U(x,s) - su(x,0) - s^3\frac{\partial u}{\partial t}(x,0). \end{aligned}$$

□

In the following proposition, the Kharrat-Toma transform of the derivatives $u(x,t)$ respect to variable x has been calculated.

Proposition 1. *Let* $B[u(x,t)] = U(x,s)$. *Then*

- *The Kharrat-Toma transform for first-order partial derivative* $u(x,t)$ *respect to* x *is*

$$\begin{aligned} B\Big[\frac{\partial u(x,t)}{\partial x}\Big] &= s^3\int_0^\infty \frac{\partial u(x,t)}{\partial x}e^{-\frac{t}{s^2}}dt \\ &= \frac{\partial}{\partial x}\left[s^3\int_0^\infty u(x,t)e^{-\frac{t}{s^2}}dt\right] \\ &= \frac{d}{dx}U(x,s). \end{aligned}$$

- *The Kharrat-Toma transform for second-order derivative $u(x,t)$ respect to x is*

$$\begin{aligned} B\Big[\frac{\partial^2 u(x,t)}{\partial x^2}\Big] &= s^3\int_0^\infty \frac{\partial^2 u(x,t)}{\partial x^2}e^{-\frac{t}{s^2}}dt \\ &= \frac{\partial^2}{\partial x^2}\left[s^3\int_0^\infty u(x,t)e^{-\frac{t}{s^2}}dt\right] \\ &= \frac{d^2}{dx^2}U(x,s). \end{aligned}$$

- *The Kharrat-Toma transform for n-order derivative $u(x,t)$ respect to x is*

$$\begin{aligned} B\Big[\frac{\partial^n u(x,t)}{\partial x^n}\Big] &= s^3\int_0^\infty \frac{\partial^n u(x,t)}{\partial x^n}e^{-\frac{t}{s^2}}dt \\ &= \frac{\partial^n}{\partial x^n}\left[s^3\int_0^\infty u(x,t)e^{-\frac{t}{s^2}}dt\right] \\ &= \frac{d^n}{dx^n}U(x,s). \end{aligned}$$

3 Homotopy perturbation method

For the first time, the homotopy perturbation method (HPM) proposed by Liao (1995) to solve the linear and nonlinear differential and integral equations. Consider the following nonlinear partial differential equation

$$A(u) - f(r) = 0, \qquad r \in \Omega \subseteq \mathbb{R}, \tag{8}$$

with initial conditions

$$I(u) = 0, \tag{9}$$

in which A is a general partial differential operator and I is a initial operator. Now, we divided A into two parts L and N that L is linear and N is nonlinear operators. Therefore

$$L(u) + N(u) - f(r) = 0. \tag{10}$$

Homotopy method construct a map $v(r,p) : \Omega \times [0,1] \longrightarrow \mathbb{R}$ which satisfies

$$\begin{aligned} H(v,p) &= (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] \\ &= (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] \\ &= L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \end{aligned} \tag{11}$$

in which $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of (8). Furthermore

$$H(v,0) = L(v) - L(u_0) = 0, \tag{12}$$

$$H(v,1) = A(v) - f(r) = 0. \tag{13}$$

Hence, if p goes from zero to one, then $H(v,p)$ goes from $L(v) - L(u_0)$ to $A(v) - f(r)$. For nonlinear operator N, we have (Liao, 1995, 2007)

$$N(u) = \sum_{k=0}^{\infty} p^k H_k(u),$$

where H_k is calculated like Adomian polynomials as

$$H_k(u_0, ..., u_k) = \frac{1}{n!}\frac{\partial}{\partial p^n}\Big[N(\sum_{i=0}^{\infty} p^i u_i)\Big], \qquad k = 1, 2, ...$$

Now, using the perturbation technique the solution of the nonlinear partial differential equation (8) is as follows

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{14}$$

and for $p \to 1$, we get

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots$$

The series (14) is convergent and the rate of convergence depends on $A(u)$ (He , 1999).

4 Applications

To demonstrate the efficiency of the Kharrat-Toma transform and homotopy perturbation method, we solve the Klein-Gordon equations in this section.

4.1 Linear Klein-Gordon equations

In this subsection, we use the two-variable Kharrat-Toma transform (6) and inverse Kharrat-Toma transform (7) for solving the linear Klein-Gordon equations. First, we consider the linear partial differential equation.

Example 1. *Consider the following linear partial differential equation*

$$u_x - 2u_t - u = 0,$$

subject to the initial condition

$$u(x, 0) = 6e^{-3x},$$

The exact solution is $u(x, t) = 6e^{-3x-2t}$.

We apply the Kharrat-Toma transform on both side of this equation. It leads

$$\begin{aligned}
&B[u_x] - 2B[u_t] - B[u] = 0\\
&\Longrightarrow \frac{d}{dx}U(x, s) - 2\Big(\frac{1}{s^2}U(x, s) - s^3 u(x, 0)\Big) - U(x, s) = 0\\
&\Longrightarrow \frac{d}{dx}U(x, s) - \Big(\frac{2}{s^2} + 1\Big)U(x, s) = -12s^3 e^{-3x}.
\end{aligned}$$

For simplicity, let $U = B[u(x, t)] = U(x, s)$. *Then*

$$U' - \Big(\frac{2}{s^2} + 1\Big)U = -12s^3 e^{-3x}.$$

Now, we solve the first order linear differential equation. So

$$\begin{aligned}
U &= e^{\int \left(\frac{2}{s^2}+1\right)dx}\left[\int e^{-\int \left(\frac{2}{s^2}+1\right)dx}\Big(-12s^3e^{-3x}\Big)dx + c\right]\\
&= e^{\left(\frac{2}{s^2}+1\right)x}\left[-12s^3\int e^{-\left(\frac{2}{s^2}+4\right)x}dx + c\right]\\
&= e^{\left(\frac{2}{s^2}+1\right)x}\left[12s^3\frac{s^2}{2+4s^2}e^{-\left(\frac{2}{s^2}+4\right)x} + c\right]\\
&= e^{\left(\frac{2}{s^2}+1\right)x}\left[\frac{6s^5}{1+2s^2}e^{-\left(\frac{2}{s^2}+4\right)x} + c\right]\\
&= \frac{6s^5}{1+2s^2}e^{-3x} + ce^{\left(\frac{2}{s^2}+1\right)x}.
\end{aligned}$$

For $c = 0$, we have

$$U(x,s) = \frac{6s^5}{1+2s^2}e^{-3x}.$$

Finally

$$u(x,t) = 6e^{-3x}B^{-1}\left[\frac{s^5}{1+2s^2}\right] = 6e^{-3x}e^{-2t} = 6e^{-3x-2t}.$$

Example 2. *Consider the linear Klein-Gordon equation (Ravi Kanth & Aruna , 2009)*

$$u_{tt} - u_{xx} - u = 0,$$

subject to the initial conditions

$$u(x,0) = 1 + \sin x, \qquad u_t(x,0) = 0,$$

in which the exact solution is $u(x,t) = \sin x + \cosh t$. Using the Kharrat-Toma transform (6), *we obtain*

$$\begin{aligned}
&B[u_{tt}] - B[u_{xx}] - B[u] = 0\\
&\Longrightarrow \frac{1}{s^4}U(x,s) - su(x,0) - s^3\frac{du}{dt}(x,0) - \frac{d^2}{dx^2}U(x,s) - U(x,s) = 0\\
&\Longrightarrow \frac{1}{s^4}U(x,s) - s(1+\sin x) - \frac{d^2}{dx^2}U(x,s) - U(x,s) = 0.
\end{aligned}$$

Let $U = B[u(x,t)] = U(x,s)$. Then

$$U'' - \Big(\frac{1}{s^4} - 1\Big)U = -s(1+\sin x).$$

First, we solve the homogeneous equation $U'' - \Big(\frac{1}{s^4} - 1\Big)U = 0$. Hence

$$\lambda^2 - \Big(\frac{1}{s^4} - 1\Big) = 0 \Longrightarrow \lambda = \pm\frac{\sqrt{1-s^4}}{s^2} \Longrightarrow U_g(x,s) = c_1e^{-\frac{\sqrt{1-s^4}}{s^2}x} + c_2e^{\frac{\sqrt{1-s^4}}{s^2}x}.$$

The private solution is as follows

$$\begin{aligned}
U_p &= A_1 + A_2\sin x + A_3\cos x\\
U_p' &= A_2\cos x - A_3\sin x\\
U_p'' &= -A_2\sin x - A_3\cos x
\end{aligned}$$

Hence

$$-A_2\sin x - A_3\cos x - \Big(\frac{1}{s^4}-1\Big)(A_1 + A_2\sin x + A_3\cos x) = -s(1+\sin x).$$

So

$$A_1 = \frac{s^5}{1-s^4}, \qquad A_2 = s^5, \qquad A_3 = 0.$$

For $c_1 = c_2 = 0$*, we have*

$$u(x,t) = B^{-1}\left[\frac{s^5}{1-s^4} + s^5\sin x\right] = \cosh t + \sin x.$$

Example 3. *Consider the linear Klein-Gordon equation (Chowdhury & Hashim, 2009)*

$$u_{tt} - u_{xx} + u = 2\sin x,$$

subject to the initial conditions

$$u(x,0) = \sin x, \qquad u_t(x,0) = 1,$$

in which the exact solution is $u(x,t) = \sin x + \sin t$*. Using the Kharrat-Toma transform* (6)*, we obtain*

$$\begin{aligned}
&B[u_{tt}] - B[u_{xx}] + B[u] = 2B[\sin x]\\
&\Longrightarrow \frac{1}{s^4}U(x,s) - su(x,0) - s^3\frac{du}{dt}(x,0) - \frac{d^2}{dx^2}U(x,s) + U(x,s) = 2s^5\sin x\\
&\Longrightarrow \Big(\frac{1}{s^4}+1\Big)U(x,s) - s\sin x - s^3 - U''(x,s) = 2s^5\sin x.
\end{aligned}$$

Let $U = B[u(x,t)] = U(x,s)$*. Then*

$$U'' - \Big(\frac{1}{s^4}+1\Big)U = -(s+2s^5)\sin x - s^3.$$

First, we solve the homogeneous equation $U'' - \Big(\dfrac{1}{s^4}+1\Big)U = 0$.

$$\lambda^2 - \Big(\frac{1}{s^4}+1\Big) = 0 \Longrightarrow \lambda = \pm\frac{\sqrt{1+s^4}}{s^2} \Longrightarrow U_g(x,s) = c_1 e^{-\frac{\sqrt{1+s^4}}{s^2}x} + c_2 e^{\frac{\sqrt{1+s^4}}{s^2}x}.$$

The private solution is as follows

$$\begin{aligned}
U_p &= A_1 + A_2\sin x + A_3\cos x\\
U_p' &= A_2\cos x - A_3\sin x\\
U_p'' &= -A_2\sin x - A_3\cos x
\end{aligned}$$

Hence

$$-A_2\sin x - A_3\cos x - \Big(\frac{1}{s^4}+1\Big)(A_1 + A_2\sin x + A_3\cos x) = -(s+2s^5)\sin x - s^3.$$

So

$$A_1 = \frac{s^7}{1+s^4}, \qquad A_2 = s^5, \qquad A_3 = 0.$$

For $c_1 = c_2 = 0$*, we have*

$$u(x,t) = B^{-1}\left[\frac{s^7}{1+s^4} + s^5\sin x\right] = \sin t + \sin x.$$

4.2 Kharrat-Toma perturbation method for Nonlinear Klein-Gordon equations

In this subsection, we solve the nonlinear Klein-Gordon equation by combining Kharrat-Toma transform and homotopy perturbation method.

Example 4. *Consider the nonlinear Klein-Gordon equation (Hosseinzadeh et al., 2010)*

$$u_{tt} - u_{xx} + u^2 = x^2t^2,$$

subject to the initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = x,$$

in which the exact solution is $u(x,t) = xt$. The Kharrat-Toma transform (6), gives us

$$\begin{aligned}
&B[u_{tt}] - B[u_{xx}] + B[u^2] = B[x^2t^2] \\
&\Longrightarrow \frac{1}{s^4}U(x,s) - su(x,0) - s^3u_t(x,0) - U''(x,s) + B[u^2] = 2x^2s^9 \\
&\Longrightarrow \frac{1}{s^4}U(x,s) - s^3x - U''(x,s) + B[u^2] = 2x^2s^9.
\end{aligned}$$

Hence

$$U(x,s) - s^7x - s^4U''(x,s) - 2x^2s^{13} + s^4B[u^2] = 0.$$

Let $B[v_i(x,t)] = V_i(x,s)$. Now, the homotopy perturbation method gives

$$\begin{aligned}
H(v,p) &= \Big(V_0 + V_1p + V_2p^2 + \cdots\Big) - s^7x - 2x^2s^{13} - U_0 + pU_0 \\
&\quad + ps^4\left(B\Big[H_0(u) + pH_1(u) + p^2H_2(u) + \cdots\Big] - \frac{d^2}{dx^2}\Big(V_0 + V_1p + V_2p^2 + \cdots\Big)\right).
\end{aligned}$$

If $u_0(x,t) = xt$, then $U_0(x,s) = s^7x$.
First iteration:

$$\begin{cases}
p^0 : V_0 - s^7x - 2x^2s^{13} - U_0 = 0 \\
V_0(x,s) = 2s^7x + 2x^2s^{13} \\
v_0(x,t) = 2xt + \frac{1}{12}x^2t^4
\end{cases}$$

Second iteration:

$$\begin{cases}
p^1 : V_1 + U_0 + s^4\Big(B[H_0(u)] - \frac{d^2V_0}{dx^2}\Big) = 0 \\
H_0(u) = \Big(u_0 + pu_1 + p^2u_2 + \cdots\Big)^2\Big|_{p=0} = u_0^2 = x^2t^2 \Longrightarrow B[H_0(u)] = 2x^2s^9 \\
V_1(x,s) + s^7x + s^4(2x^2s^9 - 4s^{13}) = 0 \\
V_1(x,s) = -s^7x - 2x^2s^{13} + 4s^{17} \\
v_1(x,t) = -xt - \frac{1}{12}x^2t^4 + \frac{1}{180}t^6
\end{cases}$$

Third iteration:

$$\begin{cases}
p^2 : V_2 + s^4\Big(B[H_1(u)] - \frac{d^2V_1}{dx^2}\Big) = 0 \\
H_1(u) = \frac{\partial}{\partial p}\Big(u_0 + pu_1 + p^2u_2 + \cdots\Big)^2\Big|_{p=0} = 2u_0^2u_1 = 4x^3t^3 + \frac{1}{6}x^6t^6 \\
B[H_1(u)] = 24x^3s^{11} + 120x^6s^{17} \\
V_2(x,s) + 24x^3s^{15} + 120x^6s^{21} - 4s^{17} = 0 \\
v_2(x,t) = -\frac{1}{5}x^3t^5 - \frac{1}{336}x^6t^8 + \frac{1}{180}t^6
\end{cases}$$

Fourth iteration:

$$\begin{cases} p^3: V_3 + s^4\Big(B[H_2(u)] - \dfrac{d^2V_2}{dx^2}\Big) = 0 \\ H_2(u) = \dfrac{1}{2}\dfrac{\partial^2}{\partial p^2}\Big(u_0 + pu_1 + p^2u_2 + \cdots\Big)^2\Big|_{p=0} = u_1^2 + 2u_0u_2 \\ \qquad\qquad = 4x^2t^2 + \dfrac{1}{144}x^4t^8 + \dfrac{1}{3}x^3t^5 - \dfrac{2}{5}x^4t^6 - \dfrac{1}{168}x^7t^9 + \dfrac{1}{90}xt^7 \\ B[H_2(u)] = 8x^2s^9 + 280x^4s^{21} + 40x^3s^{15} - 288x^4s^{17} - 2160x^7s^{23} + 56xs^{19} \\ V_3(x,s) + 8x^2s^{13} + 280x^4s^{25} + 40x^3s^{19} - 288x^4s^{21} - 2160x^7s^{27} + 56xs^{23} - 144xs^{19} - 3600x^4s^{25} = 0 \\ v_3(x,t) = -\dfrac{1}{3}x^2t^4 - 28x^4t^{10} - \dfrac{1}{128}x^3t^7 + \dfrac{1}{140}x^4t^8 + \dfrac{1}{18480}x^7t^{11} - \dfrac{1}{6480}xt^9 + \dfrac{1}{70}xt^7 + \dfrac{1}{1008}x^4t^{10} \end{cases}$$

Finally

$$u(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t).$$

The error of the approximate solution $v(x,t)$ for different values of t is presented in Table 2.

Table 2: Comparison the exact and the approximate solutions for $x = 0.5$ in Example 4.

t	Approximate	Exact	Error
0	0	0	0
0.1	0.05	0.05	0
0.2	0.1	0.1	4.221×10^{-13}
0.3	0.15	0.15	3.069×10^{-12}
0.4	0.2	0.2	1.520×10^{-10}
0.5	0.25	0.25	1.211×10^{-9}
0.6	0.3	0.3	3.012×10^{-8}
0.7	0.35	0.35	1.511×10^{-8}
0.8	0.4	0.4	1.287×10^{-7}
0.9	0.45	0.45	2.285×10^{-7}
1	0.5	0.5	3.165×10^{-6}

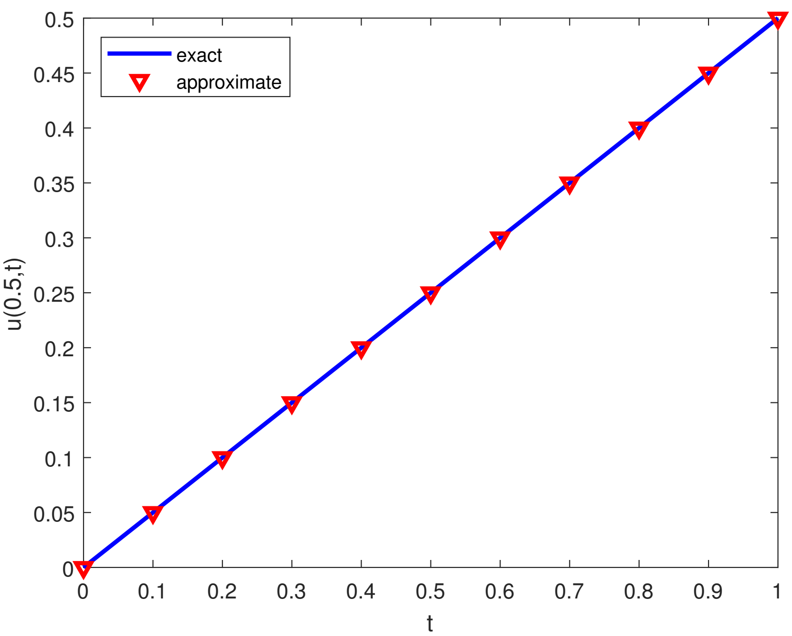

Figure 1: The exact and the approximate solutions for $x = 0.5$ in Example 4

Example 5. *Consider the nonlinear Klein-Gordon equation (Gungor, 2020)*

$$u_{tt} - u_{xx} - u + u^2 = xt + x^2t^2,$$

subject to the initial conditions

$$u(x,0) = 1, \qquad u_t(x,0) = x,$$

in which the exact solution is $u(x,t) = 1 + xt$. From Kharrat-Toma transform (6), we obtain

$$\begin{aligned}
&B[u_{tt}] - B[u_{xx}] - B[u] + B[u^2] = B[xt + x^2t^2] \\
&\Longrightarrow \frac{1}{s^4}U(x,s) - su(x,0) - s^3u_t(x,0) - U''(x,s) - U(x,s) + B[u^2] = xs^7 + 2x^2s^9 \\
&\Longrightarrow \frac{1}{s^4}U(x,s) - s - xs^3 - U''(x,s) - U(x,s) + B[u^2] = xs^7 + 2x^2s^9.
\end{aligned}$$

Hence

$$U(x,s) - s^5 - xs^7 - s^4U''(x,s) - s^4U(x,s) + s^4B[u^2] - xs^{11} - 2x^2s^{13} = 0.$$

Let $B[v_i(x,t)] = V_i(x,s)$. Now, the homotopy perturbation method gives

$$\begin{aligned}
H(v,p) = &\Big(V_0 + V_1p + V_2p^2 + \cdots\Big) - s^5 - xs^7 - xs^{11} - 2x^2s^{13} - U_0 + pU_0 - s^4\Big(V_0 + V_1p + V_2p^2 + \cdots\Big) \\
&+ ps^4\left(B\Big[H_0(u) + pH_1(u) + p^2H_2(u) + \cdots\Big] - \frac{d^2}{dx^2}\Big(V_0 + V_1p + V_2p^2 + \cdots\Big)\right).
\end{aligned}$$

If $u_0(x,t) = 1 + xt$, then $U_0(x,s) = s^5 + s^7x$. Hence

$$p^0 : V_0 - s^5 - xs^7 - xs^{11} - 2x^2s^{13} - U_0 - s^4V_0 = 0.$$

So

$$(1 - s^4)V_0 = 2s^5 + 2s^7x + xs^{11} + 2x^2s^{13},$$

or

$$V_0 = 2\frac{s^5}{1-s^4} + 2x\frac{s^7}{1-s^4} + x\Big(-s^7 + \frac{s^7}{1-s^4}\Big) + 2x^2\Big(-s^9 - s^5 + \frac{s^5}{1-s^4}\Big).$$

Using Kharrat-Toma inverse integral transform, we have

$$v_0(x,t) = 2\cosh t + 3x\sinh t - xt + x^2t^2 - 2x^2 + 2x^2\cosh t.$$

Also

$$p^1 : V_1 + U_0 - s^4V_1 + s^4\Big(B[H_0(u)] - \frac{d^2V_0}{dx^2}\Big) = 0.$$

Since

$$H_0(u) = \Big(u_0 + pu_1 + p^2u_2 + \cdots\Big)^2\Big|_{p=0} = u_0^2 = (1 + xt)^2.$$

Then

$$B[H_0(u)] = B[(1 + xt)^2] = B[1 + 2xt + x^2t^2] = s^5 + 2xs^7 + 2x^2s^9.$$

So

$$(1 - s^4)V_1(x,s) = -s^5 - s^7x + 5s^9 + 2xs^{11} + 2x^2s^{13} + 4s^{13} - 4\frac{s^9}{1-s^4}.$$

Using Kharrat-Toma inverse integral transform, we have

$$v_1(x,t) = 8\cosh t - 3x\sinh t - 9 + 2xt - x^2t^2 - 2x^2 - 2t^2 + 2x^2\cosh t - 4B^{-1}\left[\frac{s^9}{(1-s^4)^2}\right].$$

Now, we have

$$\begin{aligned} B^{-1}\left[\frac{s^9}{(1-s^4)^2}\right] &= B^{-1}\left[\frac{1}{s^3}\frac{s^{12}}{(1-s^4)^2}\right] \\ &= B^{-1}\left[\frac{s^7}{1-s^4}\right] * B^{-1}\left[\frac{s^5}{1-s^4}\right] \\ &= \sinh t * \cosh t \\ &= \int_0^t \sinh x \cosh(t-x) dx \\ &= \frac{1}{2} t \sinh t, \end{aligned}$$

Hence

$$v_1(x,t) = 8\cosh t + x\sinh t - 9 - 2xt - x^2t^2 - 2x^2 - 2t^2 + 2x^2\cosh t - 2t\sinh t.$$

After three iterations, the errors of the approximate solution $v(x,t)$ for different values of t are presented in Table 3.

Table 3: Comparison the exact and the approximate solution for $x = 0.2$ in Example 5

t	Exact	Approximate	Error
0	1	1	0
0.1	1.02	1.02	2.8809×10^{-5}
0.2	1.04	1.04	1.0734×10^{-4}
0.3	1.06	1.06	2.2393×10^{-4}
0.4	1.08	1.08	3.6709×10^{-4}
0.5	1.1	1.1	5.2558×10^{-4}
0.6	1.12	1.12	6.8834×10^{-4}
0.7	1.14	1.14	8.4450×10^{-4}
0.8	1.16	1.16	9.8337×10^{-4}
0.9	1.18	1.18	1.0944×10^{-3}
1	1.2	1.2	1.1673×10^{-3}

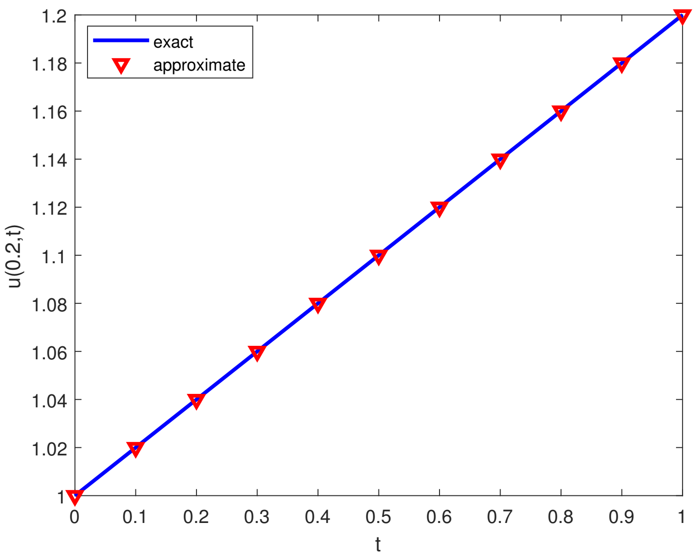

Figure 2: The exact and the approximate solutions for $x = 0.2$ in Example 4

5 Conclusion

In this paper, we solve the linear and nonlinear Klein-Gordon equations using the Kharrat-Toma transform and the homotopy perturbation method. First, we generalized the Kharrat-Toma transform to the two-dimensional case and then we solve the linear Klein-Gordon equation by it. For nonlinear Klein-Gordon equations, combined method of Kharath-Toma transform and the homotopy perturbation method has been used. Numerical examples show that the presented method has suitable accuracy.

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