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# ON AN INVERSE BOUNDARY-VALUE PROBLEM FOR A THIRD ORDER HYPERBOLIC EQUATION WITH NONCLASSICAL BOUNDARY CONDITIONS

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**Abstract.** The paper investigates the existence and uniqueness of the solution to a third-order hyperbolic equation describing the propagation of longitudinal waves in a dispersive medium with nonclassical boundary conditions. First, the original problem is transformed into an equivalent formulation with trivial boundary conditions. Then, using the Fourier method, the equivalent problem is reduced to a system of integral equations. The existence and uniqueness of the solution to this system are established using the contraction mapping principle. Finally, based on the equivalence of these problems, the existence and uniqueness of the classical solution to the original problem are demonstrated.

**Keywords**: Inverse problem, third order hyperbolic equation, existence, uniqueness, classic solution.
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## 1 Introduction

It is known that mathematical modeling of various real-world processes encountered in natural science experiments often leads to the study of inverse problems for partial differential equations. In general, inverse problems involve determining the parameters of a model when the structure of the mathematical model describing the process is already known. In other words, these problems concern the simultaneous determination of unknown coefficients and the right-hand side of partial differential equations based on additional data. Such problems are referred to as inverse problems in the theory of equations in mathematical physics. The theoretical foundations for the study of inverse problems were established and further developed in the works of Tikhonov (1943); Lavrentiev et al. (1969, 1980); Ivanov et al. (1978); Romanov (1984), and their followers (see, for example, (Isakov, 2017; Ivanchov, 2003; Kabanikhin, 2011), and others).

The wide range of applications of inverse problems - such as in seismology, mineral exploration, biology, medicine, seawater desalination, fluid flow in porous media, acoustics, electromagnetics, fluid dynamics, remote sensing, and nondestructive evaluation, among many others - underscores their significance as one of the most relevant and actively studied branches of modern mathematics.

It is worth noting that inverse boundary value problems for second-order partial differential equations have been extensively studied using various methods and under a range of boundary conditions, as seen in Kozhanov (1999); Lesnic (2021); Ramm (2021); Iskenderov & Mehraliyev (2013); Mehraliyev & Azizbayov (2021); Ismailov (2018); Azizbayov & Mehraliyev (2017); Ramazanova & Mehraliyev (2020); Azizbayov & Mehraliyev (2019), and references therein. In contrast, inverse problems for third-order hyperbolic equations-particularly those describing the propagation of longitudinal waves in dispersive media with nonclassical boundary conditions-have received significantly less attention and remain less developed compared to their second-order counterparts (e.g. see (Megraliev & Alizade, 2019a,b)).

The purpose of this paper is to prove the uniqueness and existence of solutions of an inverse boundary value problem for the third-order hyperbolic equation describing the propagation of longitudinal waves in a dispersive medium with nonclassical boundary conditions.

## 2 Problem statement and reducing it to the equivalent problem

Let $T>0$ be a fixed number and let $D_T$ be a rectangular domain defined by the inequalities $0\le x\le 1$ and $0\le t\le T$. Consider the inverse boundary value problem of identifying an unknown pair $\{u(x,t),a(t)\}$ for the following third-order hyperbolic equation (Varlamov, 1987):

$$u_{ttt}(x,t)-u_{txx}(x,t)+u_{tt}(x,t)-\alpha u_{xx}(x,t)=a(t)u(x,t)+f(x,t) \tag{1}$$

with the initial conditions

$$u\left(x,0\right)=\varphi_0\left(x\right),\ \ u_t\left(x,0\right)=\varphi_1\left(x\right),\ \ u_{tt}\left(x,0\right)=\varphi_2\left(x\right)\ \ (0\le x\le 1) \tag{2}$$

the Dirichlet boundary conditions

$$u(0,t)=0\ \ (0\le t\le T)\ , \tag{3}$$

the nonclassical boundary condition

$$u_x(1,\ t)+du_{xx}(1,\ t)=0\ \ (0\le t\le T), \tag{4}$$

and with the additional condition

$$u(x_0,t)=h(t)\ \ (0\le t\le T), \tag{5}$$

where $\alpha>0$, $d>0$, $x_0\in(0,1)$ are known numbers, $f\left(x,t\right)$, $\varphi_i\left(x\right)(i=0,1,2)$, and $h\left(t\right)$ are given functions.

**Definition 1.** *We call the classic solution of problem* (1)-(5) *a pair* $\{u(x,t),a(t)\}$ *of functions* $u(x,t)$ *and* $a(t)$ *with the following properties:*

*1) the function* $u(x,t)$ *and its derivatives* $u_t(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$,$u_{tt}(x,t)$, $u_{txx}(x,t)$ *are continuous in* $D_T$*;*

*2) the function* $a(t)$ *are continuous on* $[0,T]$*;*

*3) all conditions* (1)-(5) *are satisfied in the usual (classical) sense.*

At first, problem (1)-(5) will be reduced to an equivalent problem. We will consider the following spectral problem

$$y''(x)+\lambda\,y\left(x\right)=0,\ \ 0\le x\le 1,\ \ y(0)=0,\ \ y'(1)-d\lambda\,y\left(1\right)=0,\ \ d>0. \tag{6}$$

This problem has only eigen-functions $y_k(x)=\sqrt{2}\sin\left(\sqrt{\lambda_k}x\right)$, $k=0,\ 1,\ 2,\ ...$, with positive eigenvalues $\lambda_k$ from the equation $ctg\sqrt{\lambda}=d\sqrt{\lambda}$. We assign a zero index to any eigenfunction and number the remaining ones in ascending order of eigenvalues.

The following theorem is proved using an approach similar to the proof of Theorem 1.2 in Azizbayov & Mehraliyev (2025).

**Theorem 1.** *Let* $f(x,t) \in C(D_T)$, $\varphi_i(x) \in C[0,\ 1]$ $(i=0,1,2)$,$h(t) \in C^3\,[0,T]$, $h\,(t) \neq 0$ *for* $t \in [0,\ T]$,

$$\varphi_i(1) + \frac{1}{d\sin\sqrt{\lambda_0}} \int_0^1 \varphi_i(x)\sin(\sqrt{\lambda_0}x)dx = 0\ \ (i=0,1,2), \tag{7}$$

$$f(1,\ t) + \frac{1}{d\sin\sqrt{\lambda_0}} \int_0^1 f(x,t)\sin(\sqrt{\lambda_0}x)dx = 0\quad (0 \le t \le T), \tag{8}$$

*and the following agreement conditions be fulfilled*

$$\varphi_0'(1) + d\varphi_0''(1)\ = 0\,, \varphi_0\,(x_0) = h\,(0)\,,\quad \varphi_1\,(x_0) = h'\,(0)\,,\ \varphi_2\,(x_0) = h''\,(0)\ \ . \tag{9}$$

*Then the problem of finding the classic solution of problem* (1)-(5)*is equivalent to the problem of determining the function* $u(x,\ t)$ *and* $a(t)$*, possessing the properties 1) and 2) of determining the classic solution of problem* (1)-(5)*, satisfying equation* (1)*, conditions* (2), (3) *and conditions*

$$u(1,\ \ t) + \frac{1}{d\sin\sqrt{\lambda_0}} \int_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx = 0\ \ (0 \le t \le T), \tag{10}$$

$$h'''(t) - u_{txx}(x_0,t) + h''(t) - \alpha u_{xx}(x_0,t) = a\,(t)h(t) + f(x_0,t)\ \ (0 \le t \le T). \tag{11}$$

## 3 Auxiliary facts

Solving the homogeneous problem corresponding to problem (1)-(3), (10), (11), by the method of separation of variables, we arrive at the spectral problem

$$y''(x) + \lambda\,y(x) = 0\ \ (\,0 \le x \le 1),$$

$$y(0) = 0\,,\ y(1) + \frac{1}{d\sin\sqrt{\lambda_0}} \int_0^1 y(x)\sin(\sqrt{\lambda_0}x)dx = 0. \tag{12}$$

It is known that Kapustin & Moiseev (1997), spectral problem (12) is equivalents to spectral problem (6) without eigen function corresponding to the eigen-value $\lambda_0$. Consequently, the spectral problem (12) has only eigen-functions $y_k(x) = \sqrt{2}\sin\left(\sqrt{\lambda_k}x\right)$, $k=1,\ 2,$ ...with positive eigen-values $\lambda_k$, determined from the equation $ctg\sqrt{\lambda} = d\sqrt{\lambda}$,and numbered in the ascending order.

The following statements were formulated and justified in Kapustin & Moiseev (1997, 2000).

**Lemma 1.** *The system* $\{z_k(x)\}$ *biorthogonally conjugated to the system* $\{y_k(x)\}$, $k=1,2,3,...,$ *is determined by the following formula*

$$z_k(x) = \sqrt{2}(\sin(\sqrt{\lambda_k}x) - \sin\sqrt{\lambda_k}(\sin\sqrt{\lambda_0}x)/(\sin\sqrt{\lambda_0}))/(1 + d\sin^2\sqrt{\lambda_k}).$$

**Theorem 2.** *The system* $\{y_k(x)\}$, $k=1,2,...,$ *forms Riesz basis in the space*$L_2(0,1)$.

Since the functions $\{y_k(x)\}$, $k=1,2,3,...,$ form Riesz basics in the space $L_2(0,1)$, then it is known that for any function$g(x) \in L_2(0,1)$ the equality

$$g(x) = \sum_{k=1}^{\infty} g_k \cdot y_k(x), \tag{13}$$

is valid, where

$$g_k = \int_0^1 g(x)z_k(x)dx\,.\ (K=1,2,....)$$

Hence we have:

$$\left(\sum_{k=1}^{\infty} g_k^2\right)^{1/2} \leq M_0 \, \|g(x)\|_{L_2(0,1)}\,, \tag{14}$$

where $M = \left[\sum\limits_{k=1}^{N} \int_0^1 y_k^2(x)dx + 2/\left(9d^2\right) + 2\right]^{1/2}$,$M_0 = \left[M + \frac{1}{d\left|\sin\sqrt{\lambda_0}\right|}\left(\sum\limits_{k=1}^{\infty} \frac{1}{\lambda_k}\right)^{1/2}\right]\sqrt{2}.$

Assume that $g(x) \in C[0,1],\ g'(x) \in L_2(0,1),\ g(0) = 0$ and

$$J(g) \equiv g(1) + \frac{1}{d\sin\sqrt{\lambda_0}} \int_0^1 g(x)\sin(\sqrt{\lambda_0}x)dx = 0.$$

Then

$$\left(\sum_{k=1}^{\infty} (\sqrt{\lambda_k}\ |g_k|\,)^2\right)^{1/2} \leq M \left\|g'(x)\right\|_{L_2(0,1)}\,. \tag{15}$$

Let $g(x) \in C^1[0,1],\ g''(x) \in L_2(0,1),\ g(0) = 0$ and $J(g) = 0$. Hence we find:

$$\left(\sum_{k=1}^{\infty} (\lambda_k\ |g_k|\,)^2\right)^{1/2} \leq m\left|g'(1)\right| + \sqrt{2}M\left\|g''(x)\right\|_{L_2(0,1)}\,, \tag{16}$$

where

$$m = \frac{\sqrt{2}}{d}\left(\sum_{k=1}^{\infty} \frac{1}{\lambda_k}\right)^{1/2}.$$

Now assume that, $g(x) \in C^2[0,1],\ g'''(x) \in L_2(0,1),\ g(0) = 0,\ J(g) = 0,\ g''(0) = 0$ and $dg''(1) + g'(1) = 0$. Then we obtain:

$$\left(\sum_{k=1}^{\infty} (\lambda_k\sqrt{\lambda_k}\ |g_k|\,)^2\right)^{1/2} \leq M \left\|g'''(x)\right\|_{L_2(0,1)}\,. \tag{17}$$

1. Denote by $B_{2,T}^{3/2}$ [9], the totality of all functions $u(x,t)$ of the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t)y_k(x)\ ,$$

considered in $D_T$, where each of the functions $u_k(t)$ is continuous on $[0,T]$ and

$$\left\{\sum_{k=1}^{\infty} (\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]})^2\right\}^{1/2} < +\infty.$$

We define the norm in this set as follows:

$$\|u(x,t)\|_{B_{2,T}^{3/2}} = \left\{\sum_{k=1}^{\infty} (\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]})^2\right\}^{1/2}.$$

2. Denote by $E_T^{3/2}$ a space consisting of the topological product $B_{2,T}^{3/2} \times C[0,T]$. The norm of the element $z = \{u, a, b\}$ is determined by the formula

$$\|z\|_{E_T^{3/2}} = \|u(x,t)\|_{B_{2,T}^{3/2}} + \|a(t)\|_{C[0,T]}\,.$$

It is known that $B_{2,T}^{3/2}$ and $E_T^{3/2}$ are Banach spaces.

# 4 On solvability of the inverse boundary value problem

Taking into attention lemma 1 and theorem 2, we will look for the first component $u(x,t)$ of the solution $\{u(x,t), a(t)\}$ of problem (1)-(3),(10),(11) in the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) y_k(x), \tag{18}$$

where

$$u_k(t) = \int_0^1 u(x,t) z_k(x) dx, \ \ k = 1, 2, \ldots \ .$$

To determine the desired functions $u_k(t)$, $k = 1, 2, \ldots$, we apply the method of separation of variables to equation (1), by virtue of (2), we obtain the relation

$$u_k'''(t) + u_k''(t) + \lambda_k u_k'(t) + \alpha \lambda_k u_k(t) = F_k(t; u, a) \ (k = 1, 2, ...; \ \ 0 \le t \le T), \tag{19}$$

$$u_k(0) = \varphi_{0k}, \ \ u_k'(0) = \varphi_{1k}, \ \ u_k''(0) = \varphi_{2k} \ \ (k = 1, 2, ...), \tag{20}$$

where

$$F_k(t; u, a) = f_k(t) + a(t) u_k(t), f_k(t) = 2 \int_0^1 f(x,t) z_k(x)\, dx,$$

$$\varphi_{ik} = 2 \int_0^1 \varphi_i(x) z_k(x) dx \quad\ (i = 0, 1, 2; \ \ k = 1, 2, ...).$$

Solving problem (19), (20), we find:

$$\begin{gathered} u_k(t) = \frac{1}{b_k} \left\{ \left[ \left( \gamma_k^2 + \beta_k^2 \right) \ e^{\alpha_k t} + e^{\gamma_k t} \left[ \alpha_k \left( \alpha_k - 2\gamma_k \right) \cos \beta_k t + \right. \right. \right. \\ \left. \left. + \frac{1}{\beta_k} \left( \gamma_k^3 + \alpha_k \gamma_k^2 - \alpha_k \beta_k^2 - \alpha_k^2 \gamma_k \right) \sin \beta_k t \right] \right] \varphi_{0k} + \\ + \left[ -2\gamma_k e^{\alpha_k t} + e^{\gamma_k t} \left[ 2\gamma_k \cos \beta_k t + \frac{1}{\beta_k} \left( \alpha_k^2 + \beta_k^2 - \gamma_k^2 \right) \sin \beta_k t \right] \right] \varphi_{1k} + \\ + \left[ e^{\alpha_k t} + e^{\gamma_k t} \left[ - \cos \beta_k t + \frac{1}{\beta_k} \left( \gamma_k - \alpha_k \right) \sin \beta_k t \right] \right] \varphi_{2k} + \\ + \int_0^t F_k(\tau; u, a) \left[ e^{\alpha_k (t - \tau)} + \right. \\ \left. \left. + e^{\gamma_k (t - \tau)} \left[ \frac{\gamma_k - \alpha_k}{\beta_k} \sin \beta_k \left( t - \tau \right) - \cos \beta_k \left( t - \tau \right) \right] \right] d\tau \right\} \ (k = 1, 2, ...), \end{gathered} \tag{21}$$

where

$$\alpha_k = \alpha_{1k} + \beta_{1k} - \frac{1}{3}, \ \beta_k = \frac{\sqrt{3}}{2} \left( \alpha_{1k} - \beta_{1k} \right), \ \ \gamma_k = -\frac{1}{3} - \frac{1}{2} \left( \alpha_{1k} + \beta_{1k} \right),$$

$$b_k = \alpha_k^2 + \beta_k^2 + \gamma_k^2 - 2\alpha_k \gamma_k,$$

$$\alpha_{1k} = \left\{ -\frac{1}{2} \left( \left( \alpha - \frac{1}{3} \right) \lambda_k + \frac{2}{27} \right) + \left[ \frac{1}{4} \left( \left( \alpha - \frac{1}{3} \right) \lambda_k + \frac{2}{27} \right)^2 + \frac{1}{27} \left( \lambda_k - \frac{1}{3} \right)^3 \right]^{\frac{1}{2}} \right\}^{\frac{1}{3}},$$

$$\beta_{1k} = \left\{ -\frac{1}{2} \left( \left( \alpha - \frac{1}{3} \right) \lambda_k + \frac{2}{27} \right) - \left[ \frac{1}{4} \left( \left( \alpha - \frac{1}{3} \right) \lambda_k + \frac{2}{27} \right)^2 + \frac{1}{27} \left( \lambda_k - \frac{1}{3} \right)^3 \right]^{\frac{1}{2}} \right\}^{\frac{1}{3}}. \tag{22}$$

After substituting the expressions $u_k(t) \quad (k=1,2,...)$ in (18), to determine the component $u(x,t)$ of the solution of problem (1)-(3) ,(10),(11) we obtain:

$$u(x,t)=\sum_{k=1}^{\infty}\left\{\frac{1}{b_k}\left\{\left[\left(\gamma_k^2+\beta_k^2\right)\ e^{\alpha_k t}+e^{\gamma_k t}\left[\alpha_k\left(\alpha_k-2\gamma_k\right)\cos\beta_k t+\right.\right.\right.\right.$$

$$\left.\left.+\frac{1}{\beta_k}\left(\gamma_k^3+\alpha_k\gamma_k^2-\alpha_k\beta_k^2-\alpha_k^2\gamma_k\right)\sin\beta_k t\right]\right]\varphi_{0k}+$$

$$+\left[-2\gamma_k e^{\alpha_k t}+e^{\gamma_k t}\left[2\gamma_k\cos\beta_k t+\frac{1}{\beta_k}\left(\alpha_k^2+\beta_k^2-\gamma_k^2\right)\sin\beta_k t\right]\right]\varphi_{1k}+$$

$$+\left[e^{\alpha_k t}+e^{\gamma_k t}\left[-\cos\beta_k t+\frac{1}{\beta_k}\left(\gamma_k-\alpha_k\right)\sin\beta_k t\right]\right]\varphi_{2k}+\int_0^t F_k(\tau;u,a)\left[e^{\alpha_k(t-\tau)}+\right.$$

$$\left.\left.\left.+e^{\gamma_k(t-\tau)}\left[\frac{\gamma_k-\alpha_k}{\beta_k}\sin\beta_k\left(t-\tau\right)-\cos\beta_k\left(t-\tau\right)\right]\right]d\tau\right\}\right\}\sin\lambda_k x. \tag{23}$$

Differentiating (23) , twice, we find:

$$u_k'(t)=\frac{1}{b_k}\left\{\left[\alpha_k\left(\gamma_k^2+\beta_k^2\right)\ e^{\alpha_k t}+e^{\gamma_k t}\left[-\alpha_k\left(\gamma_k^2+\beta_k^2\right)\cos\beta_k t+\right.\right.\right.$$

$$\left.\left.+\frac{\alpha_k}{\beta_k}\left(\gamma_k-\alpha_k\right)\left(\gamma_k^2+\beta_k^2\right)\sin\beta_k t\right]\right]\varphi_{0k}+$$

$$+\left[-2\alpha_k\gamma_k e^{\alpha_k t}+e^{\gamma_k t}\left[\left(\alpha_k^2+\beta_k^2+\gamma_k^2\right)\cos\beta_k t+\frac{\gamma_k}{\beta_k}\left(\alpha_k^2-\beta_k^2-\gamma_k^2\right)\sin\beta_k t\right]\right]\varphi_{1k}+$$

$$+\left[\alpha_k e^{\alpha_k t}+e^{\gamma_k t}\left[-\alpha_k\cos\beta_k t+\frac{1}{\beta_k}\left(\beta_k^2+\gamma_k^2-\alpha_k\gamma_k\right)\sin\beta_k t\right]\right]\varphi_{2k}+$$

$$+\int_0^t F_k(\tau;u,a)\left[\alpha_k e^{\alpha_k(t-\tau)}+e^{\gamma_k(t-\tau)}\times\right.$$

$$\left.\times\left[\left(\frac{\gamma_k}{\beta_k}\left(\gamma_k-\alpha_k\right)+\beta_k\right)\sin\beta_k\left(t-\tau\right)-\alpha_k\cos\beta_k\left(t-\tau\right)\right]\right]d\tau\right\}(k=1,2,...)\ , \tag{24}$$

$$u_k''(t)=\frac{1}{b_k}\left\{\left[\alpha_k^2\left(\gamma_k^2+\beta_k^2\right)\ e^{\alpha_k t}+\alpha_k\left(\gamma_k^2+\beta_k^2\right)e^{\gamma_k t}\left[-\alpha_k\cos\beta_k t+\right.\right.\right.$$

$$\left.\left.+\frac{1}{\beta_k}\left(\gamma_k^2+\beta_k^2-\alpha_k\gamma_k\right)\sin\beta_k t\right]\right]\varphi_{0k}+$$

$$+\left[-2\alpha_k^2\gamma_k e^{\alpha_k t}+e^{\gamma_k t}\left[2\alpha_k^2\gamma_k\cos\beta_k t+\frac{1}{\beta_k}\left(\alpha_k^2\gamma_k^2-\alpha_k^2\beta_k^2-\gamma_k^4-\beta_k^4\right)\sin\beta_k t\right]\right]\varphi_{1k}+$$

$$+\left[\alpha_k^2 e^{\alpha_k t}+e^{\gamma_k t}\left[\left(\beta_k^2+\gamma_k^2-2\alpha_k\gamma_k\right)\cos\beta_k t+\right.\right.$$

$$\left.\left.+\ \frac{1}{\beta_k}\left(\gamma_k\beta_k^2+\gamma_k^3-\alpha_k\gamma_k^2+\alpha_k\beta_k^2\right)\sin\beta_k t\right]\right]\varphi_{2k}+$$

$$+\int_0^t F_k(\tau;u,a)\left[\alpha_k^2 e^{\alpha_k(t-\tau)}\ +e^{\gamma_k(t-\tau)}\left[\left(\beta_k^2+\gamma_k^2-2\alpha_k\gamma_k\right)\cos\beta_k\left(t-\tau\right)\ +\right.\right.$$

$$\left.\left.\left.+\ \frac{1}{\beta_k}\left(\gamma_k\beta_k^2+\gamma_k^3-\alpha_k\gamma_k^2+\alpha_k\beta_k^2\right)\sin\beta_k\left(t-\tau\right)t\right]\right]\ d\tau\right\}\ (k=1,2,...). \tag{25}$$

Now, from (21) and (24) we have:

$$u_k'(t)+\alpha u_k(t)=$$

$$= \frac{1}{b_k}\left\{\left[(\alpha+\alpha_k)\left(\gamma_k^2+\beta_k^2\right)\ e^{\alpha_k t}+e^{\gamma_k t}\left[\alpha_k\left(\alpha\alpha_k-2\alpha\gamma_k-\gamma_k^2-\beta_k^2\right)\cos\beta_k t+\right.\right.\right.$$

$$\left.\left.+\frac{\alpha_k}{\beta_k}\left(\left(\gamma_k-\alpha_k\right)\left(\gamma_k^2+\beta_k^2\right)+\alpha\left(\gamma_k^2-\beta_k^2-\alpha_k\gamma_k\right)\right)\sin\beta_k t\right]\right]\varphi_{0k}+$$

$$+\left[-2\left(\alpha+\alpha_k\right)\gamma_k e^{\alpha_k t}+e^{\gamma_k t}\left[\left(2\alpha\gamma_k+\alpha_k^2+\beta_k^2+\gamma_k^2\right)\cos\beta_k t+\right.\right.$$

$$\left.\left.+\frac{1}{\beta_k}\left(\alpha\left(\alpha_k^2+\beta_k^2-\gamma_k^2\right)+\gamma_k\left(\alpha_k^2-\beta_k^2-\gamma_k^2\right)\right)\sin\beta_k t\right]\right]\varphi_{1k}+\left[(\alpha+\alpha_k)\,e^{\alpha_k t}+\right.$$

$$\left.+\,e^{\gamma_k t}\left[-\left(\alpha+\alpha_k\right)\cos\beta_k t+\frac{1}{\beta_k}\left(\alpha\gamma_k-\alpha\alpha_k+\beta_k^2+\gamma_k^2-\alpha_k\gamma_k\right)\sin\beta_k t\right]\right]\varphi_{2k}+$$

$$+\int_0^t F_k(\tau;u,a)\left[(\alpha+\alpha_k)\,e^{\alpha_k(t-\tau)}+e^{\gamma_k(t-\tau)}\left[(-\alpha+\alpha_k)\cos\beta_k\left(t-\tau\right)+\right.\right.$$

$$\left.\left.\left.+\ \left(\frac{\gamma_k+\alpha}{\beta_k}\left(\gamma_k-\alpha_k\right)+\beta_k\right)\sin\beta_k\left(t-\tau\right)\right]\right]d\tau\right\}\left(k=1,2,...\right). \tag{26}$$

To obtain an equation for the second component $a(t)$ of the solution $\{u(x,t),a(t)\}$ of problem (1) - (3) , (10), (11) ,we substitute expression (18) in (11):

$$a(t)=[h(t)]^{-1}\left\{h^{'''}(t)+h^{''}(t)-f(x_0,t)+\ \sum_{k=1}^{\infty}\lambda_k\left(u_k'(t)+\alpha u_k(t)\right)y_k(x_0)\right\}.$$

Here, allowing for (26) we find:

$$a(t)=[h(t)]^{-1}\left\{h^{'''}(t)+h^{''}(t)-f(x_0,t)+\right.$$

$$+\sum_{k=1}^{\infty}\frac{\lambda_k}{b_k}\left\{\left[(\alpha+\alpha_k)\left(\gamma_k^2+\beta_k^2\right)\ e^{\alpha_k t}+e^{\gamma_k t}\left[\alpha_k\left(\alpha\alpha_k-2\alpha\gamma_k-\gamma_k^2-\beta_k^2\right)\cos\beta_k t+\right.\right.\right.$$

$$\left.\left.+\frac{\alpha_k}{\beta_k}\left(\left(\gamma_k-\alpha_k\right)\left(\gamma_k^2+\beta_k^2\right)+\alpha\left(\gamma_k^2-\beta_k^2-\alpha_k\gamma_k\right)\right)\sin\beta_k t\right]\right]\varphi_{0k}+$$

$$+\left[-2\left(\alpha+\alpha_k\right)\gamma_k e^{\alpha_k t}+e^{\gamma_k t}\left[\left(2\alpha\gamma_k+\alpha_k^2+\beta_k^2+\gamma_k^2\right)\cos\beta_k t+\right.\right.$$

$$\left.\left.+\frac{1}{\beta_k}\left(\alpha\left(\alpha_k^2+\beta_k^2-\gamma_k^2\right)+\gamma_k\left(\alpha_k^2-\beta_k^2-\gamma_k^2\right)\right)\sin\beta_k t\right]\right]\varphi_{1k}+\left[(\alpha+\alpha_k)\,e^{\alpha_k t}+\right.$$

$$\left.+\,e^{\gamma_k t}\left[-\left(\alpha+\alpha_k\right)\cos\beta_k t+\frac{1}{\beta_k}\left(\alpha\gamma_k-\alpha\alpha_k+\beta_k^2+\gamma_k^2-\alpha_k\gamma_k\right)\sin\beta_k t\right]\right]\varphi_{2k}+$$

$$+\int_0^t F_k(\tau;u,a)\left[(\alpha+\alpha_k)\,e^{\alpha_k(t-\tau)}+e^{\gamma_k(t-\tau)}\left[(-\alpha+\alpha_k)\cos\beta_k\left(t-\tau\right)+\right.\right.$$

$$\left.\left.\left.\left.+\ \left(\frac{\gamma_k+\alpha}{\beta_k}\left(\gamma_k-\alpha_k\right)+\beta_k\right)\sin\beta_k\left(t-\tau\right)\right]\right]d\tau\right\}y_k(x_0)\right\}. \tag{27}$$

Thus, the solution of problem (1)-(3) ,(10), (11) is reduced to the solution of the system (23), (27) with respect to unknown functions $u(x,t)$ and $a(t)$.

We can prove the following lemma.

**Lemma 2.** *If* $\{u(x,t),a(t)\}$ *is any solution of problem* (1)-(3) ,(10), (11) *then the function*

$$u_k(t)=\int_0^1 u(x,t)z_k(x)dx\ \ (k=1,2,...)$$

*Satisfy the system* (27).

From lemma 2 we have the following corollary

**Corollary 1.** *Let the system have a unique solution then problem* (23), (27) *can have at most one solution, i.e. if problem* (1)-(3) ,(10), (11) *has a solution, then this solution is unique.*

Let us consider in space $E_T^{3/2}$ the operator

$$\Phi(u,a) = \{\Phi_1(u,a),\ \Phi_2(u,a)\},$$

where

$$\Phi_1(u,a) = \tilde{u}(x,t) \equiv \sum_{k=1}^{\infty} \tilde{u}_k(t) y_k(x), \Phi_2(u,a) = \tilde{a}\,(t)$$

where $\tilde{u}_k(t\,)\,(\ k = 1,2,...)$, and $\tilde{a}(t)$equal to the right hand sides of (21) , and (27) respectively

Assume that the data of problem (1)-(3) , (10), (11) satisfy the following conditions:

$Q_1)$ $\varphi_i(x) \in C^2[0,1],\ \varphi_i'''(x) \in L_2(0,1),\ \varphi_i(0) = 0,\ J(\varphi_i) = 0,$
$\varphi_i''(0) = 0,$ *and* $\varphi_i''(1) + \varphi_i'(1) = 0\ \ (i = 0,1)$.

$Q_2)$ $\varphi_2(x) \in C^1[0,1],\ \varphi_2''(x) \in L_2(0,1)\,\varphi_2(0) = 0,$ *and* $J(\varphi_2\,) = 0.$

$Q_3)$ $f(x,t),\ f_x(x,t) \in C(D_T),\ f_{xx}(x,t) \in L_2(D_T), f(0,t) = 0,$ *and* $J(f) = 0\ \ (0 \le t \le T),$

$Q_4)$ $h(t) \in C^3[0,T]\ ,\ h(t) \ne 0\ (0 \le t \le T).$

Accept the denotation

$$\alpha_{2k} = -\frac{1}{2}\left(\left(\alpha - \frac{1}{3}\right)\lambda_k^2 + \frac{2}{27}\right) + \left[\frac{1}{4}\left(\left(\alpha - \frac{1}{3}\right)\lambda_k^2 + \frac{2}{27}\right)^2 + \frac{1}{27}\left(\lambda_k^2 - \frac{1}{3}\right)^3\right]^{\frac{1}{2}},$$

$$\beta_{2k} = \frac{1}{2}\left(\left(\alpha - \frac{1}{3}\right)\lambda_k^2 + \frac{2}{27}\right) + \left[\frac{1}{4}\left(\left(\alpha - \frac{1}{3}\right)\lambda_k^2 + \frac{2}{27}\right)^2 + \frac{1}{27}\left(\lambda_k^2 - \frac{1}{3}\right)^3\right]^{\frac{1}{2}}.$$

Then

$$\alpha_{1k} = \sqrt[3]{\alpha_{2k}}\ ,\ \ \beta_{1k} = -\sqrt[3]{\beta_{2k}}\ .$$

Hence we get:

$$\alpha_{1k} + \beta_{1k} = \left|\sqrt[3]{\alpha_{2k}} - \sqrt[3]{\beta_{2k}}\right| = \left|\frac{\alpha_{2k} - \beta_{2k}}{\sqrt[3]{\alpha_{2k}^2} + \sqrt[3]{\alpha_{2k}\beta_{2k}} + \sqrt[3]{\beta_{2k}^2}}\right| \le \frac{9\alpha}{2} + \frac{11}{2}\,.$$

It is not difficult to see that

$$|\alpha_k| \le \frac{9\alpha}{2} + \frac{13}{6} \equiv \varepsilon_1\,, |\gamma_k| \le \frac{9\alpha}{4} + \frac{5}{4} \equiv \varepsilon_2\,, b_k = (\alpha_k - \gamma_k)^2 + \beta_k^2 \ge \beta_k^2 \ge \varepsilon_3^2\lambda_k^2\ .$$

$$\varepsilon_3\lambda_k \equiv \frac{\sqrt{2}}{3}\lambda_k \le \beta_k \le 2\sqrt[3]{\frac{1}{2}\left(\alpha + \frac{11}{27}\right) + \sqrt{\frac{1}{4}\left(\alpha + \frac{11}{27}\right)^2 + \frac{1}{27}}}\,\lambda_k \equiv \varepsilon_4\lambda_k\,,$$

$$b_k = (\alpha_k - \gamma_k)^2 + \beta_k^2 \ge \beta_k^2 \ge \varepsilon_3^2\lambda_k^2\ .$$

Taking into account these relations, we find:

$$\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\,\|\tilde{u}_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le A_1(T) + B_1(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B_{2,T}^{3/2}}\,, \tag{28}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leq A_2(T) + B_2(T) \|a(t)\|_{C[0,T]} \|u(x,t)\|_{B^{3/2}_{2,T}} , \tag{29}$$

where

$$A_1(T) = \rho_0(T)M \left\|\varphi_0'''(x)\right\|_{L_2(0,1)} + \rho_1(T)(m \left|\varphi_1'(1)\right| + \sqrt{2}M \left\|\varphi_1''(x)\right\|_{L_2(0,1)})+$$

$$+\rho_2(T)M \left\|\varphi_2'(x)\right\|_{L_2(0,1)} + \rho_2(T)\sqrt{T}M \|f_x(x,t)\|_{L_2(D_T)} ,$$

$$B_1(T) = \rho_2(T)T ,$$

$$A_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]} \left\{\left\|h'''(t) + h''(t) - f(x_0,t)\right\|_{C[0,T]} +\right.$$

$$+\sqrt{2}\left(\sum_{k=1}^{\infty} \lambda_k^{-1}\right)^{\frac{1}{2}} \left[\rho_3(T)M \left\|\varphi_0'''(x)\right\|_{L_2(0,1)} + \rho_4(T)M \left\|\varphi_1'''(x)\right\|_{L_2(0,1)} +\right.$$

$$+\rho_5(T)(m \left|\varphi_2'(1)\right| + \sqrt{2}M \left\|\varphi_2''(x)\right\|_{L_2(0,1)})+$$

$$+\rho_5(T)\sqrt{T}\left(m \|f_x(1,t)\|_{C[0,T]} + \sqrt{2}M \|f_{xx}(x,t)\|_{L_2(D_T)}\right),$$

$$B_2(T) = \sqrt{2}\left(\sum_{k=1}^{\infty} \lambda_k^{-1}\right)^{\frac{1}{2}} \left\|[h(t)]^{-1}\right\|_{C[0,T]} T,$$

Moreover,

$$\rho_0(T) = \frac{\sqrt{5}}{\varepsilon_3^2}\left\{\left(\varepsilon_2^2 + \varepsilon_4^2\right)e^{\varepsilon_1 T} + \varepsilon_1 e^{\varepsilon_2 T}\left[\varepsilon_1 + 2\varepsilon_2 + \frac{1}{\varepsilon_3}\left(\varepsilon_2^2 + \varepsilon_4^2 + \varepsilon_1\varepsilon_2\right)\right]\right\},$$

$$\rho_1(T) = \frac{\sqrt{5}}{\varepsilon_3^2}\left\{2\varepsilon_2 e^{\varepsilon_1 T} + e^{\varepsilon_2 T}\left[2\varepsilon_2 + \frac{1}{\varepsilon_3}\left(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_4^2\right)\right]\right\},$$

$$\rho_2(T) = \frac{\sqrt{5}}{\varepsilon_3^2}\left\{e^{\varepsilon_1 T} + e^{\varepsilon_2 T}\left[1 + \frac{1}{\varepsilon_3}\left(\varepsilon_1 + \varepsilon_2\right)\right]\right\},$$

$$\rho_3(T) = \frac{1}{\varepsilon_3^2}\left\{(\alpha + \varepsilon_1)\left(\varepsilon_2^2 + \varepsilon_4^2\right)e^{\varepsilon_1 T} + e^{\varepsilon_2 T}\left[\varepsilon_1\left(\alpha\,\varepsilon_1 + 2\alpha\varepsilon_2 +\right.\right.\right.$$

$$\left.\left.\left.\varepsilon_2^2 + \varepsilon_4^2 + \frac{\varepsilon_1}{\varepsilon_3}\left((\varepsilon_1 + \varepsilon_2)\left(\varepsilon_2^2 + \varepsilon_4^2\right) + \alpha\left(\varepsilon_2^2 + \varepsilon_4^2 + \varepsilon_1\varepsilon_2\right)\right)\right]\right\} ,$$

$$\rho_4(T) = \frac{1}{\varepsilon_3^2}\left\{2\varepsilon_2(\alpha + \varepsilon_1)e^{\varepsilon_1 T} + e^{\varepsilon_2 T}\left[\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_4^2 +\right.\right.$$

$$\left.\left.+2\alpha\varepsilon_2 + \frac{1}{\varepsilon_3}\left(\varepsilon_2\left(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_4^2\right) + \alpha\left(\varepsilon_1^2 + \varepsilon_4^2 + \varepsilon_1\varepsilon_2\right)\right)\right]\right\} ,$$

$$\rho_5(T) = \frac{1}{\varepsilon_3^2}\left\{(\alpha + \varepsilon_1)e^{\varepsilon_1 T} + e^{\varepsilon_2 T}\left[\alpha + \varepsilon_1 + \frac{1}{\varepsilon_3}\left(\alpha\varepsilon_2 + \alpha\varepsilon_1 + \varepsilon_2^2 + \varepsilon_4^2 + \varepsilon_1\varepsilon_2\right)\right]\right\} .$$

From inequality (28) ,(29) we deduce:

$$\|\tilde{u}(x,t)\|_{B^{3/2}_{2,T}} + \|\tilde{a}(t)\|_{C[0,T]} \leq A(T) + B(T) \|a(t)\|_{C[0,T]} \|u(x,t)\|_{B^{3/2}_{2,T}}, \tag{30}$$

where

$$A(T) = A_1(T) + A_2(T) \ , B(T) = B_1(T) + B_2(T).$$

Thus, we proved the following theorem.

**Theorem 3.** *Let conditions $Q_1) - Q_4)$ be fulfilled, and*

$$B(T)\left(A(T)+2\right)^2 < 1. \tag{31}$$

**Remark 1.** *The inequality* (31) *is satisfied for sufficiently small values of $T$.*

*Then problem* (1)-(3), (10), (11) *has a unique solution in the sphere* $K = K_R\left(\|z\|_{E_T^{3/2}} \leq R = A(T)+2\right)$ *from* $E_T^{3/2}$

*Proof.* In the space $E_T^{3/2}$ we consider the equation

$$z = \Phi z, \tag{32}$$

where $z = \{u, a\}$, the components $\Phi_i(u, a)$ $(i = 1, 2)$ of the operator $\$(u, a)$ are determined by the right sides of equations (31), (32) , respectively.

Consider the operator $\$(u, a)$ in the sphere $K = K_R$ from $E_T^{3/2}$. Similar to (30) we obtain that for any $z = \{u, a\}$, $z_1 = \{u_1, a_1\}$, $z_2 = \{u_2, a_2\} \in K_R$ the following estimations are valid:

$$\|\Phi z\|_{E_T^{3/2}} \leq A(T) + B(T)\, \|a(t)\|_{C[0,T]}\, \|u(x,t)\|_{B_{2,T}^{3/2}}\,, \tag{33}$$

$$\|\Phi z_1 - \Phi z_2\|_{E_T^{3/2}} \leq B(T)R\left(\|a_1(t) - a_2(t)\|_{C[0,T]} + \|u_1(x,t) - u_2(x,t)\|_{B_{2,T}^{3/2}}\,.\right). \tag{34}$$

Then, allowing for (31), from estimates (33), (34) it follows that the operator $\Phi$ acts in the sphere $K = K_R$ and is compressive. Therefore, in the sphere $K = K_R$ , the operator $\Phi$ has a unique fixed point $\{u, a\}$, that is the solution of the equation (32), i.e. it is a unique solution of the system (23),(24) in the sphere $K = K_R$. Similarly to Khudaverdiev & Veliyev (2010), it can show that the function $u(x,t)$ is continuous and has continuous derivatives $u_x(x,t)$, $u_{xx}(x,t)$ , $u_t(x,\, t)$,, $u_{tx}(x,\, t)$ , $u_{txx}(x,\, t)$ , $u_{tt}(x,\, t)$ , $u_{ttt}(x,t)$ in $D_T$.

It is easy to verify that equation (1), conditions (2), (3) and (10), (11) are satisfied in the usual sense. So, $\{u(x,t), a(t)\}$ is the solution of problem (1)-(3), (10), (11). By virtue of the corollary of Lemma 2 it is unique in the sphere $K = K_R$. The theorem is proved. □

By means of Theorem 1, a unique solvability of the input problem (1)-(5) follows from the last theorem.

**Theorem 4.** *Let all the conditions of Theorem 2 be fulfilled, and*

$$\varphi_0(x_0) = h(0)\,, \quad \varphi_1(x_0) = h'(0)\,,\ \varphi_2(x_0) = h''(0)$$

*Then problem* (1)-(5) *has a unique classic solution in the sphere* $K = K_R\left(\|z\|_{E_T^{3/2}} \leq R = A(T)+2\right)$ *from* $E_T^{3/2}$ .

# 5 Conclusions

In this paper, the existence and uniqueness of the solution to a third-order hyperbolic equation describing the propagation of longitudinal waves in a dispersive medium with non classical boundary conditions are studied. First, the original problem is reduced, in a certain sense, to an auxiliary inverse boundary value problem. Then, using the Fourier method and the contraction mapping principle, an existence and uniqueness theorem for the auxiliary problem is proven. Finally, based on the equivalence of the two problems, the existence and uniqueness of a classical solution to the original inverse coefficient problem are established.

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