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ON COMPARING PONTRYAGIN'S MAXIMUM PRINCIPLE WITH THE BASIC NECESSARY CONDITIONS OF VARIATIONAL CALCULUS

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Abstract. This article provides a detailed comparison of Pontryagin's maximum principle with the basic necessary conditions of the classical variational calculus. Through a specific analysis and a concrete example, it is shown that, apart from the Euler equation, the more general formulations of the Weierstrass and Legendre conditions, as well as the Weierstrass-Erdmann and Jacobi conditions, are not consequences of Pontryagin's maximum principle. Hence, it can be stated that the Weierstrass-Erdmann conditions are not a consequence of Pontryagin's maximum principle. Therefore, Jacobi's condition is not a consequence of Pontryagin's Maximum Principle, and vice versa.

Keywords: Pontryagin's maximum principle, variational calculus, Euler's equation, Weierstrass and Legendre conditions, Weierstrass-Erdmann and Jacobi conditions.

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1 Introduction and Problem Statement

Let us consider a classic problem of variational calculus.

$$S(x(\cdot)) = \int_{t_o}^{t_1} L(t, x(t), \dot{x}(t))dt \to \min_{x(\cdot)}, \tag{1}$$

$$x(t_o) = x_0,\ x(t_1) = x_1, \quad x_0,\ x_1 \in R^n. \tag{2}$$

Here, the function $L(t, x, \dot{x}) : I \times R^n \times R^n \to R = (-\infty, +\infty)$ (i.e. the integrand) is at least continuously differentiable with respect to all variables, where $I = [t_0, t_1]$, R^n- is an $n-$dimensional Euclidean space, and the sought vector function $x(\cdot) : I \to R^n$ is piecewise smooth, i.e., it is continuous, and its derivative is continuous everywhere on $I := [t_0,\ t_1]$ except for a finite number of points $\tau_i \in (t_0,\ t_1)$, $i = \overline{1, m}$, and at the points τ_i the derivative $\dot{x}(\cdot)$ has discontinuities of the first type (at the points t_0 and t_1 the values of the derivative $\dot{x}(\cdot)$ are understood as the finite derivatives from the right and left, respectively). In this case, each point $\tau_i,\ i = \overline{1, m}$ is usually called a corner point of the function $\dot{x}(\cdot)$. The set of all piecewise smooth vector functions $x(\cdot) : I \to R^n$ is denoted by $KC^1(I,\ R^n)$.

Functions $x(\cdot) \in KC^1(I,\ R^n)$, that satisfy the boundary conditions (2) are called admissible functions. The set of admissible functions is denoted by D. Following Weierstrass (see, for example, Ioffe & Tikhomirov (1974), p.107) an admissible function $\bar{x}(\cdot)$ is called a strong (weak) local minimum in problem (1), (2), if there exists a number $\bar{\delta} > 0$ ($\tilde{\delta} > 0$) such that the inequality $S(x(\cdot)) \geq S(\bar{x}(\cdot))$ holds for all admissible functions $x(\cdot)$, for which

$$||x(\cdot) - \bar{x}(\cdot)||_{C(I,R^n)} \leqslant \bar{\delta}\,(max\{||x(\cdot) - \bar{x}(\cdot)||_{C(I,R^n)},\ ||\dot{x}(\cdot) - \dot{\bar{x}}(\cdot)||_{L_\infty(I,R^n)}\} \leqslant \tilde{\delta}).$$

It is obvious that if an admissible function $\bar{x}(\cdot)$ gives a strong minimum in problem (1), (2), then it also gives a weak minimum. Therefore, for such functions, the necessary condition for a weak local minimum is also a necessary condition for a strong local minimum. The converse statement is not always true (see, for example, Lavrentiev & Lyusternik (1950)).

It should also be noted that if the inequality $S(x(\cdot)) \geq S(\bar{x}(\cdot))$ holds for all admissible functions $x(\cdot) \in D$, then the admissible function $\bar{x}(\cdot)$ is called the absolute minimum in problem (1), (2) and is denoted by $\bar{x}(\cdot) \in \{absmin\}$.

It is known that, in the study of problem (1), (2) and more general problems in the calculus of variations, both necessary and sufficient conditions for an extremum have been obtained along with a number of important qualitative results (see for example, Bliss (1950); Hestenes (1966); Young (1969); Ioffe (1972); Ioffe & Tikhomirov (1968, 1974); Alekseev et al. (1979); Mayer (1896); Mardanov et al. (2022, 2023); Agrachev & Beschastnyi (2022); Dmitruk (1981); Milnor (1965); Mordukhovich (2024); Zelikin (1998, 2004); Morrey (1967))

It should also be noted here that in the works Isayeva (2022); Mardanov et al. (2025), various necessary conditions for a minimum were obtained for variational problems involving time delays.

It is known that (see, for example, (Alekseev et al., 1979)) the problem (1), (2) can be reduced to an optimal control problem of the following form:

$$S(x(\cdot)) = \int_{t_o}^{t_1} L(t, x(t), u(t))dt \to min, \tag{3}$$

$$\dot{x} = u,\ u \in R^n,\ x(t_0) = x_0,\ x(t_1) = x_1. \tag{4}$$

Let us recall that if the functions $L(t, x, x(\cdot))$ and $L_x(t, x, x(\cdot))$ are continuous with respect to all variables, and furthermore $\bar{u}(\cdot) = \dot{\bar{x}}(\cdot)$ is a solution to problem (3), (4), then Pontryagin's maximum principle (PMP) for problem (3), (4) has the following form (see, for example Alekseev et al. (1979), p. 372):

$$\max_{u\in R^n}[L(t, \bar{x}(t), u) - \bar{p}(t)u] = L(t, \bar{x}(t), \dot{\bar{x}}(t)) - \bar{p}(t)\dot{\bar{x}}(t), \tag{5}$$

$$\dot{\bar{p}}(t) = \bar{L}_x(t),\ t \in I\backslash\{\tau\},$$

where $\bar{L}_x(t) = L_x(t, \bar{x}(t), \dot{\bar{x}}(t))$ (similarly for $\bar{L}(t)$, $\bar{L}_{\dot{x}}(t)$, $\bar{L}_{\dot{x}\dot{x}}(t)$), and $\{\tau\} \subset (t_0, t_1)$ is the set of corner points of $\bar{x}(\cdot) \in D$.

We also recall the following well-known necessary condition, which we will formulate in the form of a theorem:

Theorem 1. *(Hestenes, 1966) Let the integrand $L(t, x, \dot{x})$ be continuously differentiable with respect to all variables. Then: (i) if the function $\bar{x}(\cdot) \in D$ gives at least a weak local minimum to the functional $S(x(\cdot))$, $x(\cdot) \in D$ in the problem (1), (2), then there exists a constant vector $c \in R^n$ and a number $\alpha \in R$, such that the following equalities hold:*

$$\bar{L}_{\dot{x}}(t) = \int_{t_0}^{t} \bar{L}_x(\mu)d\mu + c,\ \forall t \in I\backslash\{\tau\}; \tag{6}$$

$$\bar{L}(t) - \dot{\bar{x}}^T\bar{L}_{\dot{x}}(t) = \int_{t_0}^{t} \bar{L}_t(\mu)d\mu + \alpha,\ t \in I \setminus \{\tau\}, \tag{7}$$

where $\{\tau\}$ is the set of corner points of the function $\bar{x}(\cdot)$; (ii) if $\bar{x}(\cdot) \in D$ is a strong local minimum in problem (1), (2), then the Weierstrass condition is satisfied, i.e., the following inequalities hold:

$$E(\bar{L})(t,\,\xi) \geq 0, \quad \forall t \in I \setminus \{\tau\}, \quad \forall \xi \in R^n,$$

$$E(\bar{L})(\tilde{\tau}\pm,\,\xi) \geq 0, \quad \forall \tilde{\tau} \in \{\tau\}, \quad \forall \xi \in R^n. \tag{8}$$

Here $\tilde{\tau}\pm := \tilde{\tau} \pm 0$,

$$E(\bar{L})(t,\,\xi) = E(L)(t, \bar{x}(t), \dot{\bar{x}}(t), \dot{\bar{x}}(t) + \xi) =$$

$$= \bar{L}(t, \bar{x}(t), \dot{\bar{x}}(t) + \xi) - \bar{L}(t) - \bar{L}_{\dot{x}}(t)\xi, \tag{9}$$

where

$$E(L)(t, x, \dot{x}, q) = L(t, x, q) - L(t, x, \dot{x}) - L_{\dot{x}}^T(t, x, \dot{x})(q - \dot{x}). \tag{10}$$

If we assume that not only $L(\cdot)$, $L_x(\cdot)$, $L_{\dot{x}}(\cdot)$, but also $L_{\dot{x}\dot{x}}(\cdot)$ are continuous with respect to all variables and $\bar{x}(\cdot) \in D$ is at least a weak local minimum, then from (8) there follows, as a consequence, the necessary Legendre condition in the form:

$$\xi^T \bar{L}_{\dot{x}\dot{x}}(t)\xi \geq 0, \quad \forall t \in [t_0,\,t_1] \setminus \{\tau\}, \forall \xi \in R^n,$$

$$\xi^T \bar{L}_{\dot{x}\dot{x}}(t_0+0)\xi \geq 0,\ \xi^T \bar{L}_{\dot{x}\dot{x}}(t_1-0)\xi \geq 0,\ \xi^T \bar{L}_{\dot{x}\dot{x}}(t \pm 0)\xi \geq 0, \tag{11}$$

$$\forall t \in \{\tau\},\, \forall \xi \in R^n.$$

It is known (see, for example, Alekseev et al. (1979); Ioffe & Tikhomirov (1974)) that the necessary conditions (6), (8) and (11), i.e., the Euler equation, the Weierstrass and Legendre conditions, are consequences of Pontryagin's maximum principle (5).

The work Ioffe & Tikhomirov (1974) shows that the necessary Weierstrass-Erdmann condition is a consequence of Pontryagin's maximum principle (5) only for a strong local minimum.

It is important to note that in the work Mardanov et al. (2022), without assuming the existence of $L_x(\cdot)$, the validity of inequalities (8) and (11) is proven. Therefore, the corresponding statements in the work (Mardanov et al. (2022)) can be called the generalized Weierstrass and Legendre conditions.

Analyzing the above, we conclude that generalization and refinement of the conditions presented in Alekseev et al. (1979); Ioffe & Tikhomirov (1974) remains of current interest. Therefore, a more detailed study in this direction is the subject of this work.

2 On the comparison of the generalized Weierstrass, Legendre, and Weierstrass-Erdmann conditions with Pontryagin's maximum principle

Since, unlike the conditions obtained in Mardanov et al. (2022) (i.e., the generalized Weierstrass and Legendre conditions), the validity of Pontryagin's Maximum Principle (5) requires the continuity of the derivative $L_x(\cdot)$ with respect to all variables. Therefore, it is not difficult to assert that the generalized Weierstrass and Legendre conditions are not consequences of Pontryagin's Maximum Principle (5).

It should be noted that the necessary Weierstrass-Erdmann condition is important from both a theoretical and practical point of view. This condition provides additional information about the minimum only at the corner points of the extremal $\bar{x}(\cdot) \in D$. By virtue of Hilbert's Theorem (Hestenes, 1966) it is easily asserted that the extremal $\bar{x}(\cdot) \in D$ can have an corner point only at those points $t \in (t_0,\,t_1)$, where $det(\bar{L}_{\dot{x}\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t))) = 0$.

The following statement can be derived from the equations (6) and (7).

Theorem 2. *Let all the assumptions of Theorem 1 be satisfied along the weak local minimum $\bar{x}(\cdot) \in D$. Then, at each point $\tilde{\tau} \in \{\tau\}$ the following system of equalities holds:*

$$\begin{cases} \bar{L}_{\dot{x}}(t)|_{t=\tilde{\tau}-0}^{t=\tilde{\tau}+0} = 0, \\ \\ (\bar{L}(t) - \dot{\bar{x}}^T(t)\bar{L}_{\dot{x}}(t))|_{t=\tilde{\tau}-0}^{t=\tilde{\tau}+0} = 0, \end{cases} \tag{12}$$

where $\bar{L}(t) = L(t, \bar{x}(t), \dot{\bar{x}}(t))$ and $\bar{L}_{\dot{\bar{x}}}(\cdot)$ is defined similarly.

In order to prove Theorem 2, it is sufficient to take into account that the right-hand sides of equalities (6) and (7) are continuous on I.

In Ioffe & Tikhomirov (1974) it is shown that, from Pontryagin's Maximum Principle (5) the Weierstrass-Erdmann conditions follow only in the case of a strong local minimum.

By virtue of Theorem 2, the Weierstrass-Erdmann condition was proved to be necessary not only for a strong local minimum, but also for a weak local minimum. Hence, it can be argued that the Weierstrass-Erdmann conditions are not a consequence of Pontryagin's maximum principle in problem (1),(2).

3 Jacobi's condition on the conjugate point

In this section, it is assumed that the functions $L(\cdot)$, $L_x(\cdot)$, $L_{\dot{x}}(\cdot)$, $L_{\dot{x}\dot{x}}(\cdot)$, $L_{x\dot{x}}(\cdot)$ and $L_{\dot{x}x}(\cdot)$ are continuous with respect to all variables. In addition, along the extremal $\bar{x}(\cdot) \in D$ the following inequality holds:

$$\xi^T \bar{L}_{\dot{x}\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t))\xi > 0, \ \forall \xi \in R^n \backslash \{0\}, \ \forall t \in I. \tag{13}$$

Let us consider the quadratic functional, which is the second variation of the functional $S(x(\cdot))$, $x(\cdot) \in D$:

$$K(h(\cdot)) := \delta^2 S(h(\cdot); \bar{x}(\cdot)) =$$

$$= \int_{t_0}^{t_1} \{\dot{h}^T(t)\bar{L}_{\dot{x}\dot{x}}(t)\dot{h}(t) + 2\dot{h}^T(t)\bar{L}_{\dot{x}x}(t)\dot{h}(t) + h^T(t)\bar{L}_{xx}(t)h(t)\}dt, \tag{14}$$

$$h(\cdot) \in KC^1(I, R^n), \ \ h(t_0) = h(t_1) = 0,$$

where $\bar{x}(\cdot) \in KC^1(I, R^n)$ is an extremal of problem (1)-(2), i.e., it is a solution to equation (6).

It is clear that the integral form of the Euler equation of the functional $K(h(\cdot))$ has the form:

$$\bar{L}_{\dot{x}\dot{x}}(t)\dot{h}(t) + \bar{L}_{\dot{x}x}(t)h(t) = \int_{t_0}^{t} [\bar{L}_{\dot{x}x}(\mu)\dot{h}(\mu) + \bar{L}_{xx}(\mu)h(\mu)]d\mu + c \tag{15}$$

where $c \in R^n$ is some constant vector and

$$h(\cdot) \in KC^1(I, R^n), \ h(t_0) = h(t_1) = 0.$$

Equation (15) is called the Jacobi equation in integral form for the problem (1),(2).

Since (13) is satisfied, according to Hilbert's theorem (see Hestenes (1966)), equation (15) yields Jacobi's equations (see Alekseev et al. (1979)) in differential form:

$$\frac{d}{dt}[\bar{L}_{\dot{x}\dot{x}}(t)\dot{h}(t) + \bar{L}_{\dot{x}x}(t)h(t)] = \bar{L}_{x\dot{x}}(t)\dot{h}(t) + \bar{L}_{xx}(t)h(t).$$

It is known that (see, for example, (Alekseev et al., 1979 p. 374)) if the extremal $\bar{x}(\cdot)$ gives a weak minimum in problem (1), (2), then the following inequality holds:

$$K(h(\cdot)) \geq 0, \ \ \forall h(\cdot) \in KC^1(I, R^n), \ h(t_0) = h(t_1) = 0. \tag{16}$$

From this, taking (14) into account, we obtain that the function $\bar{h}(\cdot) = 0$ gives a strong minimum in the problem

$$K(h(\cdot)) \to min, \;\; h(\cdot) \in KC^1(I, R^n), \, h(t_0) = h(t_1) = 0. \tag{17}$$

Definition 1. (see, for example, Ioffe & Tikhomirov (1974)). Let condition (13) be satisfied on the extremal $\bar{x}(\cdot) \in D$. The point t_* is called a conjugate to the point t_0 (with respect to the functional $K(\cdot)$) if there exists a nontrivial solution $\bar{h}(\cdot) \in KC^1(I, R^n)$ of equation (15) such that $\bar{h}(t_0) = \bar{h}(t_*) = 0$.

Theorem 3. ***(Jacobi)*** (see, for example, Ioffe & Tikhomirov (1974)). If the extremal $\bar{x}(\cdot) \in D$ gives at least a weak minimum in problem (1),(2) and condition (13) is satisfied along it, then it is necessary that there are no conjugate points to the point t_0 on the interval $(t_0, \, t_1)$.

As a rule, the statement of Theorem 3 is called the Jacobi condition.

It is important to note that in Alekseev et al. (1979), it is claimed that all necessary conditions for an extremum in problem (1), (2), including Jacobi's necessary condition, are consequences of Pontryagin's Maximum Principle. However, the following example shows that the Jacobi's condition, i.e., the statement of Theorem 3, is not a consequence of Pontryagin's Maximum Principle.

Example 1. Let us consider the following problem

$$S(x(\cdot)) = \int_0^T (\dot{x}^2(t) - x^2(t))dt \to min, \; x(0) = x(T) = 0, \; x(\cdot) \in KC^1(I, \, R). \tag{18}$$

It is not difficult to see that if, in problem (18), $T = k\pi$, where k is an integer and $k > 1$, then the nontrivial solutions of the Euler equation of problem (18), i.e., the set of nontrivial solutions of the equation $\ddot{x} + x = 0$ with the condition $x(0) = x(T) = 0$ have the form $\bar{x}(t) = c\,sint$, $t \in [0, \, T]$, $c = const \neq 0$. Since $T = k\pi$, $k > 1$ is an integer, it follows from Theorem 3 that none of these solutions yields either a strong or a weak minimum in problem (18).

Now, to investigate the minimum of the functions $\bar{x}(t) = c\,sint$, $t \in [0, \, T]$, $c \in R\backslash\{0\}$ in problem (18) taking into account (3), (4) we will apply Pontryagin's Maximum Principle (5). As a result, we find that along $\bar{x}(t) = c\,sint$, $t \in [0, \, T]$, $c \in R$ the maximum principle (5) is satisfied, and therefore the functions $\bar{x}(t) = c\,sint$, $t \in T$, $c \in R\backslash\{0\}$ remain among the candidates for minimum.

Continuing the comparison, we observe that if, in problem (1), (2), only the integrand $L(\cdot)$ and its derivative $L_x(\cdot)$ are continuous, then clearly Jacobi's condition cannot be applied.

Consequently, taking into account Example 1 and reasoning presented above, it can be argued that Jacobi's necessary condition and Pontryagin's Maximum Principle have different domains of application. Therefore, Jacobi's condition is not a consequence of Pontryagin Maximum Principle, and vice versa.

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