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ON THE QUADRATIC PENCIL OF THE STURM-LIOUVILLE EQUATION ON THE HALF LINE

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Abstract. In this paper, we examine a quadratic pencil for the Sturm-Liouville equation with the discontinuous coefficient. This equation arises in quantum scattering theory as part of solving an inverse problem related to the massless Klein-Gordon equation with a static potential. Many other physical scattering scenarios also reduce to this equation for instance, problems in absorbing media within transmission line theory, electromagnetism, and elasticity. We construct integral representations for the solution satisfying the Jost condition at infinity and investigate the properties, related with the coefficients of the considered equation. We also study the solution of the initial value problem wich has an important role in the investigation of the inverse scattering problem for the quadratic pencil of the Sturm-Liouville equation. Assuming a potential function belongs to class of bounded functions we give a new integral representation for the Jost solution of and prove some interesting properties for the kernel function wich can be applied for the solution of the inverse scattering problem.

Keywords: Quadratic pencil of the Sturm-Liouville equation, Schrödinger equation, scattering data, Jost solution, integral representation, transformation operator.

AMS Subject Classification: Primary 34A55, Secondary 34B24, 34L05.

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1 Introduction

Consider the energy-dependent Schrödinger equation.

$$-y'' + q(x)y + 2\lambda p(x)y = \lambda^2 \rho(x)y \ , \ 0 < x < +\infty, \tag{1}$$

where

$$\rho(x) = \begin{cases} 1, & x \geq a, \\ \alpha^2, & 0 < x < a, \end{cases}$$

$\alpha \neq 1, \alpha > 0,\ a \in (0, +\infty)$, λ is a spectral parameter, $q(x)$ and $p(x)$ are real-valued functions such that

$$\int_0^{+\infty} (1 + |x|) \left(|q(x)|\right) dx < +\infty, \ \int_0^{+\infty} |p(x)|\, dx < \infty \tag{2}$$

and $p(x) \in BC[0,+\infty)$. Here $BC[0,+\infty)$ is the space of bounded continuous functions on the half line $[0,+\infty)$.

Equation (1) arises in quantum scattering theory as part of solving an inverse problem related to the massless Klein-Gordon equation with a static potential (see Cornille (1970), Kaup (1975), Weiss & Scharf (1971)). Many other physical scattering scenarios also reduce to this equation—for instance, problems in absorbing media within transmission line theory, electromagnetism, and elasticity (see Jaulent (1976)).

In spectral theory, particularly concerning Sturm-Liouville and Dirac operators, the method of transformation operators has been foundational for inverse spectral and scattering problems on both finite and infinite intervals. Initially introduced by V.A. Marchenko Marchenko (1950), Marchenko (1955), Marchenko (1986), the method was later expanded by Faddeev, Levitan, Gasymov, Gelfand, and others Faddeev (1958), Faddeev (1959), Gasymov (1964), Gasymov (1968), Gasymov & Guseinov (1989), Gasymov & Levitan (1966), Gelfand & Levitan (1951), Levitan (1979), Levitan & Gasymov (1964), leading to substantial advancements in the field. These contributions notably propelled the development of inverse problems, especially the full-line inverse scattering problem for the one-dimensional Schrödinger equation. For detailed discussions, see Chadan & Sabatier (1977), Deift et al. (1979), Guseinov (1985), Levitan (1975), Levitan (1987), Marchenko (1986). Let us also note that in recent years these fundamental studies have been followed up by applying them to operators with different singular potentials and parameter-dependent boundary conditions (Chitorkin et al., 2025; Kuznetsova, 2023).

The full-line inverse scattering problem for an energy-dependent (or generalized) Schrödinger equation, as an extension of the Marchenko approach, was first explored by Jaulent and Jean Jaulent (1976), Jaulent & Jean (1976), and independently by Kaup Kaup (1975) in the context of nonlinear evolution equations (see also Chadan & Sabatier (1977), Cornille (1970), Weiss & Scharf (1971)). Subsequently, the same class of equations considered by Kaup was further investigated in Tsutsmi (1981) by reducing the inverse problem to one involving a matrix-valued, energy-dependent Schrödinger equation. A more systematic approach was later offered by Maksudov and Guseinov Maksudov & Guseinov (1986), Maksudov & Guseinov (1989) for the case $\rho(x) = 1$, where they addressed both the direct and inverse scattering problems for Eq. (1), assuming the absence of a discrete spectrum. Their method enables the reconstruction of the potential functions $q(x)$ and $p(x)$ from scattering data. More recently, the half-line inverse scattering problem for Eq. (1) with $\rho(x) = 1$, and without requiring differentiability of $p(x)$, has been studied in Kamimura (2007), Kamimura (2008). It is remarkable to note the study of Hyrnyv et al. (2020) and Korotyaev et al. (2023) where the inverse scattering problems for energy dependent Schrödinger equations are considered on the half-line with integrable potentials $p(x)$, highly singular potentials $q(x)$, and some boundary conditions at the origin. Related direct and inverse spectral problems have also been tackled in a series of influential papers Gasymov (1964), Gasymov (1968), Gasymov (1982), Gasymov & Guseinov (1981), Gasymov & Guseinov (1989), Gasymov & Levitan (1966), who made notable contributions to the spectral theory of ordinary differential operators.

Recent works have shown that Sturm-Liouville operators with piecewise constant coefficients or internal discontinuities allow for integral representations of their special solutions, analogous to transformation operators (Guseinov et al., 2002; Huseynov et al., 2013; Nabiev et al., 2015; Nabiev et al., 2020). This has revitalized interest in inverse problems for such operators using modified Marchenko-type methods. On the half-line, the inverse scattering problem for Sturm-Liouville equations with piecewise constant coefficients was addressed in Darwish (1994), Gasymov (1977), where the interval $[0,\infty)$ was split into $[0,a]$ and $[a,\infty)$ to facilitate the solution. Further treatment was provided in Guseinov & Pashaev (2002) and Mamedov (2010), which derived new integral representations for the Jost solutions of the Schrödinger equation and applied them to inverse problems. The inverse spectral problem for second-order differential operator pencils on the half-line with turning points has been analyzed under certain smoothness conditions for complex potentials in Yurko (2006). Additional work on the real line, as in Mamedov & Nabiev (2019), revealed that discontinuities in $\rho(x)$ significantly affect both the form of the Jost solutions and the integral equations central to the inverse problem (see also Mamedov & Cetinkaya (2015), Mamedov & Kosar (2011)).

The work in Nabiev & Amirov (2015) considered Eq. (1) on a finite interval under given initial conditions and derived integral representations for two linearly independent solutions. Studies Nabiev (2024), Nabiev & Cücen (2023) focused on the full-line inverse scattering problem for Eq. (1), while Nabiev (2014) explored both direct and inverse problems for Eq. (1) with $\rho(x) = 1$ and internal discontinuities on the real line.

In this paper, we examine the Sturm-Liouville quadratic pencil (1) with the discontinuous coefficient, assuming potential functions belong to class (2). In Section 1 we construct the integral representation for the solution satisfying the Jost condition at infinity and investigate the properties related with the coefficients of the equation (1) and the kernel function of the integral representaion. In Section 2 we study the solution of the initial value problem for the equation (1) with locally integrable potential functions, which has an important role in the investigation of the inverse problem. Finally, in Section 3 we give a new integral representation for the Jost solution of the equation (1) under integrability conditions (2) and the boundness condition on the function $p(x)$, and prove some interesting properties for the kernel function which can be also applied to the solution of the inverse scattering problem for the equation (1).

2 Integral representations for solutions

Let $f(x, \lambda)$ be solution of (1) satisfying the condition at infinity

$$\lim_{x\to\infty} f(x,\lambda)e^{-i\lambda x} = 1. \tag{3}$$

The solution $f(x, \lambda)$ will be called the Jost-type solution of the equation (1). If $q(x) \equiv p(x) \equiv 0$, then the Jost-type solution is

$$f_0(x,\lambda) = r^+(x)e^{i\lambda\mu^+(x)} + r^-(x)e^{i\lambda\mu^-(x)}, \tag{4}$$

where

$$\begin{aligned} \mu^\pm(x) &= \pm x\sqrt{\rho(x)} + a\left(1 \mp \sqrt{\rho(x)}\right), \\ r^\pm(x) &= \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{\rho(x)}}\right). \end{aligned}$$

Then it is easy to show that the Jost solution $f(x, \lambda)$ is the solution of the integral equation

$$f(x,\lambda) = f_0(x,\lambda) + \int_x^\infty \Phi(x,t,\lambda)\,(q(t) + 2\lambda)f(t,\lambda)dt, \tag{5}$$

where

$$\Phi(x,t,\lambda) = \frac{f_0(x,\lambda)\,f_0(t,-\lambda) - f_0(t,\lambda)\,f_0(x,-\lambda)}{2i\lambda}, t \geq x.$$

By using (4) it is easily obtained that

$$\Phi(x,t,\lambda) = p^+(x,t)\frac{\sin\lambda\sigma^+(x,t)}{\lambda} - p^-(x,t)\frac{\sin\lambda\sigma^-(x,t)}{\lambda},\ t \geq x, \tag{6}$$

where

$$p^\pm(x,t) = \frac{1}{2}\left(\frac{1}{\sqrt{\rho(x)}} \pm \frac{1}{\sqrt{\rho(t)}}\right) \text{and } \sigma^\pm(x,t) = \mu^+(t) - \mu^\pm(x).$$

We have

$$2i\lambda\Phi(x,t,\lambda)e^{i\lambda\mu^{\pm}(t)} = p^{\pm}(x,t)\left[e^{i\lambda\left(2\mu^{\pm}(t)-\mu^{+}(x)\right)} - e^{i\lambda\mu^{+}(x)}\right] -$$

$$p^{\mp}(x,t)\left[e^{i\lambda\left(2\mu^{\pm}(t)-\mu^{-}(x)\right)} - e^{i\lambda\mu^{-}(x)}\right]. \qquad (7)$$

The formula (6) is also written as

$$\Phi(x,t,\lambda)e^{i\lambda\mu^{\pm}(t)} = \frac{1}{2}p^{\pm}(x,t)\int\limits_{\mu^{+}(x)}^{2\mu^{\pm}(t)-\mu^{+}(x)} e^{i\lambda s}ds - \frac{1}{2}p^{\mp}(x,t)\int\limits_{\mu^{-}(x)}^{2\mu^{\pm}(t)-\mu^{-}(x)} e^{i\lambda s}ds. \qquad (8)$$

Now consider the integral equation (5) and let us make the substitution

$$f(x,\lambda) = R^{+}(x)e^{i\lambda\mu^{+}(x)} + R^{-}(x)e^{i\lambda\mu^{-}(x)} + z(x,\lambda), \qquad (9)$$

where $R^{\pm}(x)$ will be defined later and $z(x,\lambda)$ is a new unknown function. We have

$$R^{+}(x)e^{i\lambda\mu^{+}(x)} + R^{-}(x)e^{i\lambda\mu^{-}(x)} + z(x,\lambda) = r^{+}(x)e^{i\lambda\mu^{+}(x)} + r^{-}(x)e^{i\lambda\mu^{-}(x)} +$$

$$\int\limits_{x}^{\infty}\Phi(x,t,\lambda)q\,(t)\left[R^{+}(t)e^{i\lambda\mu^{+}(t)} + R(t)e^{i\lambda\mu^{-}(t)}\right]dt +$$

$$2\lambda\int\limits_{x}^{\infty}\Phi(x,t,\lambda)p\,(t)\left[R^{+}(t)e^{i\lambda\mu^{+}(t)} + R(t)e^{i\lambda\mu^{-}(t)}\right]dt +$$

$$\int\limits_{x}^{\infty}\Phi(x,t,\lambda)\left[q\,(t) + 2\lambda p\,(t)\right]z(t,\lambda)dt. \qquad (10)$$

Taking into our account the formula (7) and the second integral in the right hand side of the equation (10) we require

$$R^{+}(x)e^{i\lambda\mu^{+}(x)} + R^{-}(x)e^{i\lambda\mu^{-}(x)} = r^{+}(x)e^{i\lambda\mu^{+}(x)} + r^{-}(x)e^{i\lambda\mu^{-}(x)} +$$

$$ie^{i\lambda\mu^{+}(x)}\int\limits_{x}^{\infty}p\,(t)\,R^{+}(t)p^{+}(x,t)dt - ie^{i\lambda\mu^{-}(x)}\int\limits_{x}^{\infty}p\,(t)\,R^{+}(t)p^{-}(x,t)dt +$$

$$ie^{i\lambda\mu^{+}(x)}\int\limits_{x}^{\infty}p\,(t)\,R^{-}(t)p^{-}(x,t)dt - ie^{i\lambda\mu^{-}(x)}\int\limits_{x}^{\infty}p\,(t)\,R^{-}(t)p^{+}(x,t)dt$$

to be satisfied. Obviously, the last equality will be satisfied if we choose

$$R^{\pm}(x) = r^{\pm}(x) \pm i\int\limits_{x}^{\infty}p\,(t)\,R^{+}(t)p^{\pm}(x,t)dt \pm i\int\limits_{x}^{\infty}p\,(t)\,R^{-}(t)p^{\mp}(x,t)dt. \qquad (11)$$

From (11) we immediately have

$$R^{\pm}(x) = r^{\pm}(x)e^{i\int\limits_{x}^{\infty} sgn(t\pm a)\frac{p(t)}{\sqrt{\rho(t)}}dt}.$$

Then from the equation (10) we find for the unknown function $z(x,\lambda)$ defined by (9) the integral equation

$$z(x,\lambda) = \int\limits_{x}^{\infty}\Phi(x,t,\lambda)q\,(t)\left[R^{+}(t)e^{i\lambda\mu^{+}(t)} + R^{-}(t)e^{i\lambda\mu^{-}(t)}\right]dt -$$

$$i\int_x^\infty p^+(x,t)p\,(t)\,R^+(t)e^{i\lambda(2\mu^+(t)-\mu^+(x))}dt+$$

$$i\int_x^\infty p^-(x,t)p\,(t)\,R^+(t)e^{i\lambda(2\mu^+(t)-\mu^-(x))}dt-$$

$$i\int_x^\infty p^-(x,t)p\,(t)\,R^-(t)e^{i\lambda(2\mu^-(t)-\mu^+(x))}dt+$$

$$i\int_x^\infty p^+(x,t)p\,(t)\,R^-(t)e^{i\lambda(2\mu^-(t)-\mu^-(x))}dt+$$

$$\int_x^\infty \Phi(x,t,\lambda)\left([q\,(t)+2\lambda p\,(t)]\right)z(t,\lambda)dt. \tag{12}$$

If $x<a$, then $p^-(x,t)=0$, $p^+(x,t)=\frac{1}{\alpha}$, when $t<a$ and $p^\pm(x,t)=\pm\alpha_\pm$,when $t>a$, where $\alpha_\pm=\frac{1}{2}(1\pm\frac{1}{\alpha})$. So in the case $x<a$ we have the following expressions for the known integrals in the right hand side of the last equation:

$$\int_x^\infty \Phi(x,t,\lambda)q\,(t)\left[R^+(t)e^{i\lambda\mu^+(t)}+R^-(t)e^{i\lambda\mu^-(t)}\right]dt=$$

$$\int_{\mu^+(x)}^{\mu^-(x)}\left[\frac{1}{2\alpha}\int_{a+\frac{t-\mu^-(x)}{2\alpha}}^{a} q(s)R^+(s)ds+\frac{1}{2\alpha}\int_{a+\frac{\mu^+(x)-t}{2\alpha}}^{a} q(s)R^-(s)ds+\right.$$

$$\left.\frac{\alpha_+}{2}\int_a^\infty q(s)R^+(s)ds-\frac{\alpha_-}{2}\int_a^{\frac{t+\mu^-(x)}{2}} q(s)R^+(s)ds\right]e^{i\lambda t}dt+$$

$$\int_{\mu^-(x)}^\infty\left[\frac{\alpha_+}{2}\int_{\frac{t+\mu^+(x)}{2}}^\infty q(s)R^+(s)ds+\frac{\alpha_-}{2}\int_{\frac{t+\mu^-(x)}{2}}^\infty q(s)R^+(s)ds\right]e^{i\lambda t}dt,$$

$$-i\int_x^\infty p^+(x,t)p\,(t)\,R^+(t)e^{i\lambda(2\mu^+(t)-\mu^+(x))}dt=$$

$$\frac{-i}{2\alpha^2}\int_{\mu^+(x)}^{\mu^-(x)} p(\frac{t-\mu^-\,(x)}{2\alpha}+a)R^+(a+\frac{t-\mu^-\,(x))}{2\alpha}e^{i\lambda t}dt-$$

$$\frac{i\alpha_+}{2}\int_{\mu^-(x)}^\infty p(\frac{t+\mu^+\,(x)}{2})R^+(\frac{t+\mu^+\,(x)}{2})e^{i\lambda t}dt,$$

$$i\int_x^\infty p^-(x,t)p\,(t)\,R^+(t)e^{i\lambda(2\mu^+(t)-\mu^-(x))}dt=\frac{-i\alpha_-}{2}\int_{\mu^+(x)}^\infty p(\frac{t+\mu^-\,(x)}{2})R^+(\frac{t+\mu^-\,(x)}{2})e^{i\lambda t}dt,$$

$$-i\int_x^\infty p^-(x,t)p\,(t)\,R^-(t)e^{i\lambda(2\mu^-(t)-\mu^+(x))}dt=0,$$

$$i\int_x^\infty p^+(x,t)p\,(t)\,R^-(t)e^{i\lambda(2\mu^-(t)-\mu^-(x))}dt=$$

$$\frac{i}{2\alpha^2}\int_{\mu^+(x)}^{\mu^-(x)} p(a-\frac{t-\mu^+(x)}{2\alpha})R^-(a-\frac{t-\mu^+(x))}{2\alpha}e^{i\lambda t}dt.$$

Substituting last equalities in Eq.(12) we have

$$z(x,\lambda)=\int_{\mu^+(x)}^\infty K_0(x,t)e^{i\lambda t}dt+\int_x^\infty \Phi(x,t,\lambda)\left([q\,(t)+2\lambda p\,(t)]\right)z(t,\lambda)dt, \tag{13}$$

where

$$K_0(x,t)=\frac{1}{2\alpha}\int_{a+\frac{t-\mu^-(x)}{2\alpha}}^{a} q(s)R^+(s)ds+\frac{1}{2\alpha}\int_{a+\frac{\mu^+(x)-t}{2\alpha}}^{a} q(s)R^-(s)ds+\frac{\alpha_+}{2}\int_a^\infty q(s)R^+(s)ds-$$

$$\frac{\alpha_-}{2}\int_a^{\frac{t+\mu^-(x)}{2}} q(s)R^+(s)ds-\frac{i}{2\alpha^2}p(a+\frac{t-\mu^-(x)}{2\alpha})R^+(a+\frac{t-\mu^-(x)}{2\alpha})+$$

$$\frac{i}{2\alpha^2}p(a-\frac{t-\mu^+(x)}{2\alpha})R^-(a-\frac{t-\mu^+(x)}{2\alpha})-$$

$$\frac{i\alpha_-}{2}p(\frac{t+\mu^-(x)}{2})R^+(\frac{t+\mu^-(x)}{2}),\ \ \mu^+(x)\le t<\mu^-(x)\,,\ \ 0\le x<a,$$

and

$$K_0(x,t)=\frac{\alpha_+}{2}\int_{\frac{t+\mu^+(x)}{2}}^\infty q(s)R^+(s)ds+\frac{\alpha_-}{2}\int_{\frac{t+\mu^-(x)}{2}}^\infty q(s)R^+(s)ds-$$

$$\frac{i\alpha_+}{2}p(\frac{t+\mu^+(x)}{2})R^+(\frac{t+\mu^+(x)}{2})-\frac{i\alpha_-}{2}p(\frac{t+\mu^-(x)}{2})R^+(\frac{t+\mu^-(x)}{2}),t>\mu^-(x)\,,\,0\le x<a.$$

Now we require that the integral equation (13) has a solution

$$z(x,\lambda)=\int_{\mu^+(x)}^\infty K(x,t)e^{i\lambda t}dt,$$

where $K(x,t)$ is an unknown function. Substituting this expression for $z(x,\lambda)$ in the equation (13) we have

$$\int_{\mu^+(x)}^\infty K(x,t)e^{i\lambda t}dt=\int_{\mu^+(x)}^\infty K_0(x,t)e^{i\lambda t}dt+$$

$$\int_x^\infty [q\,(t)+2\lambda p\,(t)]\,dt\int_{\mu^+(t)}^\infty K(t,s)\Phi(x,t;\lambda)e^{i\lambda s}ds. \tag{14}$$

Further using (7) and (8) again we can transform the second integral in the right hand -side of Eq. (14) to form of the integral standing in the left hand-side of Eq.(14) . Namely we have

$$\int_x^\infty q(t)\,dt\int_{\mu^+(t)}^\infty K(t,s)\Phi(x,t;\lambda)e^{i\lambda s}ds=\int_{\mu^+(x)}^{\mu^-(x)}e^{i\lambda t}dt\left[\frac{1}{2\alpha}\int_{a+\frac{t-\mu^-(x)}{2\alpha}}^{a}q(s)ds\int_{\mu^+(s)}^{t+\alpha(s-x)}K(s,\xi)d\xi+\right.$$

$$\frac{1}{2\alpha}\int_x^{a+\frac{t-\mu^-(x)}{2\alpha}}q(s)ds\int_{t-\alpha(s-x)}^{t+\alpha(s-x)}K(s,\xi)d\xi+\frac{\alpha_+}{2}\int_a^\infty q(s)\,ds\int_s^{t+s-\mu^+(x)}K(s,\xi)d\xi-$$

$$\left.\frac{\alpha_-}{2}\int_a^{\frac{t+\mu^-(x)}{2}}q(s)\,ds\int_s^{t-s+\mu^-(x)}K(s,\xi)d\xi\right]+$$

$$\int_{\mu^-(x)}^\infty e^{i\lambda t}dt\left[\frac{1}{2\alpha}\int_x^a q(s)ds\int_{t-\alpha(s-x)}^{t+\alpha(s-x)}K(s,\xi)d\xi+\frac{\alpha_+}{2}\int_a^{\frac{t+\mu^+(x)}{2}}q(s)\,ds\int_{t-s+\mu^+(x)}^{t+s-\mu^+(x)}K(s,\xi)d\xi+\right.$$

$$\frac{\alpha_+}{2}\int_{\frac{t+\mu^+(x)}{2}}^\infty q(s)\,ds\int_s^{t+s-\mu^+(x)}K(s,\xi)d\xi-\frac{\alpha_-}{2}\int_a^{\mu^-(x)}q(s)\,ds\int_{t+s-\mu^-(x)}^{t-s+\mu^-(x)}K(s,\xi)d\xi+$$

$$\left.\frac{\alpha_-}{2}\int_{\frac{t+\mu^-(x)}{2}}^\infty q(s)\,ds\int_s^{t+s-\mu^-(x)}K(s,\xi)d\xi+\frac{\alpha_-}{2}\int_{\mu^-(x)}^{\frac{t+\mu^-(x)}{2}}q(s)\,ds\int_{t-s+\mu^-(x)}^{t+s-\mu^-(x)}K(s,\xi)d\xi\right]$$

and

$$2\lambda\int_x^\infty p(t)\,dt\int_{\mu^+(t)}^\infty K(t,s)\Phi(x,t;\lambda)e^{i\lambda s}ds=\int_{\mu^+(x)}^{\mu^-(x)}e^{i\lambda t}dt\left[\frac{1}{i\alpha}\int_x^{a+\frac{t-\mu^-(x)}{2\alpha}}p(s)K(s,t-\alpha(s-x))ds-\right.$$

$$\frac{1}{i\alpha}\int_x^a p(s)K(s,t+\alpha(s-x))ds+i\alpha_+\int_a^\infty p(s)\,K(s,t+s-\mu^+(x))ds-$$

$$\left.i\alpha_-\int_a^{\frac{t+\mu^-(x)}{2}}p(s)\,K(s,t-s+\mu^-(x))ds+\right.$$

$$\int_{\mu^-(x)}^\infty e^{i\lambda t}dt\left[\frac{1}{i\alpha}\int_x^a p(s)K(s,t-\alpha(s-x))ds-\frac{1}{i\alpha}\int_x^a p(s)K(s,t+\alpha(s-x))ds-\right.$$

$$i\alpha_+\int_a^{\frac{t+\mu^+(x)}{2}}p(s)\,K(s,t-s+\mu^+(x))ds+i\alpha_+\int_a^\infty p(s)\,K(s,t+s-\mu^+(x))ds-$$

$$\left. i\alpha_- \int\limits_a^{\frac{t+\mu^-(x)}{2}} p(s)\,K(s,t-s+\mu^-(x))ds + i\alpha_- \int\limits_a^{\infty} p(s)\,K(s,t+s-\mu^-(x))ds \right] .$$

Taking into out account the last two equalities in the right hand-side of Eq.(14) we obtain the following integral equation for the determination of the unknown kernel function $K(x,t)(0 \leq x < a)$:

$$K(x,t) = K_0(x,t) + \frac{1}{2\alpha} \int\limits_{a+\frac{t-\mu^-(x)}{2\alpha}}^{a} q(s)ds \int\limits_{\mu^+(s)}^{t+\alpha(s-x)} K(s,\xi)d\xi +$$

$$\frac{1}{2\alpha} \int\limits_{x}^{a+\frac{t-\mu^-(x)}{2\alpha}} q(s)ds \int\limits_{t-\alpha(s-x}^{t+\alpha(s-x)} K(s,\xi)d\xi + \frac{\alpha_+}{2} \int\limits_a^{\infty} q(s)\,ds \int\limits_{s}^{t+s-\mu^+(x)} K(s,\xi)d\xi -$$

$$\frac{\alpha_-}{2} \int\limits_a^{\frac{t+\mu^-(x)}{2}} q(s)\,ds \int\limits_{s}^{t-s+\mu^-(x)} K(s,\xi)d\xi + \frac{1}{i\alpha} \int\limits_{x}^{a+\frac{t-\mu^-(x)}{2\alpha}} p(s)K(s,t-\alpha(s-x))ds -$$

$$\frac{1}{i\alpha} \int\limits_x^a p(s)K(s,t+\alpha(s-x))ds + i\alpha_+ \int\limits_a^{\infty} p(s)\,K(s,t+s-\mu^+(x))ds -$$

$$i\alpha_- \int\limits_a^{\frac{t+\mu^-(x)}{2}} p(s)\,K(s,t-s+\mu^-(x))ds,\ \ \mu^+(x) \leq t < \mu^-(x)\,, \tag{15}$$

$$K(x,t) = K_0(x,t) + \frac{1}{2\alpha} \int\limits_x^a q(s)ds \int\limits_{t-\alpha(s-x)}^{t+\alpha(s-x)} K(s,\xi)d\xi + \frac{\alpha_+}{2} \int\limits_a^{\frac{t+\mu^+(x)}{2}} q(s)\,ds \int\limits_{t-s+\mu^+(x)}^{t+s-\mu^+(x)} K(s,\xi)d\xi +$$

$$\frac{\alpha_+}{2} \int\limits_{\frac{t+\mu^+(x)}{2}}^{\infty} q(s)\,ds \int\limits_{s}^{t+s-\mu^+(x)} K(s,\xi)d\xi - \frac{\alpha_-}{2} \int\limits_a^{\mu^-(x)} q(s)\,ds \int\limits_{t+s-\mu^-(x)}^{t-s+\mu^-(x)} K(s,\xi)d\xi +$$

$$\frac{\alpha_-}{2} \int\limits_{\frac{t+\mu^-(x)}{2}}^{\infty} q(s)\,ds \int\limits_{s}^{t+s-\mu^-(x)} K(s,\xi)d\xi + \frac{\alpha_-}{2} \int\limits_{\mu^-(x)}^{\frac{t+\mu^-(x)}{2}} q(s)\,ds \int\limits_{t-s+\mu^-(x)}^{t+s-\mu^-(x)} K(s,\xi)d\xi +$$

$$\frac{1}{i\alpha} \int\limits_x^a p(s)K(s,t-\alpha(s-x))ds - \frac{1}{i\alpha} \int\limits_x^a p(s)K(s,t+\alpha(s-x))ds -$$

$$i\alpha_+ \int\limits_a^{\frac{t+\mu^+(x)}{2}} p(s)\,K(s,t-s+\mu^+(x))ds + i\alpha_+ \int\limits_a^{\infty} p(s)\,K(s,t+s-\mu^+(x))ds -$$

$$i\alpha_- \int\limits_a^{\frac{t+\mu^-(x)}{2}} p(s)\,K(s,t-s+\mu^-(x))ds + i\alpha_- \int\limits_a^{\infty} p(s)\,K(s,t+s-\mu^-(x))ds, t > \mu^-(x)\,. \tag{16}$$

Analogously, if $x > a$ then Eq. (13) has taken the form

$$z(x,\lambda) = \int_x^\infty K_0(x,t)e^{i\lambda t}dt + \int_x^\infty \Phi(x,t,\lambda)\left([q(t) + 2\lambda p(t)]\right) z(t,\lambda)dt, \tag{17}$$

where

$$K_0(x,t) = \frac{1}{2}\int_{\frac{x+t}{2}}^{\infty} q(s)R^+(s)ds - \frac{i}{2}p(\frac{x+t}{2})R^+(\frac{x+t}{2}),\ x > a.$$

Seeking the solution of the form

$$z(x,\lambda) = \int_x^\infty K(x,t)e^{i\lambda t}dt$$

for the integral equation (17) we have

$$K(x,t) = K_0(x,t) + \frac{1}{2}\int_x^\infty q(s)ds \int_{t-s+x}^{t+s-x} K(s,\xi)d\xi -$$

$$i\int_x^{\frac{x+t}{2}} p(s)\, K(s,t-s+x)ds + i\int_x^\infty p(s)\, K(s,t+s-x)ds, x > a. \tag{18}$$

Now by the method of successive approximation it is easy to prove the following theorem.

Theorem 1. *If the conditions* (2) *are satisfied then the equation* (1) *for all* λ *from the closed upper halfplane* $Im\lambda \geq 0$ *has the Jost solution* $f(x,\lambda)$ *satisfying the condition* (3) *at infinity can be represented in the form*

$$f(x,\lambda) = R^+(x)e^{i\lambda\mu^+(x)} + R^-(x)e^{i\lambda\mu^-(x)} + \int_{-\mu^+(x)}^{+\infty} K(x,t)e^{i\lambda t}dt, \tag{19}$$

where

$$\mu^\pm(x) = \pm x\sqrt{\rho(x)} + a\left(1 \mp \sqrt{\rho(x)}\right),$$

$$R^\pm(x) = r^\pm(x)e^{i\int_x^\infty sgn(t\pm a)\frac{p(t)}{\sqrt{\rho(t)}}dt},\ r^\pm(x) = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{\rho(x)}}\right)$$

and the kernel $K(x,t)$ *satisfies*

$$\int_{-\mu^+(x)}^{+\infty} |K(x,t)|\, dt \leq C_0\left[e^{\int_x^\infty [(\mu(t)-\mu(x))|q(t)|+|p(t)|]dt} - 1\right],$$

with $C_0 > 0$.

From the integral equations (15),(16) and (18) we immediately get the following relationships between the kernel function and the coefficients of the equation (1).

Theorem 2. *The function $K(x,t)$, where $x > 0$ and $t \geq \mu^+(x)$, is continuous at $t \neq \mu^-(x)$, $x \neq a$ and the following relations are satisfied:*

$$K\left(x,\mu^+(x)\right) =$$

$$R_+(x)\left\{\frac{1}{2}\int\limits_x^{+\infty}\left(\frac{q(s)}{\sqrt{\rho(s)}}+\frac{p^2(s)}{\rho(s)\sqrt{\rho(s)}}\right)ds-\frac{i}{2}\frac{p(x)}{\rho(x)}-\frac{i}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)^2 p(a)\right\}, \tag{20}$$

$$K\left(x,\mu^-(x)+0\right)-K\left(x,\mu^-(x)-0\right)R^-(x)\left\{\frac{1}{2}\int\limits_x^{+\infty}\left(\frac{q(s)}{\sqrt{\rho(s)}}+\frac{p^2(s)}{\rho(s)\sqrt{\rho(s)}}\right)sgn(s-a)ds-\right.$$

$$\left.\frac{i}{2}\frac{p(x)}{\rho(x)}+\frac{i}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)^2 p(a)\right\}. \tag{21}$$

Note that from the integral equations (15),(16) and (18) we obtain that if $p(x)$ is a differentiable function then the function $K(x,t)$ has its first order partial derivatives with respect to each argument when $t \neq \mu^-(x)$ and $x \neq a$. We also have provided that if $q(x)$ is differentiable and $p(x)$ is twice differentiable, then the kernel function $K(x,t)$ satisfy the partial differential equation

$$D_{xx}K(x,t)-\rho(x)D_{tt}K(x,t)=q(x)K(x,t)+2ip(x)D_tK(x,t) \tag{22}$$

for all $t \neq \mu^-(x)$ and $x \neq a$. In this case from the relations (19), (20) and (21) we have

$$\frac{d}{dx}\left(K(x,\mu(x))e^{-i\omega(x)}\right)=-\frac{1}{4}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)\left(\frac{q(x)}{\sqrt{\rho(x)}}+\frac{p^2(x)}{\rho(x)\sqrt{\rho(x)}}+i\frac{p'(x)}{\rho(x)}\right), \tag{23}$$

$$\frac{d}{dx}\left(\left(K^+(x,-\mu(x)+0)-K^+(x,-\mu(x)-0)\right)e^{-i(2\omega(a)-\omega(x))}\right)=$$

$$\frac{1}{4}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)\left(\frac{q(x)}{\sqrt{\rho(x)}}+\frac{p^2(x)}{\rho(x)\sqrt{\rho(x)}}-i\frac{p'(x)}{\rho(x)}\right), \tag{24}$$

where

$$\omega(x)=\int\limits_x^{\infty}\frac{p(t)}{\sqrt{\rho(t)}}dt. \tag{25}$$

By setting $Re\,K(x,t)=A(x,t)$ and $ImK(x,t)=B(x,t)$ from (23) and (24)we obtain

$$\frac{1}{2\sqrt{\rho(x)}}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)\left(q(x)+\frac{p^2(x)}{\rho(x)}\right)=$$

$$2\frac{d}{dx}\left[A(x,\mu^+(x))\cos\omega(x)+B(x,\mu^+(x))\sin\omega(x)\right], \tag{26}$$

$$\omega^+(x)=\int\limits_x^{+\infty}\left(1+\sqrt{\rho(t)}\right)\left[A(t,\mu^+(t))\sin\omega(t)-B(t,\mu^+(t))\cos\omega(t)\right]dt-$$

$$\int\limits_x^{+\infty}\left(1-\sqrt{\rho(t)}\right)\left[PA(t,\mu^-(t))\sin\widetilde{\omega}(t)-PB(t,\mu^-(t))\cos\widetilde{\omega}(t)\right]dt, \tag{27}$$

where $Pg(t,z)=g(t,z+0)-g(t,z-0)$ denotes the jumping of the function $g(t,z)$ with respect to z and $\widetilde{\omega}(x)=2\omega(a)-\omega(x)$.

Theorem 3. *If $q(x)$ is differentiable and $p(x)$ is twice differentiable, then the kernel function $K(x,t)$ satisfies the partial differential equation* (22) *with conditions* (23) *and* (24) *where* $\omega(x)$ *is defined as* (25) *.Moreover, equations* (23) *and* (24) *imply the relations* (26) *and* (27) *respectively.*

The results of this theorem we will use in another paper.

3 Solution of the initial value problem

Now we consider the solution $\varphi(x,\lambda)$ of the equation (1) with initial conditions

$$\varphi(0,\lambda)=0, \varphi'(0,\lambda)=1.$$

Then

$$\varphi(x,\lambda)=\varphi_0(x,\lambda)+\int_0^x S(x,t,\lambda)(q(t)+2\lambda p(t))\varphi(t,\lambda)dt, \tag{28}$$

where

$$\varphi_0(x,\lambda)=\begin{cases} \frac{\sin\alpha\lambda x}{\alpha\lambda},\ 0\le x\le a,\\ \alpha_+\frac{\sin\lambda(x+\alpha a-a)}{\lambda}+\alpha_-\frac{\sin\lambda(x-\alpha a-a)}{\lambda},\ x\ge a,\end{cases}$$

with

$$\alpha_\pm=\frac{1}{2}(1\pm\frac{1}{\alpha})$$

and

$$S(x,t,\lambda)=\begin{cases} \frac{\sin\alpha\lambda(x-t)}{\alpha\lambda},\ 0<t\le x<a,\\ \alpha_+\frac{\sin\lambda(x-\mu^+(t))}{\lambda}+\alpha_-\frac{\sin\lambda(x-\mu^-(t))}{\lambda},\ 0<t<a<x,\\ \frac{\sin\lambda(x-t)}{\lambda}, a<t\le x.\end{cases}$$

Clearly

$$\varphi_0(x,\lambda)=\frac{1}{2}\left(\frac{1}{\sqrt{\rho(x)}}+\frac{1}{\alpha}\right)\frac{\sin\lambda(\mu^+(x)-\mu^+(0))}{\lambda}-$$

$$\frac{1}{2}\left(\frac{1}{\sqrt{\rho(x)}}-\frac{1}{\alpha}\right)\frac{\sin\lambda(\mu^-(x)-\mu^+(0))}{\lambda}$$

and

$$S(x,t,\lambda)=p^+(x,t)\frac{\sin\lambda\sigma_1^+(x,t)}{\lambda}+p^-(x,t)\frac{\sin\lambda\sigma_1^-(x,t)}{\lambda},\ 0\le t\le x,$$

where

$$p^\pm(x,t)=\frac{1}{2}\left(\frac{1}{\sqrt{\rho(x)}}\pm\frac{1}{\sqrt{\rho(t)}}\right) \text{ and } \sigma_1^\pm(x,t)=\mu^+(x)-\mu^\pm(t).$$

We seek the solution of equation (28) in the form of

$$\varphi(x,\lambda)=\varphi_0(x,\lambda)\psi_0(x)+\frac{1}{2\sqrt{\rho(x)}}\int_{-\nu(x)}^{\nu(x)}\Phi(x,t)e^{i\lambda t}dt, \tag{29}$$

where $\nu(x)=x\sqrt{\rho(x)}+a(\alpha-\sqrt{\rho(x)})$, $\psi_0(x)$ is a solution of $\psi_0''=q(x)\psi_0$ satisfying initial conditions

$$\psi_0(0)=1, \psi_0'(0)=0$$

and jump conditions at $x=a$

$$\psi_0(a^+)=2-\psi_0(a^-)+a\psi'(a^-),\ \psi'(a^+)=0. \tag{30}$$

If $0 \leq x \leq a$, then we have

$$\varphi(x,\lambda) = \frac{\sin \alpha\lambda x}{\alpha\lambda}\psi_0(x) + \frac{1}{2\alpha}\int\limits_{-\alpha x}^{\alpha x} \Phi(x,t)e^{i\lambda t}dt$$

(see Kamimura (2008)), where

$$\Phi(x,t) = -\int\limits_{-0}^{\frac{\alpha x - t}{2\alpha}} \left(\frac{\alpha x - t}{2\alpha} - s\right) q(s)\psi_0(s)ds - \int\limits_{-0}^{\frac{\alpha x + t}{2\alpha}} \left(\frac{\alpha x + t}{2\alpha} - s\right) q(s)\psi_0(s)ds -$$

$$\frac{i}{\alpha}\int\limits_{\frac{\alpha x - t}{2}}^{\frac{\alpha x + t}{2\alpha}} p(s)\psi_0(s)ds + \frac{1}{\alpha^2}\int\limits_{-0}^{\frac{\alpha x + t}{2\alpha}} du \int\limits_{-0}^{\frac{\alpha x - t}{2\alpha}} q(u+v)\Phi(u+v, \alpha\,(u-v))dv -$$

$$\frac{i}{\alpha}\int\limits_{0}^{\frac{\alpha x + t}{2\alpha}} p(u + \frac{\alpha x - t}{2\alpha})\Phi(u + \frac{\alpha x - t}{2\alpha}, \alpha\left(u - \frac{\alpha x - t}{2\alpha}\right))du +$$

$$\frac{i}{\alpha}\int\limits_{0}^{\frac{\alpha x - t}{2\alpha}} p(\frac{\alpha x + t}{2\alpha} + v)\Phi(\frac{\alpha x + t}{2\alpha} + v, \alpha\left(\frac{\alpha x + t}{2\alpha} - v\right))dv$$

is a continuous function with boundary values

$$\begin{aligned} \Phi(x,\alpha x) &= -\psi_0(x) + \exp\left(\frac{-i}{\alpha}\int\limits_{-0}^{x} p(s)ds\right), \\ \Phi(x,-\alpha x) &= -\psi_0(x) + \exp\left(\frac{i}{\alpha}\int\limits_{-0}^{x} p(s)ds\right). \end{aligned}$$

Consider more complicated case $x \geq a$. Writing the formula (29) in the integral equation (28) we have

$$\varphi_0(x,\lambda)\psi_0(x) + \frac{1}{2}\int\limits_{-x-\alpha a+a}^{x+\alpha a-a} \Phi(x,t)e^{i\lambda t}dt = \varphi_0(x,\lambda) +$$

$$\int\limits_{0}^{a} \left(\alpha_+\frac{\sin\lambda(x-\mu^+(t))}{\lambda} + \alpha_-\frac{\sin\lambda(x-\mu^-(t))}{\lambda}\right)[q(t) +$$

$$2\lambda p(t)]\left[\frac{\sin\alpha\lambda t}{\alpha\lambda}\psi_0(t) + \frac{1}{2\alpha}\int\limits_{-\alpha t}^{\alpha t}\Phi(t,s)e^{i\lambda s}ds\right] +$$

$$\int\limits_{a}^{x}\frac{\sin\lambda(x-t)}{\lambda}\left[q(t)+2\lambda p(t)\right]\left[\varphi_0(t,\lambda)\psi_0(t) + \frac{1}{2}\int\limits_{-t-\alpha a+a}^{t+\alpha a-a}\Phi(t,s)e^{i\lambda s}ds\right]dt. \qquad (31)$$

Note that

$$\psi_0(x) = 1 + \int\limits_{0}^{x}(x-s)q(s)\psi_0(s)ds, 0 \leq x \leq a. \qquad (32)$$

We transform right hand-sided integrals in the equation (31) by the following manner:

$$J_1=\int_0^a\left(\alpha_+\frac{\sin\lambda(x-\mu^+(t))}{\lambda}+\alpha_-\frac{\sin\lambda(x-\mu^-(t))}{\lambda}\right)q(t)\frac{\sin\alpha\lambda t}{\alpha\lambda}\psi_0(t)dt=$$

$$\frac{i\alpha_+}{4\alpha\lambda}\int_0^a q(t)\psi_0(t)\left[\int_{x-\mu^+(t)+\alpha t}^{x-\mu^+(t)-\alpha t}e^{i\lambda s}ds-\int_{\mu^+(t)+\alpha t-x}^{\mu^+(t)-\alpha t-x}e^{i\lambda s}ds\right]+$$

$$\frac{i\alpha_-}{4\alpha\lambda}\int_0^a q(t)\psi_0(t)\left[\int_{x-\mu^-(t)+\alpha t}^{x-\mu^-(t)-\alpha t}e^{i\lambda s}ds-\int_{\mu^-(t)+\alpha t-x}^{\mu^-(t)-\alpha t-x}e^{i\lambda s}ds\right]=$$

$$\frac{i\alpha_+}{4\alpha\lambda}\int_{x_2}^{x_1}e^{i\lambda t}dt\int_{\frac{x_1-t}{2\alpha}}^{a}q(s)\psi_0(s)ds+\frac{i\alpha_+}{4\alpha\lambda}\int_{-x_1}^{-x_2}e^{i\lambda t}dt\int_{\frac{x_1+t}{2\alpha}}^{a}q(s)\psi_0(s)ds-$$

$$\frac{i\alpha_-}{4\alpha\lambda}\int_{x_2}^{x_1}e^{i\lambda t}dt\int_{\frac{t-x_2}{2\alpha}}^{a}q(s)\psi_0(s)ds+\frac{i\alpha_-}{4\alpha\lambda}\int_{-x_1}^{-x_2}e^{i\lambda t}dt\int_{-\frac{x_2+t}{2\alpha}}^{a}q(s)\psi_0(s)ds,$$

where $x_1=x+\alpha a-a$ and $x_2=x-\alpha a-a$. Further,

$$J_2=2\int_0^a\left(\alpha_+\sin\lambda(x-\mu^+(t))+\alpha_-\sin\lambda(x-\mu^-(t))\right)p(t)\frac{\sin\alpha\lambda t}{\alpha\lambda}\psi_0(t)dt=$$

$$\frac{i\alpha_+}{2\alpha}\int_{x_2}^{x_1}e^{i\lambda t}dt\int_{\frac{x_1-t}{2\alpha}}^{a}p(s)\psi_0(s)ds+\frac{i\alpha_+}{2\alpha}\int_{-x_1}^{-x_2}e^{i\lambda t}dt\int_{\frac{x_1+t}{2\alpha}}^{a}p(s)\psi_0(s)ds-$$

$$\frac{i\alpha_-}{2\alpha}\int_{x_2}^{x_1}e^{i\lambda t}dt\int_{\frac{t-x_2}{2\alpha}}^{a}p(s)\psi_0(s)ds+\frac{i\alpha_-}{2\alpha}\int_{-x_1}^{-x_2}e^{i\lambda t}dt\int_{-\frac{x_2+t}{2\alpha}}^{a}p(s)\psi_0(s)ds,$$

$$J_3=\int_0^a\left(\alpha_+\frac{\sin\lambda(x-\mu^+(t))}{\lambda}+\alpha_-\frac{\sin\lambda(x-\mu^-(t))}{\lambda}\right)q(t)\frac{1}{2\alpha}\int_{-\alpha t}^{\alpha t}\Phi(t,s)e^{i\lambda s}ds=$$

$$\frac{\alpha_+}{4\alpha}\int_0^a q(t)dt\int_{-\alpha t}^{\alpha t}\Phi(t,s)\frac{e^{i\lambda(x-\mu^+(t)+s)}-e^{i\lambda(-x+\mu^+(t)+s)}}{i\lambda}ds+$$

$$\frac{\alpha_-}{4\alpha}\int_0^a q(t)dt\int_{-\alpha t}^{\alpha t}\Phi(t,s)\frac{e^{i\lambda(x-\mu^-(t)+s)}-e^{i\lambda(-x+\mu^-(t)+s)}}{i\lambda}ds=$$

$$\frac{\alpha_+}{4\alpha}\int_0^a q(t)dt\int_{-\alpha t}^{\alpha t}\Phi(t,s)ds\int_{-x+\mu^+(t)+s}^{x-\mu^+(t)+s}e^{i\lambda\xi}d\xi+\frac{\alpha_-}{4\alpha}\int_0^a q(t)dt\int_{-\alpha t}^{\alpha t}\Phi(t,s)ds\int_{-x+\mu^-(t)+s}^{x-\mu^-(t)+s}e^{i\lambda\xi}d\xi=$$

$$\frac{\alpha_+}{4\alpha}\int_{-x_1}^{x_1}e^{i\lambda t}dt\int_0^a q(s)ds\int_{-x+\mu^+(s)+t}^{x-\mu^+(s)+t}\Phi(s,\xi)dsd\xi+$$

$$\frac{\alpha_-}{4\alpha}\int\limits_{-x_1}^{x_1} e^{i\lambda t}dt\int\limits_0^a q(s)ds\int\limits_{-x+\mu^-(s)+t}^{x-\mu^-(s)+t}\Phi(s,\xi)dsd\xi,$$

where we suppose $\Phi(x,t)=0$ for $|x|>\alpha\,|t|$. Note that if $x_2=x-\alpha a-a<0$, then the last integral is changed as

$$\frac{\alpha_-}{4\alpha}\int\limits_{-x_1}^{x_1} e^{i\lambda t}dt\int\limits_0^{-\frac{x_2}{\alpha}} q(s)ds\int\limits_{x-\mu^-(s)+t}^{-x+\mu^-(s)+t}\Phi(s,\xi)dsd\xi+\frac{\alpha_-}{4\alpha}\int\limits_{-x_1}^{x_1} e^{i\lambda t}dt\int\limits_{-\frac{x_2}{\alpha}}^{a} q(s)ds\int\limits_{-x+\mu^-(s)+t}^{x-\mu^-(s)+t}\Phi(s,\xi)dsd\xi.$$

The next integral is

$$J_4=\frac{1}{\alpha}\int\limits_0^a\left(\alpha_+\sin\lambda(x-\mu^+(t))+\alpha_-\sin\lambda(x-\mu^-(t))\right)p(t)dt\int\limits_{-\alpha t}^{\alpha t}\Phi(t,s)e^{i\lambda s}ds=$$

$$-\frac{i\alpha_+}{2\alpha}\int\limits_{x_2}^{x_1} e^{i\lambda t}dt\int\limits_{\frac{x_1-t}{2\alpha}}^{a} p(s)\Phi(s,t-x+\mu^+(s))ds+\frac{i\alpha_+}{2\alpha}\int\limits_{-x_1}^{-x_2} e^{i\lambda t}dt\int\limits_{\frac{x_1+t}{2\alpha}}^{a} p(s)\Phi(s,t+x-\mu^+(s))ds-$$

$$\frac{i\alpha_-}{2\alpha}\int\limits_{x_2}^{x_1} e^{i\lambda t}dt\int\limits_{\frac{t-x_2}{2\alpha}}^{a} p(s)\Phi(s,t-x+\mu^-(s))ds+\frac{i\alpha_-}{2\alpha}\int\limits_{-x_1}^{-x_2} e^{i\lambda t}dt\int\limits_{-\frac{x_2+t}{2\alpha}}^{a} q(s)\Phi(s,t+x-\mu^-(s))ds.$$

Further, since

$$\frac{\sin\lambda u\sin\lambda v}{\lambda}=\frac{\cos\lambda\,(u-v)-\cos\lambda(u+v)}{2\lambda}=\frac{i}{4}\int\limits_{-u-v}^{u-v} e^{i\lambda s}ds-\frac{i}{4}\int\limits_{-u+v}^{u+v} e^{i\lambda s}ds$$

we have

$$J_5=\int\limits_a^x\frac{\sin\lambda(x-t)}{\lambda}q(t)\varphi_0(t,\lambda)\psi_0(t)dt=\int\limits_a^x\frac{\sin\lambda(x-t)}{\lambda}q(t)\psi_0(t)\left[\alpha_+\frac{\sin\lambda(t+\alpha a-a)}{\lambda}+\right.$$

$$\left.\alpha_-\frac{\sin\lambda(t-\alpha a-a)}{\lambda}\right]dt=\frac{i\alpha_+}{4\lambda}\int\limits_a^x q(t)\psi_0(t)dt\int\limits_{-x-\alpha a+a}^{x-\alpha a+a-2t} e^{i\lambda s}ds-$$

$$\frac{i\alpha_+}{4\lambda}\int\limits_a^x q(t)\psi_0(t)dt\int\limits_{-x+\alpha a-a+2t}^{x+\alpha a-a} e^{i\lambda s}ds+\frac{i\alpha_-}{4\lambda}\int\limits_a^x q(t)\psi_0(t)dt\int\limits_{-x+\alpha a+a}^{x+\alpha a+a-2t} e^{i\lambda s}ds-$$

$$\frac{i\alpha_-}{4\lambda}\int\limits_a^x q(t)\psi_0(t)dt\int\limits_{-x-\alpha a-a+2t}^{x-\alpha a-a} e^{i\lambda s}ds=$$

$$\frac{i\alpha_+}{4\lambda}\int\limits_{-x_1}^{x_2} e^{i\lambda t}dt\int\limits_a^{\frac{x_2-t}{2}+a} q(s)\psi_0(s)ds-\frac{i\alpha_+}{4\lambda}\int\limits_{-x_2}^{x_1} e^{i\lambda t}dt\int\limits_a^{\frac{x_2+t}{2}+a} q(s)\psi_0(s)ds+$$

$$\frac{i\alpha_-}{4\lambda}\int\limits_{-x_2}^{x_1} e^{i\lambda t}dt\int\limits_a^{\frac{x_1-t}{2}+a} q(s)\psi_0(s)ds-\frac{i\alpha_-}{4\lambda}\int\limits_{-x_1}^{x_2} e^{i\lambda t}dt\int\limits_a^{\frac{x_1+t}{2}+a} q(s)\psi_0(s)ds,$$

$$J_6 = 2\int_a^x \sin\lambda(x-t)p(t)\varphi_0(t,\lambda)\psi_0(t)dt = 2\int_a^x \sin\lambda(x-t)p(t)\psi_0(t)\left[\alpha_+\frac{\sin\lambda(t+\alpha a-a)}{\lambda}+\right.$$

$$\left.\alpha_-\frac{\sin\lambda(t-\alpha a-a)}{\lambda}\right]dt = \frac{i\alpha_+}{2}\int_{-x_1}^{x_2} e^{i\lambda t}dt \int_a^{\frac{x_2-t}{2}+a} p(s)\psi_0(s)ds-$$

$$\frac{i\alpha_+}{2}\int_{-x_2}^{x_1} e^{i\lambda t}dt \int_a^{\frac{x_2+t}{2}+a} p(s)\psi_0(s)ds + \frac{i\alpha_-}{2}\int_{-x_2}^{x_1} e^{i\lambda t}dt \int_a^{\frac{x_1-t}{2}+a} p(s)\psi_0(s)ds-$$

$$\frac{i\alpha_-}{2}\int_{-x_1}^{x_2} e^{i\lambda t}dt \int_a^{\frac{x_1+t}{2}+a} p(s)\psi_0(s)ds,$$

$$J_7 = \frac{1}{2}\int_a^x \frac{\sin\lambda(x-t)}{\lambda}q(t)dt \int_{-t-\alpha a+a}^{t+\alpha a-a} \Phi(t,s)e^{i\lambda s}ds =$$

$$\frac{1}{4}\int_a^x q(t)dt \int_{-t-\alpha a+a}^{t+\alpha a-a} \Phi(t,s)ds \int_{s-x+t}^{s+x-t} e^{i\lambda\xi}d\xi = \frac{1}{4}\int_{-x_1}^{x_1} e^{i\lambda t}dt \int_a^x q(s)ds \int_{s-x+t}^{s+x-t} \Phi(s,\xi)d\xi,$$

where it is supposed $\Phi(x,t)=0$ for $|t|>x+\alpha a-a=x_1$. Finally we have

$$J_8 = \int_a^x \sin\lambda(x-t)p(t)dt \int_{-t-\alpha a+a}^{t+\alpha a-a} \Phi(t,s)e^{i\lambda s}ds =$$

$$\frac{1}{2i}\int_a^x p(t)dt \int_{x_2-2t+2a}^{x_1} \Phi(t,s-x+t)e^{i\lambda s}ds - \frac{1}{2i}\int_a^x p(t)dt \int_{-x_1}^{2t-2a-x_2} \Phi(t,s+x-t)e^{i\lambda s}ds =$$

$$\frac{1}{2i}\int_{-x_1}^{x_1} e^{i\lambda t}dt \left(\int_{\max(a,\frac{x_2-t}{2}+a)}^{x} p(s)\Phi(s,t-x+s)ds - \int_{\max(a,\frac{x_2+t}{2}+a)}^{x} p(s)\Phi(s,t+x-s)ds \right).$$

Now using integration by part we find

$$J_1 = -\frac{i\alpha_+}{2\lambda}e^{i\lambda x_1}\int_0^a sq(s)\psi_0(s)ds - \frac{\alpha_+}{2}\int_{x_2}^{x_1} e^{i\lambda t}dt \int_{\frac{x_1-t}{2\alpha}}^{a} \left(s-\frac{x_1-t}{2\alpha}\right)q(s)\psi_0(s)ds+$$

$$\frac{i\alpha_+}{2\lambda}e^{-i\lambda x_1}\int_0^a sq(s)\psi_0(s)ds - \frac{\alpha_+}{2}\int_{-x_1}^{-x_2} e^{i\lambda t}dt \int_{\frac{x_1+t}{2\alpha}}^{a} \left(s-\frac{x_1+t}{2\alpha}\right)q(s)\psi_0(s)ds-$$

$$\frac{i\alpha_-}{2\lambda}e^{i\lambda x_2}\int_0^a sq(s)\psi_0(s)ds + \frac{\alpha_-}{2}\int_{x_2}^{x_1} e^{i\lambda t}dt \int_{\frac{t-x_2}{2\alpha}}^{a} \left(s-\frac{t-x_2}{2\alpha}\right)q(s)\psi_0(s)ds+$$

$$\frac{i\alpha_-}{2\lambda}e^{-i\lambda x_2}\int_0^a sq(s)\psi_0(s)ds + \frac{\alpha_-}{2}\int_{-x_1}^{-x_2} e^{i\lambda t}dt \int_{-\frac{x_2+t}{2\alpha}}^{a} \left(s+\frac{x_2+t}{2\alpha}\right)q(s)\psi_0(s)dsdt,$$

therefore

$$J_1 = \varphi_0(x,\lambda)\int\limits_0^a sq(s)\psi_0(s)ds - \frac{\alpha_+}{2}\int\limits_{x_2}^{x_1} e^{i\lambda t}dt \int\limits_{\frac{x_1-t}{2\alpha}}^{a}\left(s-\frac{x_1-t}{2\alpha}\right)q(s)\psi_0(s)ds-$$

$$\frac{\alpha_+}{2}\int\limits_{-x_1}^{-x_2} e^{i\lambda t}dt \int\limits_{\frac{x_1+t}{2\alpha}}^{a}\left(s-\frac{x_1+t}{2\alpha}\right)q(s)\psi_0(s)ds+$$

$$\frac{\alpha_-}{2}\int\limits_{x_2}^{x_1} e^{i\lambda t}dt \int\limits_{\frac{t-x_2}{2\alpha}}^{a}\left(s-\frac{t-x_2}{2\alpha}\right)q(s)\psi_0(s)ds+$$

$$\frac{\alpha_-}{2}\int\limits_{-x_1}^{-x_2} e^{i\lambda t}dt \int\limits_{-\frac{x_2+t}{2\alpha}}^{a}\left(s+\frac{x_2+t}{2\alpha}\right)q(s)\psi_0(s)ds.$$

Similarly,

$$J_5 = \varphi_0(x,\lambda)\int\limits_a^x (x-s)\,q(s)\psi_0(s)ds - \frac{\alpha_+}{2}\int\limits_{-x_1}^{x_2} e^{i\lambda t}dt \int\limits_{a}^{\frac{x_2-t}{2}+a}\left(\frac{x_2-t}{2}+a-s\right)q(s)\psi_0(s)ds-$$

$$\frac{\alpha_+}{2}\int\limits_{-x_2}^{x_1} e^{i\lambda t}dt \int\limits_{a}^{\frac{x_2+t}{2}+a}\left(\frac{x_2+t}{2}+a-s\right)q(s)\psi_0(s)ds-$$

$$\frac{\alpha_-}{2}\int\limits_{-x_1}^{x_2} e^{i\lambda t}dt \int\limits_{a}^{\frac{x_1+t}{2}+a}\left(\frac{x_1+t}{2}+a-s\right)q(s)\psi_0(s)ds-$$

$$\frac{\alpha_-}{2}\int\limits_{-x_2}^{x_1} e^{i\lambda t}dt \int\limits_{a}^{\frac{x_1-t}{2}+a}\left(\frac{x_1-t}{2}+a-s\right)q(s)\psi_0(s)ds.$$

Taking into our account the equation (32) suppose that $\psi_0(x)(x \geq a)$ satisfies the equation

$$\psi_0(x) = 1 + \int\limits_0^a sq(s)\psi_0(s)ds + \int\limits_a^x (x-s)q(s)\psi_0(s)ds \ (x \geq a).$$

Since

$$\int\limits_0^a sq(s)\psi_0(s)ds = \int\limits_0^a s\psi''(s)ds = s\psi_0'(s)\mid_0^a - \int\limits_0^a \psi_0'(s)ds = a\psi_0'(a) - \psi_0(a) + 1$$

we have

$$\psi_0(x) = 2 + a\psi_0'(a) - \psi_0(a) + \int\limits_a^x (x-s)q(s)\psi_0(s)ds \ (x \geq a).$$

Clearly, $\psi_0(x)$ is the solution of the equation $\psi_0^{''} = q(x)\psi_0\,(x \geq a)$ satisfying jump conditions (30). Finally, considering the equation (31) and expressions for right handed integrals from

J_1 to J_8 we find that (28) is a solution of (31) when $x \geq a$ if and only if the kernel function $\Phi(x,t)\,(x \geq a)$satisfies the following integral equation:

$$\frac{1}{2}\Phi(x,t) = -\frac{\alpha_+}{2}\int\limits_{\frac{x_1+t}{2\alpha}}^{a}\left(s-\frac{x_1+t}{2\alpha}\right)q(s)\psi_0(s)ds + \frac{\alpha_-}{2}\int\limits_{-\frac{x_2+t}{2\alpha}}^{a}\left(s+\frac{x_2+t}{2\alpha}\right)q(s)\psi_0(s)ds +$$

$$\frac{i\alpha_+}{2\alpha}\int\limits_{\frac{x_1+t}{2\alpha}}^{a}p(s)\psi_0(s)ds + \frac{i\alpha_-}{2\alpha}\int\limits_{-\frac{x_2+t}{2\alpha}}^{a}p(s)\psi_0(s)ds - \frac{\alpha_+}{2}\int\limits_{a}^{\frac{x_2-t}{2}+a}\left(\frac{x_2-t}{2}+a-s\right)q(s)\psi_0(s)ds -$$

$$\frac{\alpha_-}{2}\int\limits_{a}^{\frac{x_1+t}{2}+a}\left(\frac{x_1+t}{2}+a-s\right)q(s)\psi_0(s)ds + \frac{i\alpha_+}{2}\int\limits_{a}^{\frac{x_2-t}{2}+a}p(s)\psi_0(s)ds -$$

$$\frac{i\alpha_-}{2}\int\limits_{a}^{\frac{x_1+t}{2}+a}p(s)\psi_0(s)ds + \frac{\alpha_+}{4\alpha}\int\limits_{0}^{\frac{x_1+t}{2\alpha}}q(s)ds\int\limits_{-\alpha s}^{\alpha s}\Phi(s,\xi)d\xi +$$

$$\frac{\alpha_+}{4\alpha}\int\limits_{\frac{x_1+t}{2\alpha}}^{a}q(s)ds\int\limits_{-\alpha s}^{t+x_1-\alpha s}\Phi(s,\xi)d\xi + \frac{\alpha_-}{4\alpha}\int\limits_{-\frac{x_2+t}{2\alpha}}^{a}q(s)ds\int\limits_{-\alpha s}^{t+x_2+\alpha s}\Phi(s,\xi)d\xi \; +$$

$$\frac{1}{4}\int\limits_{a}^{a+\frac{x_2-t}{2}}q(s)ds\int\limits_{-s-\alpha a+a}^{t+x-s}\Phi(s,\xi)d\xi + \frac{1}{4}\int\limits_{a+\frac{x_2-t}{2}}^{x}q(s)ds\int\limits_{t-x+s}^{t+x-s}\Phi(s,\xi)d\xi +$$

$$i\frac{\alpha_+}{2\alpha}\int\limits_{\frac{x_1+t}{2\alpha}}^{a}p(s)\Phi(s,t+x_1-\alpha s)ds + i\frac{\alpha_-}{2\alpha}\int\limits_{-\frac{x_2+t}{2\alpha}}^{a}p(s)\Phi(s,t+x_2+\alpha s)ds -$$

$$\frac{i}{2}\int\limits_{\frac{x_2-t}{2}+a}^{x}p(s)\Phi(s,t-x+s)ds + \frac{i}{2}\int\limits_{a}^{x}p(s)\Phi(s,t+x-s)ds,$$

$$-x_1 \leq t < -x_2,\; x_2 > 0, \tag{33}$$

$$\frac{1}{2}\Phi(x,t) = -\frac{\alpha_+}{2}\int\limits_{a}^{\frac{x_2-t}{2}+a}\left(\frac{x_2-t}{2}+a-s\right)q(s)\psi_0(s)ds -$$

$$\frac{\alpha_+}{2}\int\limits_{a}^{\frac{x_2+t}{2}+a}\left(\frac{x_2+t}{2}+a-s\right)q(s)\psi_0(s)ds -$$

$$\frac{\alpha_-}{2}\int\limits_{a}^{\frac{x_1-t}{2}+a}\left(\frac{x_1-t}{2}+a-s\right)q(s)\psi_0(s)ds - \frac{\alpha_-}{2}\int\limits_{a}^{\frac{x_1+t}{2}+a}\left(\frac{x_1+t}{2}+a-s\right)q(s)\psi_0(s)ds +$$

$$\frac{i\alpha_+}{2}\int\limits_{a}^{\frac{x_2-t}{2}+a}p(s)\psi_0(s)ds - \frac{i\alpha_+}{2}\int\limits_{a}^{\frac{x_2+t}{2}+a}p(s)\psi_0(s)ds +$$

$$\frac{i\alpha_-}{2}\int_a^{\frac{x_1-t}{2}+a} p(s)\psi_0(s)ds - \frac{i\alpha_-}{2}\int_a^{\frac{x_1+t}{2}+a} p(s)\psi_0(s)ds +$$

$$\frac{1}{4\alpha}\int_0^a q(s)ds\int_{-\alpha s}^{\alpha s}\Phi(s,\xi)d\xi + \frac{1}{4}\int_{a+\frac{x_2+|t|}{2}}^{x} q(s)ds\int_{t-x+s}^{t+x-s}\Phi(s,\xi)d\xi +$$

$$\frac{1}{4}\int_a^{a+\frac{x_2-|t|}{2}} q(s)ds\int_{-s-\alpha a+a}^{s+\alpha a-a}\Phi(s,\xi)d\xi + \frac{1}{4}\int_{a+\frac{x_2-|t|}{2}}^{a+\frac{x_2+|t|}{2}} q(s)ds\int_{\max(-s-\alpha a+a,t-x+s)}^{\min(s+\alpha a-a,t+x-s)}\Phi(s,\xi)d\xi +$$

$$\frac{i}{2}\int_{\frac{x_2+t}{2}+a}^{x} p(s)\Phi(s,t+x-s)ds - \frac{i}{2}\int_{\frac{x_2-t}{2}+a}^{x} p(s)\Phi(s,t-x+s)ds,\ -x_2 < t < x_2, \tag{34}$$

$$\frac{1}{2}\Phi(x,t) = -\frac{\alpha_+}{2}\int_{\frac{x_1-t}{2\alpha}}^{a}\left(s-\frac{x_1-t}{2\alpha}\right)q(s)\psi_0(s)ds + \frac{\alpha_-}{2}\int_{\frac{t-x_2}{2\alpha}}^{a}\left(s-\frac{t-x_2}{2\alpha}\right)q(s)\psi_0(s)ds -$$

$$\frac{i\alpha_+}{2\alpha}\int_{\frac{x_1-t}{2\alpha}}^{a} p(s)\psi_0(s)ds - \frac{i\alpha_-}{2\alpha}\int_{\frac{t-x_2}{2\alpha}}^{a} p(s)\psi_0(s)ds -$$

$$\frac{\alpha_+}{2}\int_a^{\frac{x_2+t}{2}+a}\left(\frac{x_2+t}{2}+a-s\right)q(s)\psi_0(s)ds - \frac{\alpha_-}{2}\int_a^{\frac{x_1-t}{2}+a}\left(\frac{x_1-t}{2}+a-s\right)q(s)\psi_0(s)ds -$$

$$\frac{i\alpha_+}{2}\int_a^{\frac{x_2+t}{2}+a} p(s)\psi_0(s)ds + \frac{i\alpha_-}{2}\int_a^{\frac{x_1-t}{2}+a} p(s)\psi_0(s)ds +$$

$$\frac{\alpha_+}{4\alpha}\int_0^{\frac{x_1-t}{2\alpha}} q(s)ds\int_{-\alpha s}^{\alpha s}\Phi(s,\xi)d\xi +$$

$$\frac{\alpha_+}{4\alpha}\int_{\frac{x_1-t}{2\alpha}}^{a} q(s)ds\int_{t-x_1+\alpha s}^{\alpha s}\Phi(s,\xi)d\xi + \frac{\alpha_-}{4\alpha}\int_{\frac{t-x_2}{2\alpha}}^{a} q(s)ds\int_{t-x_2-\alpha s}^{\alpha s}\Phi(s,\xi)d\xi +$$

$$\frac{1}{4}\int_a^x q(s)ds\int_{t-x+s}^{\min(t+x-s,s+\alpha a-a)}\Phi(s,\xi)d\xi -$$

$$i\frac{\alpha_+}{2\alpha}\int_{\frac{x_1-t}{2\alpha}}^{a} p(s)\Phi(s,t-x_1+\alpha s)ds - i\frac{\alpha_-}{2\alpha}\int_{\frac{t-x_2}{2\alpha}}^{a} p(s)\Phi(s,t-x_2-\alpha s)ds -$$

$$\frac{i}{2}\int_a^x p(s)\Phi(s,t-x+s)ds + \frac{i}{2}\int_{\frac{x_2+t-}{2}+a}^{x} p(s)\Phi(s,t+x-s)ds,\ x_2 < t < x_1. \tag{35}$$

By the similar arguments we have for $x_2 < 0$

$$\frac{1}{2}\Phi(x,t) = -\frac{\alpha_+}{2}\int\limits_{\frac{x_1+t}{2\alpha}}^{a}\left(s-\frac{x_1+t}{2\alpha}\right)q(s)\psi_0(s)ds + \frac{\alpha_-}{2}\int\limits_{-\frac{x_2+t}{2\alpha}}^{a}\left(s+\frac{x_2+t}{2\alpha}\right)q(s)\psi_0(s)ds+$$

$$\frac{i\alpha_+}{2\alpha}\int\limits_{\frac{x_1+t}{2\alpha}}^{a}p(s)\psi_0(s)ds + \frac{i\alpha_-}{2\alpha}\int\limits_{-\frac{x_2+t}{2\alpha}}^{a}p(s)\psi_0(s)ds - \frac{\alpha_+}{2}\int\limits_{a}^{\frac{x_2-t}{2}+a}\left(\frac{x_2-t}{2}+a-s\right)q(s)\psi_0(s)ds-$$

$$\frac{\alpha_-}{2}\int\limits_{a}^{\frac{x_1+t}{2}+a}\left(\frac{x_1+t}{2}+a-s\right)q(s)\psi_0(s)ds + \frac{i\alpha_+}{2}\int\limits_{a}^{\frac{x_2-t}{2}+a}p(s)\psi_0(s)ds-$$

$$\frac{i\alpha_-}{2}\int\limits_{a}^{\frac{x_1+t}{2}+a}p(s)\psi_0(s)ds + \frac{\alpha_+}{4\alpha}\int\limits_{0}^{\frac{x_1+t}{2\alpha}}q(s)ds\int\limits_{-\alpha s}^{\alpha s}\Phi(s,\xi)d\xi+$$

$$\frac{\alpha_+}{4\alpha}\int\limits_{\frac{x_1+t}{2\alpha}}^{a}q(s)ds\int\limits_{-\alpha s}^{t+x_1-\alpha s}\Phi(s,\xi)d\xi + \frac{\alpha_-}{4\alpha}\int\limits_{-\frac{x_2+t}{2\alpha}}^{a}q(s)ds\int\limits_{-\alpha s}^{t+x_2+\alpha s}\Phi(s,\xi)d\xi\ +$$

$$\frac{1}{4}\int\limits_{a}^{a+\frac{x_2-t}{2}}q(s)ds\int\limits_{-s-\alpha a+a}^{t+x-s}\Phi(s,\xi)d\xi+$$

$$\frac{1}{4}\int\limits_{a+\frac{x_2-t}{2}}^{x}q(s)ds\int\limits_{t-x+s}^{t+x-s}\Phi(s,\xi)d\xi + i\frac{\alpha_+}{2\alpha}\int\limits_{\frac{x_1+t}{2\alpha}}^{a}p(s)\Phi(s,t+x_1-\alpha s)ds+$$

$$i\frac{\alpha_-}{2\alpha}\int\limits_{-\frac{x_2+t}{2\alpha}}^{a}p(s)\Phi(s,t+x_2+\alpha s)ds - \frac{i}{2}\int\limits_{\frac{x_2-t}{2}+a}^{x}p(s)\Phi(s,t-x+s)ds+$$

$$\frac{i}{2}\int\limits_{a}^{x}p(s)\Phi(s,t+x-s)ds, -x_1 \le t < x_2,\ x_2 < 0, \tag{36}$$

$$\frac{1}{2}\Phi(x,t) = -\frac{\alpha_+}{2}\int\limits_{\frac{x_1+t}{2\alpha}}^{a}\left(s-\frac{x_1+t}{2\alpha}\right)q(s)\psi_0(s)ds + \frac{\alpha_-}{2}\int\limits_{-\frac{x_2+t}{2\alpha}}^{a}\left(s+\frac{x_2+t}{2\alpha}\right)q(s)\psi_0(s)ds-$$

$$\frac{\alpha_+}{2}\int\limits_{\frac{x_1-t}{2\alpha}}^{a}\left(s-\frac{x_1-t}{2\alpha}\right)q(s)\psi_0(s)ds + \frac{\alpha_-}{2}\int\limits_{\frac{t-x_2}{2\alpha}}^{a}\left(s-\frac{t-x_2}{2\alpha}\right)q(s)\psi_0(s)ds+$$

$$\frac{i\alpha_+}{2\alpha}\int\limits_{\frac{x_1+t}{2\alpha}}^{a}p(s)\psi_0(s)ds + \frac{i\alpha_-}{2\alpha}\int\limits_{-\frac{x_2+t}{2\alpha}}^{a}p(s)\psi_0(s)ds-$$

$$\frac{i\alpha_+}{2\alpha}\int\limits_{\frac{x_1-t}{2\alpha}}^{a}p(s)\psi_0(s)ds - \frac{i\alpha_-}{2\alpha}\int\limits_{\frac{t-x_2}{2\alpha}}^{a}p(s)\psi_0(s)ds\ +$$

$$\frac{\alpha_+}{4\alpha}\int\limits_{\frac{x_1+|t|}{2\alpha}}^{a} q(s)ds\int\limits_{t-x_1+\alpha s}^{t+x_1-\alpha s}\Phi(s,\xi)d\xi+\frac{\alpha_+}{4\alpha}\int\limits_{0}^{\frac{x_1-|t|}{2\alpha}} q(s)ds\int\limits_{-\alpha s}^{\alpha s}\Phi(s,\xi)d\xi+$$

$$\frac{\alpha_+}{4\alpha}\int\limits_{\frac{x_1-|t|}{2\alpha}}^{\frac{x_1+|t|}{2\alpha}} q(s)ds\int\limits_{\max(t-x_1+\alpha s,-\alpha s)}^{\min(t+x_1-\alpha s,\alpha s)}\Phi(s,\xi)d\xi-\frac{\alpha_-}{4\alpha}\int\limits_{\frac{|t|-x_2}{2\alpha}}^{a} q(s)ds\int\limits_{t+x_2+\alpha s}^{t-x_2-\alpha s}\Phi(s,\xi)d\xi-$$

$$\frac{\alpha_-}{4\alpha}\int\limits_{\frac{|t|+x_2}{2\alpha}}^{\frac{|t|-x_2}{2\alpha}} q(s)ds\int\limits_{\max(t+x_2+\alpha s,-\alpha s)}^{\min(t-x_2-\alpha s,\alpha s)}\Phi(s,\xi)d\xi-\frac{\alpha_-}{4\alpha}\int\limits_{0}^{-\frac{|t|+x_2}{2\alpha}} q(s)ds\int\limits_{-\alpha s}^{\alpha s}\Phi(s,\xi)d\xi+$$

$$\frac{1}{4}\int\limits_{a}^{x} q(s)ds\int\limits_{t-x+s}^{t+x-s}\Phi(s,\xi)d\xi+i\frac{\alpha_+}{2\alpha}\int\limits_{\frac{x_1+t}{2\alpha}}^{a} p(s)\Phi(s,t+x_1-\alpha s)ds+$$

$$i\frac{\alpha_-}{2\alpha}\int\limits_{-\frac{x_2+t}{2\alpha}}^{a} p(s)\Phi(s,t+x_2+\alpha s)ds-i\frac{\alpha_+}{2\alpha}\int\limits_{\frac{x_1-t}{2\alpha}}^{a} p(s)\Phi(s,t-x_1+\alpha s)ds-$$

$$i\frac{\alpha_-}{2\alpha}\int\limits_{\frac{t-x_2}{2\alpha}}^{a} p(s)\Phi(s,t-x_2-\alpha s)ds-\frac{i}{2}\int\limits_{a}^{x} p(s)\Phi(s,t-x+s)ds+$$

$$\frac{i}{2}\int\limits_{a}^{x} p(s)\Phi(s,t+x-s)ds,\ x_2<t<-x_2, \tag{37}$$

$$\frac{1}{2}\Phi(x,t)=-\frac{\alpha_+}{2}\int\limits_{\frac{x_1-t}{2\alpha}}^{a}\left(s-\frac{x_1-t}{2\alpha}\right)q(s)\psi_0(s)ds+\frac{\alpha_-}{2}\int\limits_{\frac{t-x_2}{2\alpha}}^{a}\left(s-\frac{t-x_2}{2\alpha}\right)q(s)\psi_0(s)ds-$$

$$\frac{i\alpha_+}{2\alpha}\int\limits_{\frac{x_1-t}{2\alpha}}^{a} p(s)\psi_0(s)ds-\frac{i\alpha_-}{2\alpha}\int\limits_{\frac{t-x_2}{2\alpha}}^{a} p(s)\psi_0(s)ds-$$

$$\frac{\alpha_+}{2}\int\limits_{a}^{\frac{x_2+t}{2}+a}\left(\frac{x_2+t}{2}+a-s\right)q(s)\psi_0(s)ds-\frac{\alpha_-}{2}\int\limits_{a}^{\frac{x_1-t}{2}+a}\left(\frac{x_1-t}{2}+a-s\right)q(s)\psi_0(s)ds-$$

$$\frac{i\alpha_+}{2}\int\limits_{a}^{\frac{x_2+t}{2}+a} p(s)\psi_0(s)ds+\frac{i\alpha_-}{2}\int\limits_{a}^{\frac{x_1-t}{2}+a} p(s)\psi_0(s)ds+$$

$$\frac{\alpha_+}{4\alpha}\int\limits_{0}^{\frac{x_1-t}{2\alpha}} q(s)ds\int\limits_{-\alpha s}^{\alpha s}\Phi(s,\xi)d\xi+$$

$$\frac{\alpha_+}{4\alpha}\int\limits_{\frac{x_1-t}{2\alpha}}^{a} q(s)ds\int\limits_{t-x_1+\alpha s}^{\alpha s}\Phi(s,\xi)d\xi+\frac{\alpha_-}{4\alpha}\int\limits_{\frac{t-x_2}{2\alpha}}^{a} q(s)ds\int\limits_{t-x_2-\alpha s}^{\alpha s}\Phi(s,\xi)d\xi+$$

$$\frac{1}{4}\int\limits_{a}^{a+\frac{x_2+t}{2}} q(s)ds \int\limits_{t-x+s}^{s+\alpha a-a} \Phi(s,\xi)d\xi + \frac{1}{4}\int\limits_{a+\frac{x_2+t}{2}}^{x} q(s)ds \int\limits_{t-x+s}^{t+x-s} \Phi(s,\xi)d\xi -$$

$$i\frac{\alpha_+}{2\alpha}\int\limits_{\frac{x_1-t}{2\alpha}}^{a} p(s)\Phi(s,t-x_1+\alpha s)ds - i\frac{\alpha_-}{2\alpha}\int\limits_{\frac{t-x_2}{2\alpha}}^{a} p(s)\Phi(s,t-x_2-\alpha s)ds-$$

$$\frac{i}{2}\int\limits_{a}^{x} p(s)\Phi(s,t-x+s)ds + \frac{i}{2}\int\limits_{\frac{x_2+t-}{2}+a}^{x} p(s)\Phi(s,t+x-s)ds, \ \ -x_2<t<x_1. \tag{38}$$

Further using the method of successive approximations it is easy to prove that, under the assumption that $q(x), p(x)$ are locally integrable functions, the integral equations $(33)-(35)$ and $(36)-(38)$ have continuous for $t \neq \pm x_2$ and bounded solution $\Phi(x,t)$. Moreover the following relations are satisfied:

$$\Phi(x,-x_1) = -\alpha_+\psi_0(x) + \alpha_+ \exp\left(i\int\limits_0^x \frac{p(s)}{\sqrt{\rho(s)}}ds\right), \tag{39}$$

$$\Phi(x,x_1) = -\alpha_+\psi_0(x) + \alpha_+ \exp\left(-i\int\limits_0^x \frac{p(s)}{\sqrt{\rho(s)}}ds\right), \tag{40}$$

$$\Phi(x,-x_2+0) - \Phi(x,-x_2-0) = \alpha_-\psi_0(x) - \alpha_- \exp\left(i\int\limits_0^x \frac{sgn(s-a)p(s)}{\sqrt{\rho(s)}}ds\right), \tag{41}$$

$$\Phi(x,x_2+0) - \Phi(x,x_2-0) = \alpha_-\psi_0(x) - \alpha_- \exp\left(-i\int\limits_0^x \frac{sgn(s-a)p(s)}{\sqrt{\rho(s)}}ds\right). \tag{42}$$

We have proved the following theorem.

Theorem 4. *Assume that $q(x)$ and $p(x)$ are locally integrable functions in the interval $[0,\infty)$. Then for any $\lambda \in \mathbb{C}$, the solution $\varphi(x,\lambda)$ of the equation (1) with initial conditions*

$$\varphi(0,\lambda) = 0, \varphi'(0,\lambda) = 1$$

can be represented as

$$\varphi(x,\lambda) = \varphi_0(x,\lambda)\psi_0(x) + \frac{1}{2\sqrt{\rho(x)}}\int\limits_{-\nu(x)}^{\nu(x)} \Phi(x,t)e^{i\lambda t}dt,$$

where $\nu(x) = x\sqrt{\rho(x)} + a(\alpha - \sqrt{\rho(x)})$, $\psi_0(x)$ is a solution of $\psi_0'' = q(x)\psi_0$ satisfying the initial conditions

$$\psi_0(0) = 1, \psi_0'(0) = 0$$

and the jump conditions at $x = a$

$$\psi_0(a^+) = 2 - \psi_0(a^-) + a\psi'(a^-), \ \psi'(a^+) = 0.$$

Moreover,the integral kernel $\Phi(x,t)$ is a continuous function for $-\nu(x) \leq t \leq \nu(x)(t \neq x_2 = x - \alpha a - a$ if $x \geq a$ and satisfies the conditions $(39)-(42)$.

From these theorem and integral equations for the kernel function $\Phi(x,t)$ we also have that the partial derivative $\Phi_t(x,t)$ belongs to $L_1(-\nu(x),\nu(x))$ with respect to t for each x.

Theorem 5. *Under the assumptions of the Theorem 4 the solution $\varphi(x,\lambda)$ can be represented as*

$$2i\lambda\varphi(x,\lambda)=i\left(\frac{1}{\sqrt{\rho(x)}}+\frac{1}{\alpha}\right)\sin(\lambda(\mu^+(x)-\mu^+(0))-\int\limits_0^x\frac{p(s)}{\rho(s)}ds)-$$

$$i\left(\frac{1}{\sqrt{\rho(x)}}-\frac{1}{\alpha}\right)\sin(\lambda(\mu^-(x)-\mu^+(0))+\int\limits_0^x sgn(s-a)\frac{p(s)}{\rho(s)}ds)-\frac{1}{\sqrt{\rho(x)}}\int\limits_{-\nu(x)}^{\nu(x)}\Phi_t(x,t)e^{i\lambda t}dt.$$

We prove this theorem applying an integration by parts in the formula (29) and then considering the expressions (39) – (42) for the boundary values of the kernel function $\Phi(x,t)$.

4 New integral representation for the Jost solution

Theorem 6. *Let $(1+x)q(x),p(x)\in L_1(0,\infty)$. Then the Jost solution $f(x,\lambda)$ is expressed as*

$$f(x,\lambda)=f(x,0)e^{i\lambda\mu^+(x)}-i\lambda\int\limits_{\mu^+(x)}^{+\infty}A(x,t)e^{i\lambda t}dt \tag{43}$$

where the kernel $A(x,t)$ defined on the region $0\le\mu^+(x)\le t<+\infty$ is a continuous bounded function. Moreover, the kernel $A(x,t)$ is uniquely determined from the coefficient $p(x)$ and $q(x)$ of the equation (1) *and the following properties are satisfied:*

1)

$$\lim_{t\to+\infty}A(x,t)=0.$$

2) For each $x\ge0$ the kernel $A(x,t)$ is differentiable with respect to $t\ne\mu^-(x)$ almost everywhere and

$$\int\limits_{-\mu^+(x)}^{+\infty}|A_t(x,t)|\,dt\le C_0\left[e^{\int\limits_x^\infty[(\mu(t)-\mu(x))|q(t)|+|p(t)|]dt}-1\right]$$

for some constant C_0.

3) $A(x,t)$ satisfies the relation

$$f(x,0)+A(x,\mu^+(x))=R^+(x). \tag{44}$$

4) If additionally $p(x)\in BC[0,+\infty)$ then $A_t(x,t)$ belongs to $L_1(\mu^+(x),\infty)$, it is a continuous bounded function for $0\le\mu^+(x)\le t<+\infty$, $t\ne\mu^-(x)$, and the following relations are satisfied:

$$A_t\left(x,\mu^+(x)\right)=R_+(x)\left\{\frac{1}{2}\int\limits_x^{+\infty}\left(\frac{q(s)}{\sqrt{\rho(s)}}+\frac{p^2(s)}{\rho(s)\sqrt{\rho(s)}}\right)ds-\frac{i}{2}\frac{p(x)}{\rho(x)}-\frac{i}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)^2p(a)\right\},$$

$$A_t\left(x,\mu^-(x)+0\right)-A_t\left(x,\mu^-(x)-0\right)=$$

$$R^{-}(x)\left\{\frac{1}{2}\int\limits_{x}^{+\infty}\left(\frac{q(s)}{\sqrt{\rho(s)}}+\frac{p^2(s)}{\rho(s)\sqrt{\rho(s)}}\right)sgn(s-a)ds-\right.$$
$$\left.\frac{i}{2}\frac{p(x)}{\rho(x)}-\frac{i}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)^2 p(a)\right\}.$$

5) The identity

$$f(x,\lambda)=R^{+}(x)e^{i\lambda\mu^{+}(x)}+R^{-}(x)e^{i\lambda\mu^{-}(x)}+\int\limits_{-\mu^{+}(x)}^{+\infty}A_t(x,t)e^{i\lambda t}dt,\ Imk\geq 0,$$

holds. In particular,

$$f(0,\lambda)=\alpha_{+}e^{i\lambda a(1-\alpha)+i\omega}+\alpha_{-}e^{i\lambda a(1+\alpha)+i\widetilde{\omega}}+\int\limits_{-a(1-\alpha)}^{+\infty}A_t(0,t)e^{i\lambda t}dt,$$

where ω *and* $\widetilde{\omega}$ *are defined by*

$$\omega=\frac{1}{\alpha}\int\limits_{0}^{a}p(s)ds+\int\limits_{a}^{+\infty}p(s)ds,\ \widetilde{\omega}=-\frac{1}{\alpha}\int\limits_{0}^{a}p(s)ds+\int\limits_{a}^{+\infty}p(s)ds.$$

Proof. Let us denote by

$$A(x,t):=-\int\limits_{t}^{+\infty}K(x,s)ds-R^{-}(x)H\left(\mu^{-}(x)-t\right),t\geq\mu^{+}(x), \tag{45}$$

where

$$H(y)=\begin{cases}1, y>0,\\ 0, y<0,\end{cases}$$

is the Heaviside function. Then by the Theorem 1 we have

$$f(x,\lambda)=R^{+}(x)e^{i\lambda\mu^{+}(x)}+R^{-}(x)e^{i\lambda\mu^{-}(x)}+\int\limits_{-\mu^{+}(x)}^{+\infty}e^{i\lambda t}d_t\left(-\int\limits_{t}^{+\infty}K(x,s)ds\right)=$$

$$R^{+}(x)e^{i\lambda\mu^{+}(x)}+R^{-}(x)e^{i\lambda\mu^{-}(x)}+e^{i\lambda\mu^{+}(x)}\int\limits_{\mu^{+}(x)}^{+\infty}K(x,s)ds+$$

$$i\lambda\int\limits_{-\mu^{+}(x)}^{+\infty}e^{i\lambda t}dt\left(\int\limits_{t}^{+\infty}K(x,s)ds\right)=\left(R^{+}(x)+R^{-}(x)+\int\limits_{\mu^{+}(x)}^{+\infty}K(x,s)ds\right)e^{i\lambda\mu^{+}(x)}+$$

$$i\lambda R^{-}(x)\int\limits_{-\mu^{+}(x)}^{\mu^{-}(x)}e^{i\lambda t}dt+i\lambda\int\limits_{-\mu^{+}(x)}^{+\infty}e^{i\lambda t}dt\left(\int\limits_{t}^{+\infty}K(x,s)ds\right)=$$

$$f(x,0)e^{i\lambda\mu^{+}(x)}-i\lambda\int\limits_{-\mu^{+}(x)}^{+\infty}e^{i\lambda t}\left(-\int\limits_{t}^{+\infty}K(x,s)ds-R^{-}(x)H\left(\mu^{-}(x)-t\right)\right)dt=$$

$$f(x,0)e^{i\lambda\mu^+(x)} - i\lambda \int\limits_{-\mu^+(x)}^{+\infty} A(x,t)e^{i\lambda t}dt,$$

that is we get (43) . Since

$$f(x,0) = R^+(x) + R^-(x) + \int\limits_{\mu^+(x)}^{+\infty} K(x,s)ds = R^+(x) - A(x,\mu^+(x)$$

we have (44) . The remained assertions of the theorem are directly results of the Theorem 1. The theorem is proved. □

From (45) it is obtained that

$$K(x,t) = A_t(x,t)(t \neq \mu^-(x)) \text{ and } K(x,2a-t) = -A_t(x,2a-t)\left(t \neq \mu^+(x)\right)$$

Also we have

$$A(x,\mu^-(x)+0) - A(x,\mu^-(x)-0) = R^-(x).$$

5 Conclusion

By establishing new integral representations of the solutions satisfying the Jost conditions as $x \to +\infty$ and also constructing a solution with initial conditions we can examine the spectral properties of the associated problem. Due to the discontinuities in the kernel functions of the integral representations for solutions, the relationship between the potential functions and kernel functions are enough complicated. However, leveraging this relationship, we can derive the main integral equation of the Faddeev-Marchenko type , which plays a central role in solving of the inverse problem. Under the assumption of an absent discrete spectrum the unique solvability of the Faddeev-Marchenko type integral equation arises.The stated problem will be very complicated and can be applied to many physical scattering scenarios if we consider the case when there is a discreet spectrum.

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