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PIECEWISE CONSTANT ARGUMENT METHOD FOR SOLVING DUFFING DIFFERENTIAL EQUATIONS

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Abstract. The purpose of this article is to solve numerically the second order differential equations of Duffing type using Piecewise constant argument method. Modelling of some differential equations under the various physical problems with the initial conditions are solved to demonstrate the effectiveness and the accuracy of the piecewise constant arguments. We compared the present work with some recently approximate results in several examples, revealing that the proposed architecture is effective and accurate.

Keywords: Approximate solution, initial value problems, duffing equation, non-linear second order ordinary differential equation, piecewise constant argument.

AMS Subject Classification: 34A06, 34A12, 34A34, 65A05.

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1 Introduction

In this paper we consider a general nonlinear oscillator system

$$\varepsilon z''(t) + (\delta + \beta z^p(t))z'(t) - \mu z(t) + \alpha z^q(t) = \phi(f, \omega, t) \tag{1}$$

with initial conditions

$$z(0) = a_0, z'(0) = a_1, \tag{2}$$

where a_0, a_1, α, β, ε, δ and μ are real numbers and $p, q \geq 0$, $\phi(f, \omega, t)$ is $\frac{2\pi}{\omega}$-periodic function of t, ω defines an angular frequency.

For the case, when $p = 2$, $q = 3$ and $\phi(f, \omega, t) = f \cos(\omega t)$ the Eq.(1) is presented to the Duffing van der Pol oscillator Mukherjee et al. (2010) of the form

$$\varepsilon z''(t) + (\delta + \beta z^2(t))z'(t) - \mu z(t) + \alpha z^3(t) = f \cos(\omega t) \tag{3}$$

with initial conditions (2). This forced equation is a damped pendulum if $f \neq 0$ and unforced damped pendulum as $f = 0$. To study the Duffing equation is interesting since it allows to investigate when we force on oscillator near its resonant frequency. Note that in most physical systems to investigate the damping or nonlinear effects are important when the amplitude grows sufficiently large.

For $\alpha = 0$, $p = 2$ and $\phi = f sin(\omega, t)$, the Eq.(1) leads to the Van der Pol oscillator Lepik (2009) of the form

$$\varepsilon z''(t) + (\delta + \beta z^2(t))z'(t) - \mu z(t) = f sin(\omega, t). \tag{4}$$

The Duffing and Van der Pol equations are solved using a linearization method in Motsa & Sibanda (2012), an algebraic method and Hermite neural network are investigated Van der Pol, Rayleigh, and Duffing equations in Akbari et al. (2014) and Mall & Chakraverty (2016), respectively. In Bota (2017), polynomial least square approach has been applied to find approximated solution for nonlinear oscillator equations. The uniform Haar wavelet approximation and quasilinearization method was introduced in Kaur et al. (2014) for solving selected nonlinear oscillator equations. Additionally, numerical inverse Laplace transform based on Bernoulli polynomials operational matrix method was presented in Rani & Mishra (2020). In Mohammadi et al. (2023), applying integral equation method finding approximated solution of the nonlinear Duffing-Van der Pol oscillator equation was reduced to study a nonlinear Volterra integral equation of the second kind. Further, to approximate integral equation authors used the hybrid Legendre polynomials and block-pulse function (HLBPFs).

In this paper we construct an approximate solution for the initial value problem (1) and (2). Firstly, we introduce the differential equation with piecewise constant arguments corresponding to the considered Duffing equation with initial conditions, which depend on a positive integer N. Then we give a definition for the solution of the introduced differential equation as a piecewise-smooth function with respect to the positive integer N. It is proved that this equation has a unique piecewise-smooth solution, which will be an approximate solution of the considered initial value problem for large N. A new calculation method is presented to find this approximate solution that reduces the simplicity of the computational process. Numerical results are presented in order to show the effectiveness of the proposed methods, and the errors between the approximate and exact solutions are estimated.

The article consists of five sections. Section 2 deals with the elementary Piecewise constant arguments, solution technique and stability of the proposed method. Section 3 are given the proof of the main results. The algorithm of the method is introduced and shows some examples and concrete calculations of solutions in Section 4. Finally, we conclude the paper in Section 5.

2 Differential equations with piecewise constant arguments

Let N be a positive integer and $t_k = \frac{k}{N}$, $k = 1, 2, ..., N$. Consider the following differential equation with piecewise constant arguments of the form

$$\begin{gathered} \varepsilon z''(t) + (\delta + \beta z^p(t_{k-1}))z'(t_{k-1}) - \mu z(t_{k-1}) + \alpha z^q(t_{k-1}) = \phi(f, \omega, t), \\ z(0) = a_0, z'(0) = a_1, \quad t \in [t_{k-1}, t_k) \qquad k = 1, 2, ..., N. \end{gathered} \tag{5}$$

A solutions of (5) is defined as following Muminov & Radjabov (2022, 2024)

Definition 1. *A function $z(t) := z_N(t)$ is called a solution of the initial value problem (5) if the following conditions are satisfied:*

(i) $z(t)$, $z'(t)$ are continuous on $[0, 1]$;

(ii) $z''(t)$ exists and continuous on $[0, 1]$ with possible exception at points t_k, $k = 0, 1, ..., N$, where one-sided derivatives exist;

(iii) $z(t)$ satisfies the initial value problem (5) in (0,1), with the possible exception at the points t_k, $k = 0, 1, ..., N$.

The equation (5) is called differential equations with the piecewise constant argument corresponding to the non-linear differential equation (1).

Theorem 1. *For any positive integer number N, the initial value problem (5) has a unique solution $z(t) := z_N(t)$ and defined as*

$$z(t) = \frac{1}{\varepsilon} \int\limits_{t_{k-1}}^{t} \int\limits_{t_{k-1}}^{x} \Big(\phi(f,\omega,s) - \alpha z^q(t_{k-1}) + \mu z(t_{k-1})$$

$$-(\delta + \beta z^p(t_{k-1}))z'(t_{k-1})\Big) ds dx + z'(t_{k-1})(t - t_{k-1}) + z(t_{k-1}), \qquad t \in [t_{k-1}, t_k), \tag{6}$$

where $z(t_{k-1}) = \lim\limits_{t \to t_{k-1}-0} z(t), \quad z'(t_{k-1}) = \lim\limits_{t \to t_{k-1}-0} z'(t), \quad k = 1, 2, ..., N.$

Further, we suppose that there exists a positive number C such that for any N and $k \in \{1, \ldots, N\}$ following the inequalities

$$|z(t_k)| < C \tag{7}$$

$$|z'(t_k)| < C \tag{8}$$

hold. We remark that for any continuous function φ, the functional sequence $\varphi_n(t_k), \quad t \in [t_{k-1}, t_k), k = 1, 2, ..., n$ approaches uniformly on $[0, 1]$. Therefore, if $z(t)$ is a solution for the initial value problem (1), then $z(t)$ will be approximate solution of the piecewise constant initial value problem (5). Therefore, it is clear to expect that the solution z_N of the initial value problem (5) is an approximate solution for the initial value problem (1).

The following theorem claims that the function z_N can be approximate solution for the initial value problem (1).

Theorem 2. *Let $z(t) := z_N(t)$ be solution of the initial value problem (5). Then for any $\varepsilon_1 > 0$ there exist a positive number $N_0 = N(\varepsilon_1)$ such that for any positive integer number N with $N > N_0$ the inequality*

$$\sup_{t \in [0,1]} |\varepsilon z''(t) + (\delta + \beta z^p(t))z'(t) - \mu z(t) + \alpha z^q(t) - \phi(f, \omega, t)| < \varepsilon_1 \tag{9}$$

holds.

We remark that if the function z is approximated solution of the initial value problem (1) then it should be uniformly bounded, since an analytic solution of the problem is bounded.

To analyze the error of the nonlinear oscillator Eq. (1), we introduce the following definition and further we discuss the convergence of the method for nonlinear oscillator problem if we know the exact solution.

Definition 2. *(Residual) According to Eq. (1), we define a residual function as follows Parand et al. (2024)*

$$Residual = |\varepsilon z''(t) + (\delta + \beta z^p(t))z'(t) - \mu z(t) + \alpha z^q(t) - \phi(f, \omega, t)|.$$

The objective is to minimize the residual function in a specific interval $[a, b]$. In addition, the residual should be close to zero, as possible.

3 Proof of the Main results

Proof of the Theorem 1. Let N be a positive integer. For $t \in [0, t_1)$ the equation (5) is

$$z''(t) = \frac{1}{\varepsilon}\Big(\phi(f, \omega, t) - \alpha z^q(0) + \mu z(0) - (\delta + \beta z^p(0))z'(0)\Big).$$

Hence

$$z(t)=\frac{1}{\varepsilon}\int\limits_0^t\int\limits_0^x\Big(\phi(f,\omega,s)-\alpha z^q(0)+\mu z(0)-(\delta+\beta z^p(0))z'(0)\Big)dsdx+z'(0)t+z(0),\quad t\in[0,t_1).$$

There exist the following limits $z(t_1)=\lim\limits_{t\to t_1-0}z(t)$, $z'(t_1)=\lim\limits_{t\to t_1-0}z'(t)$, since z and z' are continuous on $[0,t_1]$.

We solve equation (5) for $t\in[t_{k-1},t_k)$, $k=1,\dots,N$, as

$$z(t)=\frac{1}{\varepsilon}\int\limits_{t_{k-1}}^t\int\limits_{t_{k-1}}^x\Big(\phi(f,\omega,s)-\alpha z^q(t_{k-1})+\mu z(t_{k-1})$$

$$-(\delta+\beta z^p(t_{k-1}))z'(t_{k-1})\Big)dsdx+z'(t_{k-1})(t-t_{k-1})+z(t_{k-1}),\qquad t\in[t_{k-1},t_k),$$

where $z(t_{k-1})=\lim\limits_{t\to t_{k-1}-0}z(t)$, $z'(t_{k-1})=\lim\limits_{t\to t_{k-1}-0}z'(t)$.

By construction the functions $z(t)$ and $z'(t)$ are continuous on $[0,1]$ and $z''(t)$ exists on $(0,t_1)\cup(t_1,t_2)\cup\dots\cup(t_{N-1},1)$ and continuous on $[0,1]$. It is clear that this function is the unique solution of the equation (5).

Proof of the Theorem 2. Let $z(t)$ be a solution of (5) for $t\in[t_{k-1},t_k)$. Then for any $t\in[t_{k-1},t_k)$ the equality

$$\varepsilon z''(t)+(\delta+\beta z^p(t))z'(t)-\mu z(t)+\alpha z^q(t)-\phi(f,\omega,t)=R_N(t),$$

holds, where

$$R_N(t)=(\delta+\beta z^p(t))z'(t)-\mu z(t)+\alpha z^q(t)$$
$$-[(\delta+\beta z^p(t_{k-1}))z'(t_{k-1})-\mu z(t_{k-1})+\alpha z^q(t_{k-1})].$$

We rewrite $R_N(t)$ as

$$R_N(t)=\delta(z'(t)-z'(t_{k-1}))+\beta z^p(t)(z'(t)-z'(t_{k-1}))$$

$$+\beta z'(t_{k-1})(z^p(t)-z^p(t_{k-1}))-\mu(z(t)-z(t_{k-1}))+\alpha(z^q(t)-z^q(t_{k-1})).\tag{10}$$

The identity

$$b^n-a^n=n\int\limits_a^b r^{n-1}dr$$

yields the inequality

$$|b^n-a^n|\le nc^{n-1}|b-a|,\quad c=\max(|a|,|b|)\quad\text{for}\quad a,b\in\mathbb{R}\quad\text{and}\quad n>0.\tag{11}$$

This gives

$$|z^p(t)-z^p(t_{k-1})|\le pC^{p-1}|z(t)-z(t_{k-1})|,\tag{12}$$

$$|z^q(t)-z^q(t_{k-1})|\le qC^{q-1}|z(t)-z(t_{k-1})|,\tag{13}$$

Using the inequalities (12), (13) in (10) we obtain

$$|R(t)|\le(|\delta|+|\beta||z^p(t)|)|z'(t)-z'(t_{k-1})|$$

$$+(p|\beta|C^{p-1}|z'(t_{k-1})|+q|\alpha|C^{q-1}+|\mu|)|z(t)-z(t_{k-1})|.\tag{14}$$

For $t \in [t_{k-1}, t_k)$ the inequalities

$$z'(t) - z'(t_{k-1}) = \frac{1}{\varepsilon} \int\limits_{t_{k-1}}^{t} \Big(\phi(f,\omega,s) - \alpha z^q(t_{k-1}) + \mu z(t_{k-1}) - (\delta + \beta z^p(t_{k-1}))z'(t_{k-1})\Big) ds$$

and

$$z(t) - z(t_{k-1}) = \frac{1}{\varepsilon} \int\limits_{t_{k-1}}^{t} \int\limits_{t_{k-1}}^{x} \Big(\phi(f,\omega,s) - \alpha z^q(t_{k-1}) + \mu z(t_{k-1})$$

$$-(\delta + \beta z^p(t_{k-1}))z'(t_{k-1})\Big) ds dx + z'(t_{k-1})(t - t_{k-1})$$

give

$$|z(t) - z(t_{k-1})| \le \frac{1}{|\varepsilon|}\Big(|\phi| + |\alpha|C^q + |\mu|C + |\delta|C + |\beta|C^{p+1}\Big)\frac{1}{N^2} + C\frac{1}{N} \tag{15}$$

and

$$|z'(t) - z'(t_{k-1})| \le \frac{1}{|\varepsilon|}\Big(|\phi| + |\alpha|C^q + |\mu|C + |\delta|C + |\beta|C^{p+1}\Big)\frac{1}{N}, \tag{16}$$

respectively. Now using the inequalities (7), (8), (15) and (16) we evalute (14) as

$$|R| \le \frac{1}{|\varepsilon|}\frac{1}{N}\Big[(|\delta| + |\beta|C^p)\Big(|\phi| + |\alpha|C^q + |\mu|C + |\delta|C + |\beta|C^{p+1}\Big)$$

$$+(p|\beta|C^p + q|\alpha|C^{q-1} + |\mu|)\Big(|\phi| + |\alpha|C^q + |\mu|C + |\delta|C + |\beta|C^{p+1} + C\Big)\Big].$$

Thus

$$|R| \le \frac{C_1}{N}, \tag{17}$$

where

$$C_1 = \frac{1}{|\varepsilon|}\Big[(|\delta| + |\beta|C^p)\Big(|\phi| + |\alpha|C^q + |\mu|C + |\delta|C + |\beta|C^{p+1}\Big)$$

$$+(p|\beta|C^p + q|\alpha|C^{q-1} + |\mu|)\Big(|\phi| + |\alpha|C^q + |\mu|C + |\delta|C + |\beta|C^{p+1} + C\Big)\Big].$$

4 Algorithm of the method and numerical results

In order to obtain an approximate solution of equation (1) from the Theorem 2, we need to calculate $z(t) = z_N(t)$, which has the explicit form (6). Moreover, the Theorem 2 provides the convergence of algorithm, where the equation (9) represents smallness the residual error for growing larger N.

The algorithms of the Piecewise constant argument method are presented as follows:
Input: N is a positive integer number, initial conditions.
Step 1: Compute $z(t)$, $z'(t)$ for $t \in [0, t_1)$
Step 2: Find limits $z(t_1) = \lim\limits_{t \to t_1 - 0} z(t_1)$, $z'(t_1) = \lim\limits_{t \to t_1 - 0} z'(t_1)$;
Step 3: Consider $z(t_{k-1})$ and $z'(t_{k-1})$ initial values to (5) compute $z(t)$, $z'(t)$ for $t \in [t_{k-1}, t_k)$, $k = 2, \ldots, N$;
Step 4: Construct an approximate solution from (6).
Output: Residual errors are found.

In the papers Lepik (2009), Kimiaeifar et al. (2009), Sajadi et al. (2010) are studied the approximate solution of the forced Duffing -van der Pol oscillator. The authors are presented the values of the approximate solution at specific points for the several values of the parameters. In the following Examples (a), (b) and (c) we give the numerical results and residual error for

Table 1: Comparison of Piecewise constant argument method solution with those solutions available in by different methods for Example 1(a).

t	z(t) Haar wavelet solution	z(t) VIM	z(t) HPM	z(t) Numerical	z(t) Present method	Residual error Present method
0.2	1.0099	1.0099	1.0099	1.0099	1.009814	0
0.4	1.0392	1.0391	1.0391	1.0391	1.038092	0
0.6	1.0834	1.0854	1.0862	1.0854	1.082135	1×10^{-10}
0.8	1.1440	1.1453	1.1493	1.1453	1.137966	1×10^{-10}
1	1.2037	1.2138	1.2278	1.2037	1.200374	2×10^{-10}

the forced Duffing -van der Pol oscillator and comparisons with the results in these references, which show the higher accuracy residual error of the proposed method.

Example 1(a). Consider the Eq.(3) with parameters $\varepsilon = 1$, $\delta = 0.1$, $\beta = 0.1$, $\mu = 0.5$, $\alpha = 0.5$, $f = 0.5$ and $\omega = 0.79$ as below

$$\begin{gathered} z''(t) + 0.1(1 + z^2(t))z'(t) - 0.5z(t) + 0.5z^3(t) = 0.5\cos 0.79t, \\ z(0) = 1, z'(0) = 0. \end{gathered} \tag{18}$$

Comparison of the solutions with those available in Sajadi et al. (2010) and by the methods Haar wavelet solution Kaur et al. (2014), Variational iteration method (VIM) Lepik (2009), Homotopy perturbation method (HPM) Lepik (2009), Numerical Lepik (2009) are given in Table 1 for different values of parameters.

Example 1(b). Consider the Eq. (3) with the parameters $\varepsilon = 1$, $\delta = 0.1$, $\beta = 0.1$, $\mu = -0.5$, $\alpha = 0.5$, $f = 0.5$ and $\omega = 0.79$ as below

$$\begin{gathered} z''(t) + 0.1(1 + z^2(t))z'(t) + 0.5z(t) + 0.5z^3(t) = 0.5\cos 0.79t, \\ z(0) = 1, z'(0) = 0. \end{gathered} \tag{19}$$

Comparison of solutions with those available in Sajadi et al. (2010) and by different methods Haar wavelet solution Kaur et al. (2014), Variational iteration method (VIM) Lepik (2009), Homotopy perturbation method (HPM) Lepik (2009), Numerical Lepik (2009) are given in Table 2 for different values of parameters.

Example 1(c). Consider the Eq. (3) with the parameters $\varepsilon = 1$, $\delta = -0.1$, $\beta = 0.1$, $f = 0.5$ and $\omega = 0.79$.
Piecewise constant argument method solutions and Haar wavelet solution Kaur et al. (2014) are reported in Table 3, 4 and 5 for different values of parameters α and μ.

Example 2. Consider the unforced Van der Pol oscillator Lepik (2009) from Eq. (4) with the following initial conditions and parameters $\varepsilon = 0.1$, $\delta = -0.05$, $\beta = 0.05$, $\mu = -1$, $\alpha = 0$ and $f = 0$.

$$\begin{gathered} 0.1z''(t) - 0.05(1 - z^2(t))z'(t) + z(t) = 0, \\ z(0) = 0, z'(0) = 0.5. \end{gathered} \tag{20}$$

The numerical results and residual error and comparisons with the results in Kaur et al. (2014) are presented in Table 6, which show the higher accuracy residual error. The graph of the residual error in the intervals $[0, \frac{4}{N}]$, $[\frac{N-50005}{N}, \frac{N-50001}{N}]$ and $[\frac{N-5}{N}, 1]$ are showed in Fig. 1 for $N = 10^5$.

Example 3. We consider Eq. (4) with the following initial conditions and parameters $\delta = 1$, $\beta = -1$, $\mu = -1$, $\alpha = 0$ and $f = 0$, i.e.

$$\begin{gathered} \varepsilon z''(t) + (1 - z^2(t))z'(t) + z(t) = 0, \\ z(0) = 0.5, z'(0) = 1. \end{gathered} \tag{21}$$

We examine the proposed method for the specific small values of the parameter ε. The accuracy of residual error and comparisons with the results Haar wavelet solution Kaur et al. (2014) presented in the Tables 7 and 8 claims the effectiveness of the approximate solution.

Table 2: Comparison of Piecewise constant argument method solution with those solutions available in by different methods for Example 1(b).

t	z(t) Haar wavelet solution	z(t) VIM	z(t) HPM	z(t) Numerical	z(t) Present method	Residual error Present method
0.2	0.9900	0.9900	0.9900	0.9900	0.990176	2×10^{-10}
0.4	0.9609	0.9607	0.9607	0.9607	0.961721	8×10^{-10}
0.6	0.9128	0.9134	0.9138	0.9134	0.916719	5×10^{-10}
0.8	0.8502	0.8502	0.8521	0.8502	0.857705	7×10^{-10}
1	0.7766	0.7735	0.7797	0.7735	0.787295	2×10^{-10}

Table 3: Computed Piecewise constant argument method solutions z(t) for example 1(c) at different values of μ and α.

t	Haar wavelet solution $\mu = -0.5$ $\alpha = 0.5$	Present method $\mu = -0.5$ $\alpha = 0.5$	Residual error	Haar wavelet solution $\mu = -0.5$ $\alpha = 1.5$	Present method $\mu = -0.5$ $\alpha = 1.5$	Residual error
0.0156	0.999939	0.99993915	0	0.999817	0.99981744	8×10^{-10}
0.1094	0.997014	0.99701157	9×10^{-10}	0.991072	0.99106605	1.3×10^{-9}
0.2031	0.989735	0.98973523	3×10^{-10}	0.969560	0.96955924	2.4×10^{-9}
0.2969	0.978191	0.97817892	1×10^{-10}	0.936134	0.93610244	1.6×10^{-9}
0.3906	0.962516	0.96249732	1×10^{-10}	0.892052	0.89200914	1×10^{-10}
0.5156	0.935526	0.93543598	3×10^{-10}	0.819362	0.81919992	3×10^{-10}
0.6094	0.911024	0.91078391	9×10^{-10}	0.756565	0.75617694	9×10^{-10}
0.7031	0.883181	0.88269096	6×10^{-10}	0.688489	0.68781878	1.3×10^{-9}
0.7969	0.852307	0.85133753	6×10^{-10}	0.616619	0.61544202	4×10^{-10}
0.8906	0.818731	0.81703302	1×10^{-10}	0.542236	0.54046847	2×10^{-10}
0.9844	0.782791	0.77994180	4×10^{-10}	0.466374	0.46378545	5×10^{-10}

Table 4: Computed Piecewise constant argument method solutions z(t) for example 1(c) at different values of μ and α.

t	Haar wavelet solution $\mu = -0.5$ $\alpha = 2.5$	Present method $\mu = -0.5$ $\alpha = 2.5$	Residual error	Haar wavelet solution $\mu = -0.5$ $\alpha = -0.5$	Present method $\mu = -0.5$ $\alpha = -0.5$	Residual error
0.0156	0.999695	0.99969570	2.2×10^{-9}	1.00006	1.0000608	2×10^{-10}
0.1094	0.985167	0.98515542	3×10^{-10}	1.00299	1.0029931	3×10^{-10}
0.2031	0.949797	0.94978204	5×10^{-9}	1.01033	1.0103254	5×10^{-10}
0.2969	0.895886	0.89573148	1.1×10^{-9}	1.02212	1.0220978	4×10^{-10}
0.3906	0.826694	0.82620772	3.1×10^{-9}	1.03834	1.0383259	4×10^{-10}
0.5156	0.717307	0.71528798	1.6×10^{-9}	1.06693	1.0670373	2.2×10^{-9}
0.6094	0.627435	0.62276042	2×10^{-10}	1.09352	1.0940050	3.5×10^{-9}
0.7031	0.534554	0.52558688	5×10^{-10}	1.12435	1.1257327	9×10^{-10}
0.7969	0.441452	0.42582131	1×10^{-9}	1.15913	1.1624727	4×10^{-10}
0.8906	0.350018	0.32533067	8×10^{-10}	1.19739	1.2043949	9×10^{-10}
0.9844	0.261395	0.2250192	3×10^{-10}	1.23849	1.2519210	0

In the following Examples 4 and 5 we presented the values of the approximate solution and residual errors with the higher accuracy at specific points for non integer values of the parameters p and q.

Example 4. Consider nonlinear oscillator from Eq. (1) with following initial conditions and parameters $p = \frac{3}{2}$, $q = \frac{5}{2}$, $\varepsilon = 1$, $\delta = 1$, $\beta = 1$, $\mu = 1$, $\alpha = -1$, $\omega = 0.79$ and $f = 1$.

$$\begin{gathered} z''(t) + (1 + z^{\frac{3}{2}}(t))z'(t) - z(t) - z^{\frac{5}{2}}(t) = sin0.79t, \\ z(0) = 1, z'(0) = 0. \end{gathered} \tag{22}$$

The solution of Piecewise constant argument method is presented in Table 9.

Example 5. Consider the nonlinear oscillator from Eq. (1) with the following initial conditions and parameters $p = \frac{7}{8}$, $q = \frac{9}{4}$, $\varepsilon = 1$, $\delta = 1$, $\beta = 1$, $\mu = 1$, $\alpha = -1$, $\omega = 0.79$ and $f = 1$.

Table 5: Computed Piecewise constant argument method solutions z(t) for example 1(c) at different values of μ and α.

t	Haar wavelet solution $\mu = -1.5$ $\alpha = 0.5$	Present method $\mu = -1.5$ $\alpha = 0.5$	Residual error	Haar wavelet solution $\mu = -2.5$ $\alpha = 0.5$	Present method $\mu = -2.5$ $\alpha = 0.5$	Residual error
0.0156	0.999817	0.99981745	1×10^{-10}	0.999695	0.99969575	1×10^{-10}
0.1094	0.991054	0.99104825	1.7×10^{-9}	0.985106	0.98509650	9×10^{-10}
0.2031	0.96935	0.96935255	1.1×10^{-9}	0.949106	0.94910514	2×10^{-9}
0.2969	0.935213	0.93518858	1.5×10^{-9}	0.892876	0.89279541	7×10^{-10}
0.3906	0.889407	0.88938745	3×10^{-10}	0.818179	0.81799492	3×10^{-10}
0.5156	0.811893	0.81180574	3×10^{-10}	0.693792	0.69307469	1.4×10^{-9}
0.6094	0.742905	0.74267394	4×10^{-10}	0.585343	0.58365526	1.3×10^{-9}
0.7031	0.665995	0.66561726	2×10^{-10}	0.467045	0.46384756	8×10^{-10}
0.7969	0.582432	0.58171577	8×10^{-10}	0.341798	0.33609927	2×10^{-10}
0.8906	0.493472	0.49238262	6×10^{-10}	0.212456	0.2034023	2.5×10^{-9}
0.9844	0.400332	0.39863194	7×10^{-10}	0.081763	0.0682005	3.3×10^{-9}

Table 6: Comparison of Piecewise constant argument method solution z(t) with those obtained by different numerical methods for Example 2.

t	Runge Kutta method	Gear Method	Chebyshev series solution	Haar wavelet solution	Present method	Residual error
0	0	0	0	0	0	0
0.1	0.05004	0.05004	0.05004	0.05004	0.05004166	1×10^{-11}
0.2	0.09983	0.09983	0.09982	0.09983	0.09983221	2×10^{-11}
0.3	0.14886	0.14886	0.14885	0.14886	0.14886981	6×10^{-11}
0.4	0.19665	0.19665	0.19664	0.196659	0.19665661	6×10^{-11}
0.5	0.24270	0.24270	0.24275	0.24270	0.24270396	8×10^{-11}
0.6	0.28653	0.28653	0.28653	0.28653	0.28653779	1.4×10^{-10}
0.7	0.32770	0.32770	0.32771	0.32770	0.32770373	6×10^{-11}
0.8	0.36577	0.36577	0.36576	0.36577	0.36577225	6×10^{-11}
0.9	0.40034	0.40034	0.40040	0.40034	0.40034349	1.9×10^{-10}
1	0.43105	0.43105	0.43104	0.43105	0.43105180	3.07×10^{-6}

Table 7: Comparison of Piecewise constant argument method solution z(t) with those obtained by different numerical methods for Example 3.

t	Haar wavelet solution $\varepsilon = 0.0001$	Present method $\varepsilon = 0.0001$	Residual error	Haar wavelet solution $\varepsilon = 0.001$	Present method $\varepsilon = 0.001$	Residual error
0.0156	0.503418	0.48998010	0	0.515625	0.49190177	1×10^{-10}
0.1094	0.426271	0.43487530	0	0.609375	0.43635714	0
0.2031	0.346682	0.38847935	1×10^{-10}	0.703125	0.38966308	1×10^{-10}
0.2969	-0.650879	0.34852115	0	-0.301753	0.34948762	0
0.3906	12.9768	0.31370606	0	53.2588	0.31450701	0
0.5156	24.4202	0.27359827	1×10^{-10}	331.704	0.27423119	0
0.6094	-28.1094	0.24740459	0	-610.707	0.24793901	0
0.7031	42.7498	0.22403554	1×10^{-10}	881.696	0.22448877	0
0.7969	-46.5388	0.20306267	0	-1272.02	0.20344799	0
0.8906	66.4663	0.18422464	2×10^{-10}	1588.06	0.18455275	0
0.9844	-71.4282	0.16722806	0	-2117.17	0.16750753	0

$$
\begin{gathered}
z''(t) + (1 + z^{\frac{7}{8}}(t))z'(t) - z(t) - z^{\frac{9}{4}}(t) = sin0.79t, \\
z(0) = 0.5, z'(0) = 1.
\end{gathered} \tag{23}
$$

Piecewise constant argument method solutions are reported in Table 10.

In the following example we presented the values of the approximate solution and residual errors at specific points for the large integer values of the parameters p and q.

Example 6. Consider the nonlinear oscillator from Eq. (1) with the following initial conditions and parameters $p = 10$, $q = 15$, $\varepsilon = 1$, $\delta = -3$, $\beta = 3$, $\mu = 1$, $\alpha = 3$, $\omega = 0.79$ and $f = 1$.

Table 8: Comparison of Piecewise constant argument method solution z(t) with those obtained by different numerical methods for Example 3.

t	Haar wavelet solution $\varepsilon = 0.1$	Present method $\varepsilon = 0.1$	Residual error	Haar wavelet solution $\varepsilon = 1000$	Present method $\varepsilon = 1000$	Residual error
0.0156	0.515625	0.51413606	3×10^{-10}	0.515625	0.51559984	5×10^{-10}
0.1094	0.609375	0.55081700	2×10^{-11}	0.609367	0.60939251	3×10^{-10}
0.2031	0.703125	0.53455539	0	0.703098	0.70307434	3×10^{-10}
0.2969	0.796875	0.49238616	0	0.796818	0.79684553	4×10^{-10}
0.3906	0.890625	0.44108060	0	0.890525	0.89050656	1×10^{-10}
0.5156	1.01563	0.37390644	2×10^{-10}	1.01544	1.01543958	8×10^{-11}
0.6094	1.10938	0.32927453	1×10^{-10}	1.10911	1.10917927	1×10^{-10}
0.7031	1.20313	0.29050116	1×10^{-10}	1.20271	1.20281196	1×10^{-10}
0.7969	-108.566	0.25703852	0	1.29623	1.29653692	1×10^{-10}
0.8906	-1744.21	0.22818361	0	1.38947	1.39015640	0
0.9844	-106017	0.20311320	1×10^{-10}	1.48229	1.48387198	0

Table 9: Computation of Piecewise constant argument method solutions z(t) for example 4.

t	Present method	Residual error
0.0156	1.000241	1×10^{-9}
0.1094	1.011341	3.2×10^{-10}
0.2031	1.037581	6×10^{-10}
0.2969	1.077632	1×10^{-10}
0.3906	1.130519	1×10^{-10}
0.5156	1.220247	4×10^{-10}
0.6094	1.301708	1.4×10^{-9}
0.7031	1.395215	4×10^{-10}
0.7969	1.501201	5.8×10^{-9}
0.8906	1.619842	0
0.9844	1.751950	4×10^{-10}

Table 10: Computation of Piecewise constant argument method solutions z(t) for example 5.

t	Present method	Residual error
0.0156	0.515500	2.4×10^{-10}
0.1094	0.605063	2.2×10^{-10}
0.2031	0.690041	4×10^{-10}
0.2969	0.772881	1.5×10^{-9}
0.3906	0.855487	7×10^{-10}
0.5156	0.968719	7×10^{-10}
0.6094	1.058205	1.4×10^{-9}
0.7031	1.153251	4×10^{-10}
0.7969	1.255602	2×10^{-10}
0.8906	1.366570	0
0.9844	1.487967	3.4×10^{-9}

Table 11: Computation of Piecewise constant argument method solutions z(t) for example 6.

t	Present method	Residual error
0.0156	0.516156	1.3×10^{-9}
0.1094	0.639731	1×10^{-10}
0.2031	0.818809	7×10^{-10}
0.2969	1.054090	8×10^{-9}
0.3906	1.176996	3.5×10^{-8}
0.5156	1.034455	4×10^{-10}
0.6094	0.891991	6×10^{-10}
0.7031	0.731530	3.5×10^{-9}
0.7969	0.537632	4.1×10^{-9}
0.8906	0.295007	3.8×10^{-9}
0.9844	-0.016001	1×10^{-10}

$$\begin{gathered} z''(t) - 3(1 - z^{10}(t))z'(t) - z(t) + 3z^{15}(t) = cos0.79t, \\ z(0) = 0.5, z'(0) = 1. \end{gathered} \tag{24}$$

Table 11 shows the piecewise constant argument method solutions.

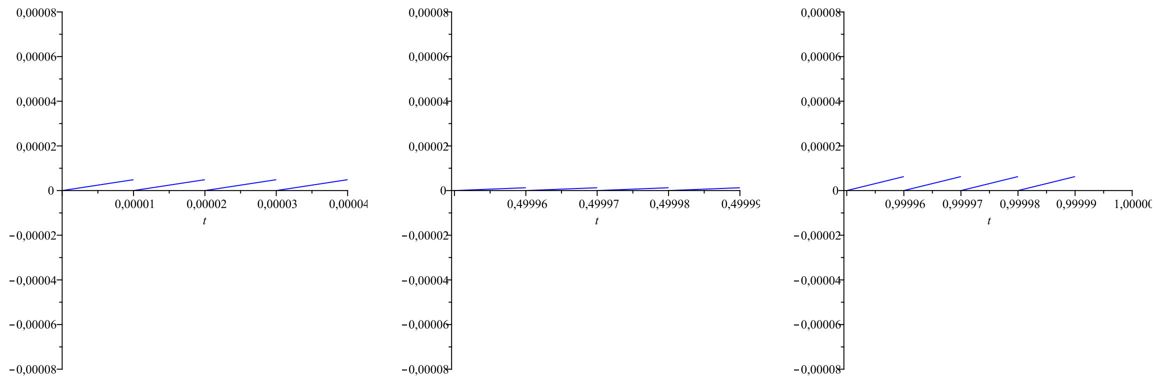

Figure 1: The graph of residual error for Example 2.

5 Conclusion

In this work, an effective numerical algorithm based on the use of constant arguments method is proposed for solving the approximate solution of nonlinear second-order differential equations with the initial value. Firstly, the differential equation with piecewise constant arguments corresponding to the considering equation has been constructed, where equation has the unique continuous and piecewise smooth function $z_N(t)$ depending on positive integer N. Moreover, the residual error for $z_N(t)$ approaches zero when N growing large enough. Numerical results were implemented to demonstrate the effectiveness of the proposed method. The considered examples illustrate the approximate solution values at specified points, along with the small residual errors (See the Figure 1). It is observed that the residual error equals zero at several selected points. Additionally, it should be noted that the values of the Haar wavelet method Kaur et al. (2014) at the given points, as shown in Tables 7 and 8, exhibit significant large differences. Due to the simplicity of the method the examples considered in Kaur et al. (2014) are solved with high accuracy for a large N and the comparisons in Tables 7 and 8 show that the proposed architecture is effective and accurate. Furthermore, realization of this method presented efficiency of finding the approximate solutions of the Duffing -van der Pol oscillator for the cases when the values of the parameters p and q are non integer (See the Examples 4 and 5) or large integer numbers (See the Example 6).

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Data Availability. The data sets generated during and analyzed during the current study are available from the corresponding author on reasonable request.

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