Advanced Mathematical Models & Applications
Vol.10, No.1, 2025, pp.81-87
https://doi.org/10.62476/amma.10181

RAPIDLY DECREASING SOLUTION OF THE INITIAL BOUNDARY-VALUE PROBLEM FOR AN INFINITE SYSTEM OF NONLINEAR EVOLUTION EQUATIONS

A.Kh. Khanmamedov[1,2,3,4,5,6], **A.F. Mamedova**[7]

[1]Institute of Mathematics and Mechanics, Baku, Azerbaijan
[2]Baku State University, Baku, Azerbaijan
[3]Azerbaijan University, Baku, Azerbaijan
[4]Baku Engineering University, Khirdalan, Azerbaijan
[5]Azerbaijan Technical University, Baku, Azerbaijan
[6]Western Caspian University, Baku, Azerbaijan
[7]Azerbaijan State University of Economics (UNEC), Baku, Azerbaijan

Abstract. An initial boundary-value problem with zero boundary condition for an infnite system of nonlinear evolution equations is considered. First, the scattering problem for a second-order difference equation, which is related to the problem under consideration, was considered. We prove the existence and uniqueness of a rapidly decreasing solution and determine a class of initial data that guarantees the existence of a rapidly decreasing solution. Then, the scattering problem is considered for a second-order difference equation whose coefficients are solutions to a nonlinear equation. A law of variation of scattering data with time has been found. Using the variation of the scattering data and by the inverse spectral method, an algorithm for constructing the solution is obtained.

Keywords: nonlinear evolution equation, Langmuir model, scattering data, method of inverse spectral problem, rapidly decreasing soluition.

AMS Subject Classification: 37K40, 35Q53.

Corresponding author: A.F. Mamedova, Azerbaijan State University of Economics (UNEC), Istiglaliyyat 6, AZ1001, Baku, Azerbaijan, e-mail: *Mammadova_Afaq@unec.edu.az*

Received: 24 December 2024; Revised: 27 January 2025; Accepted: 14 February 2025;
Published: 10 April 2025.

1 Introduction

Many applied problems are formulated as mixed problems for nonlinear differential-difference equations (see Manakov (1975)). The mixed problem for equations like the Toda lattice or the Langmuir lattice are certainly of practical interest (see, e.g., Berezanskii (1985), Osipov (1994), Vereshchagin (1997)) and has therefore been the subject of much investigation over a period of years (Makhmudova et al., 2015; Khanmamedov et al., 2019; Teschl et al., 2023; Shi et al. , 2023; Cho et al., 2023; Wang et al., 2024).

For sequence $c_n = c_n(t) > 0$ of real-valued functions $c_n \in C^{(1)}[0, \infty)$, we consider the

How to cite (APA): Khanmamedov, A.Kh., Mamedova A.F. (2025). Rapidly decreasing solution of the initial boundary-value problem for an infinite system of nonlinear evolution equations. *Advanced Mathematical Models & Applications*, *10*(1), 81-87 https://doi.org/10.62476/amma.10181

following Cauchy problem for the infinite system of nonlinear evolution equations:

$$\dot{c}_n = c_n \left(\alpha \left(c_{n+1} - c_{n-1} \right) - \beta \left(\left(c_{n+1} - c_{n-1} \right) \sum_{k=0}^{2} c_{n+k} \right) \right), c_0 = 0, \, n \in N, \, \cdot = \frac{d}{dt}, \tag{1}$$

$$c_n(0) = \hat{c}_n > 0, \; n \in N \tag{2}$$

where N is the set of natural numbers and α, β are real numbers. This system was first studied in the work of Bogoyavlenskii (1988), where it was also found there that system of equations (1) can be integrated using the method of inverse spectral problem. The system of equations (1) has various applications in crystallography and plasma physics (Manakov, 1975; Bogoyavlenskii, 1988). Moreover, this system is a generalization of the well-known Lotka-Volterra predator-prey model (Manakov, 1975). With $\alpha = 1$, $\beta = 0$, system of equations (1) represents the well-known Langmuir model, which was studied by the method of the inverse spectral problem by many authors (see Manakov (1975), Vereshchagin (2006), Makhmudova et al. (2015)). A method for the integration of Cauchy problem for equation on the whole axis based on the inverse problem for a spectral function was proposed in [1].

A solution $c\left(t\right) = \left(c_n(t)\right)_{n \in N}$ of problem (1)–(3) is called rapidly decreasing if $x_n\left(t\right) = c_n\left(t\right) - 1$ is rapidly decreasing function, i.e., a function satisfying for any $T > 0$the inequality

$$\left\| Q_1\left(t\right) \right\|_{C[0,T]} < \infty, \tag{3}$$

where $Q_1\left(t\right) = \sum\limits_{n=1}^{\infty} \left(1 + |n|\right) \left|x_n\left(t\right)\right|$.

In the present paper, using the inverse scattering transform, we prove the existence of a rapidly decreasing solution of problem (1) – (3). Note that problem (1), (2) can be considered as a discrete analogue of the initial-boundary value problem for the higher Korteweg-de Vries equation. As follows from the works of Khabibulin (1999), the initial-boundary value problem for nonlinear continuous models using the inverse scattering transform cannot be solved as effectively as the Cauchy problem. The results of this work show that for some discrete nonlinear equations, the inverse spectral problem method, unlike in the continuous models, is effectively applied to the integration of the initial-boundary value problem.

2 Preliminary information

In this section, we formulate several auxiliary facts; many of them are given in work of Makhmudova et al. (2015). Consider the boundary value problem

$$\sqrt{c_{n-1}} y_{n-1} + \sqrt{c_n} y_{n+1} = \lambda y_n, n \in N, \tag{4}$$

$$y_0 = 0, \tag{5}$$

where c_n satisfies the condition $\sum\limits_{n=1}^{\infty} n \left|c_n - 1\right| < \infty$. In Eq. (4), we set $\lambda = z + z^{-1}$. For $|z| \leq 1$, we define Jost-type solutions $f_n\left(z\right)$ of Eq. (4) by setting $\lim\limits_{n \to \infty} f_n\left(z\right) z^{-n} = 1$. A solution of this sort exists and is unique. The following representation in terms of the operator of transformation is true (Makhmudova et al., 2015):

$$f_n\left(z\right) = \alpha_n z^n \left(1 + \sum_{m=1}^{\infty} A_{nm} z^{2m}\right), \tag{6}$$

where

$$\alpha_n = 1 + o(1), n \to \infty, \; A_{nm} = O\left(\sum_{k=n+\left[\frac{m}{2}\right]}^{\infty} |\sqrt{c_n} - 1|\right), n + m \to \infty,$$

and $[x]$ is the integer part of x. The quantities α_n, A_{nm} and the coefficient $\sqrt{c_n}$ of Eq. (4) are connected by the relations

$$c_n = \left(\frac{\alpha_{n+1}}{\alpha_n}\right)^2, \; c_n = 1 + A_{n1} - A_{n-1,1}. \tag{7}$$

Let $\varphi_n = \varphi_n(z)$ denote a solution of Eq. (4) that satisfies the conditions $\varphi_0 = 0$ and $\varphi_1 = 1$. In this case, the following relation is true (Makhmudova et al., 2015):

$$\left(z^{-1} - z\right) \varphi_n(z) = f_0(z)\, \overline{f_n(z)} - \overline{f_0(z)} f_n(z). \tag{8}$$

A function $f_0(z)$ that is regular in the circle $|z| < 1$, where it can have a finite number of simple real rootsz_k, $0 < z_k^2 < 1, k = 1, 2, .., P$. The boundary-value problem (4), (5) generates a bounded self-adjoint operator Lin the space $\ell_2[1, \infty)$. Moreover, the eigenvalues of the operator L are simple and coincide with the points $\lambda_k = z_k + z_k^{-1}, k = 1, ..P$.

Denote

$$S(z) = \frac{\overline{f_0(z)}}{f_0(z)}, \; M_k^{-2} = \sum_{n=1}^{\infty} f_n^2(z_k), \; k = 1, 2, ..., P.$$

A collection of quantities $\left\{S(z), |z| = 1, \; z_k, 0 < z_k^2 < 1, \; M_k, M_k > 0, k = 1, ..., P\right\}$ is called scattering data for problem (4), (5). The inverse scattering problem for problem (4), (5) amounts to reconstructing the coefficient c_n from the scattering data.

From the scattering data, the coefficient c_n of Eq. (4) is uniquely reconstructed by the following algorithm.

Algorithm 1

Given scattering data $\left\{S(z), |z| = 1, \; z_k, 0 < z_k^2 < 1, \; M_k, M_k > 0, k = 1, ..., P\right\}$.

Step 1. Construct the function

$$F_n = \sum_{k=1}^{P} M_k^2 z_k^n - \frac{1}{2\pi i} \int_{|z|=1} S(z)\, z^{n-1} dz \tag{9}$$

Step 2. Find from the Marchenko-type equation

$$F_{2n+2m} + A_{nm} + \sum_{k=1}^{\infty} A_{nk} F_{2n+2m+2k} = 0 \;, \; n \geq 0, m \geq 1, \tag{10}$$

which has (see [4]) a unique solution.

Step 3. Calculate c_n by the formulas

$$(\alpha_n)^{-2} = 1 + F_{2n} + \sum_{k=1}^{\infty} A_{nk} F_{2n+2k} \tag{11}$$

$$c_n = \left(\frac{\alpha_{n+1}}{\alpha_n}\right)^2. \tag{12}$$

3 Solvability of problem (1), (2)

Let us study the global solvability of problem (1) and (2).

Theorem 1. *There exists a unique solution of problem* (1) *and* (2) *in class* (3), *if* $Q_1(0) < \infty$.

Proof. First, it should be noted that problem (1) and (2) is equivalent to the problem

$$\dot{x}_n = (x_n + 1)(x_{n+1} - x_{n-1})\left\{\alpha - 3\beta - \beta \sum_{k=0}^{2} x_{n+k}\right\},\ x_0 = -1, n \geq 1, \tag{13}$$

$$x_n(0) = x_n^0, \tag{14}$$

where $x_n = x_n(t) = c_n(t) - 1$. Let B be the Banach space of sequences $x = (x_n)_{n\in N}$ with the norm $\|x\|_B = \sum_{n\in N}(1 + |n|)|x_n| < \infty$. Then, the set $C([0,T];B)$ of functions $x(t)$ continuous on the interval $[0,T]$ with values in B is a Banach space with the norm $\|x(t)\|_{C([0,T]:B)} = \max\limits_{0\leq t\leq T}\|x(t)\|_B$ (see, e.g., Khanmamedov et al. (2019)).

Rewrite problem (13) and (14) in the form of an operator equation:

$$x(t) = x(0) + \int_0^T F(x(\tau))\, d\tau,$$

where $x(t) = (x_n(t))_{n\in N}$ and F is the operator generated by the right-hand side of system (11). Since the right-hand side of system (13) is a polynomial of variable $x(t) = (x_n(t))_{n\in N}$, the operator F at each $T > 0$ continuously maps the space $C([0,T];B)$ into itself. Therefore, we can apply to the last equation the principle of contracting mapping. As a result, in a certain interval $[0,\delta]$, problem (13) and (14) has a solution $x(t) = (x_n(t))_{n\in Z}$ with a finite norm $\|x(t)\|_{C([0,\delta]:B)} < \infty$. Let us prove that this solution can be continued to any finite interval $[0,T]$. Assume the contrary. Then, there is a point $t_0 < T$ such that problem (13) and (14) has a solution $x(t) = (x_n(t))_{n\in Z}$ continuous on the interval $[0,T)$ but such that $\overline{\lim\limits_{t\to t_0}}\|x(t)\|_B = +\infty$.

Using (11), we have

$$x_n(t) = x_n(0) + \int_0^t (x_n(\tau) + 1)(x_{n+1}(\tau) - x_{n-1}(\tau))\left\{\alpha - 3\beta - \beta\sum_{k=0}^{2} x_{n+k}(\tau)\right\} d\tau, t \in [0,t_0). \tag{15}$$

We introduce a family of operators $L = L(t)$ acting in the space $\ell_2[1,\infty)$ by the formula

$$(Ly)_n = \sqrt{c_{n-1}(t)}y_{n-1} + \sqrt{c_n(t)}y_{n+1},\ \ n \in N.$$

where when calculating $(Ly)_0$ it is assumed that $y_0 = 0$. Obviously, the operator $L = L(t)$ is strongly continuously differentiable only if the coefficient $c_n = c_n(t)$ satisfies system of equations (1). We also introduce the family of operators $A = A(t)$, acting in the space $\ell_2[1,\infty)$ by the formula

$$(Ay)_n = \frac{1}{2}\left(-\beta\prod_{k-0}^{3}\sqrt{c_{n-1-k}}y_{n-4}-\right.$$

$$-\sqrt{c_{n-1}c_{n-2}}\left(-\alpha + \beta\sum_{k=-1}^{2} c_{n-1-k}\right)y_{n-2}+$$

$$\left.+\sqrt{c_n c_{n+1}}\left(-\alpha + \beta\sum_{k=-1}^{2} c_{n+k}\right)y_{n+2} + \beta\prod_{k=0}^{3}\sqrt{c_{n+k}}y_{n+4}\right),$$

where when calculating $(Ay)_1, (Ay)_2, (Ay)_3, (Ay)_4$ it is assumed that $y_{-3} = 0, y_{-2} = 0$, $y_{-1} = 0, y_0 = 0$. It is easy to see that the operators L and A form a Lax pair; i.e., system (1) is equivalent to the operator equation

$$\dot{L} = [L, A] = LA - AL. \tag{16}$$

The last relationship implies (Khanmamedov et al., 2019) that the family of operators $L(t)$ is unitarily equivalent. Therefore, at all $t \in [0, t_0)$, we have the equality $\|L(t)\| = \|L(0)\|$. It is easy to verify that, for any $n \in N$, we have the estimate $|c_n(t)| \leq \|L(t)\|^2 = \|L(0)\|^2$. Using this inequality, as in as in the work Khanmamedov et al. (2019), it is established that the norm $\|x(t)\|_B$ is bounded. The latter statement is contrary to our assumption. Therefore, problem (11) and (12) on the interval has a unique solution with a finite norm. Thus, the unique solvability of problem (1) and (2) in class (3) is proven. □

4 Transformation of Scattering Data

Assume that the coefficient $c_n(t)$, $n = 1$, in Eq. (4) form a rapidly decreasing solution of problem (1), (2). Then relation (9) with functions $\varphi_n(z)$ and $f_n(z)$ depending on t (i.e., $\varphi_n = \varphi_n(z,t)$ and $f_n = f_n(z,t)$ is true.

Theorem 2. *If the coefficient $c_n = c_n(t), n = 1$, of Eq. (4) form a rapidly decreasing solution of problem (1),(2), then the evolution of scattering data is described by the relations*

$$S(z,t) = S(z,0)\exp\left\{\beta\left(z^{-4} - z^4\right)t + (4\beta - \alpha)\left(z^{-2} - z^2\right)t\right\}, \tag{17}$$

$$z_k(t) = z_k(0) = z_k, k = 1, .., P, \tag{18}$$

$$M_k^2(t) = M_k^2(0)\exp\left\{\beta\left(z_k^{-4} - z_k^4\right)t + (4\beta - \alpha)\left(z_k^{-2} - z_k^2\right)t\right\}, k = 1, ..., P. \tag{19}$$

Proof. Using Eq. (16) and the equality $\dot{L}\psi + L\dot{\psi} = \lambda\dot{\psi}$, we establish that the operator $B = \frac{d}{dt} - A$ transforms a solution of the equation $L\psi = \lambda\psi$ into a solution of the same equation. Consider equality (9). It is obvious that $\varphi_n = \{\varphi_n(z,t)\}_{n=1}^{\infty}$ is a solution of the equation $L\psi = \lambda\psi$, where $\lambda = z + z^{-1}$. On the other hand, it is easy to verify that $(B\varphi)_1 = p(\lambda, t)$, where $p(\lambda, t)$ is a polynomial of degree 5 in λ with real coefficients. Whence $B\varphi = p(\lambda, t)\varphi$. Furthermore, using (9), as $n \to \infty$ we get

$$B\left[\left(z^{-1} - z\right)\varphi_n(z,t)\right] = \left\{\dot{f}_0(z,t) - \frac{1}{2}\left(\beta\left(z^4 - z^{-4}\right) + (4\beta - \alpha)\left(z^2 - z^{-2}\right)\right)f_0(z,t)\right\}z^{-n} -$$

$$-\left\{\overline{\dot{f}_0(z,t)} + \frac{1}{2}\left(\beta\left(z^4 - z^{-4}\right) + (4\beta - \alpha)\left(z^2 - z^{-2}\right)\right)\overline{f_0(z,t)}\right\}z^n + o(1).$$

However, the equation $L\psi = \lambda\psi$ has a unique solution with this asymptotic representation. Therefore, get

$$B\left[\left(z^{-1} - z\right)\varphi_n(z,t)\right] =$$

$$= \left\{\dot{f}_0(z,t) - \frac{1}{2}\left(\beta\left(z^4 - z^{-4}\right) + (4\beta - \alpha)\left(z^2 - z^{-2}\right)\right)f_0(z,t)\right\}\overline{f_n(z,t)} -$$

$$-\left\{\overline{\dot{f}_0(z,t)} + \frac{1}{2}\left(\beta\left(z^4 - z^{-4}\right) + (4\beta - \alpha)\left(z^2 - z^{-2}\right)\right)\overline{f_0(z,t)}\right\}f_n(z,t).$$

Comparing this identity with relation (9) and the equality $B\varphi = p(\lambda, t)\varphi$, we obtain the equations

$$\dot{f}_0(z,t) - \frac{1}{2}\left(\beta\left(z^4 - z^{-4}\right) + (4\beta - \alpha)\left(z^2 - z^{-2}\right)\right) f_0(z,t) = p(\lambda, t) f_0(z,t),$$

$$\overline{\dot{f}_0(z,t)} + \frac{1}{2}\left(\beta\left(z^4 - z^{-4}\right) + (4\beta - \alpha)\left(z^2 - z^{-2}\right)\right) \overline{f_0(z,t)} = p(\lambda, t) \overline{f_0(z,t)},$$

which yield the relation (17). Furthermore, from the unitary equivalence of the family of operators $L = L(t)$ it follows that the eigenvalues $\lambda_k(t)$ do not depend on t, i.e. $z_k(t) = z_k(0) = z_k, k = 1, .., P$ and equality (18) is true.

Now consider the normalized eigenfunction $u^{(k)}(t) = \{u_n(z_k, t)\}_{k \in N}$ of the operator $L = L(t)$. Since the eigenvalues of this operator are simple, we have

$$\dot{u}^{(k)}(t) - Au^{(k)}(t) = Cu^{(k)}(t).$$

Multiplying both sides of the last equality in $\ell_2[1, \infty)$ scalarly by $u^{(k)}(t)$ and using the fact that $u^{(k)}(t)$ is the normalized eigenfunction and A is a skew-symmetric operator, we get $C = 0$. Therefore, $Bu^{(k)}(t) = 0$. On the other hand, as $n \to \infty$, we obtain

$$\left(Bu^{(k)}(t)\right)_n = \left(\dot{M}_k(t) + \frac{1}{2}\left(\beta\left(z_k^4 - z_k^{-4}\right) + (4\beta - \alpha)\left(z_k^2 - z_k^{-2}\right)\right) M_k(t)\right) z_k^n + o(1).$$

Therefore, $\dot{M}_k(t) + \frac{1}{2}\left(\beta\left(z_k^4 - z_k^{-4}\right) + (4\beta - \alpha)\left(z_k^2 - z_k^{-2}\right)\right) M_k(t) = 0$, which implies that the identity (19) is true. □

The theorem is proved.

Using relationships (17)-(19), we obtain the following algorithm for solving problem (1) and (2) by the method of the inverse spectral problem.

Algorithm 2

Given initial condition $c_n(0) = \hat{c}_n > 0$, $n \in N$.

Step 1. Construct the scattering data $\{S(z,0), \ z_k(0), \ M_k(0), k = 1, ..., P\}$.

Step 2. Calculate the set $\{S(z,t), \ z_k(t), \ M_k(t), k = 1, ..., P\}$.by formulas (17)-(19).

Step 3. Construct the solution by solving the inverse spectral problem with the set by algorithm 1, where $S(z), \ M_k^2$ in (7) are replaced with $S(z,t), \ M_k^2(t)$.

5 Conclusions

In this paper, we concentrate on studying the Cauchy problem for a semi-infinite system of nonlinear differential equations. Essentially, this system is a generalization of the Lotka-Volterra predator-prey model. We used the inverse scattering method. First, we investigated the unique solvability of the problem and proved that at any time, the solution has the same asymptotics as the initial condition. Then, we indicated an algorithm for finding a solution. The obtained results can be extended to the case of infinite lattices in both directions.

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