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RIEMANN'S METHOD FOR THE DIFFERENCE ANALOGUE OF A SECOND-ORDER HYPERBOLIC EQUATION

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Abstract. In this paper is considered the Cauchy problem for a discrete analogue of a second- order hyperbolic equation with a periodic coefficient. The value of the solution at each point is considered as the value of some linear functionality on the initial data. The concept of Riemann function is introduced for discrete hyperbolic type equations. Using the eigenfunctions of the discrete Hill equation, the Riemann function of a discrete analogue of a second-order hyperbolic equation is constructed. For this purpose, some facts related to the spectral theory of the discrete Hill equation are proved. The properties of the Riemann function are studied. A representation of the solution to the Cauchy problem through Riemann functions is found.

Keywords: hyperbolic equation, Riemann function, Riemann method, Cauchy problem, discrete Hill equation.
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1 Introduction

The second-order hyperbolic equation has a wide application. In particular, the hyperbolic one-dimensional equation describes the vibrations of a string. On the other hand, the numerical solution of the decaying wave equation is of great importance for wave phenomena and many practical problems use the hyperbolic type approximating difference equation (Copson, 1957; Khanmamedov, 2022).

The Riemann method of representing solutions to various boundary value problems is well known in the theory of linear differential equations of hyperbolic type with two independent variables Copson (1957); Riemann (1948); Korovina et al. (2021); Sadalov et al. (2020); Yarysheva (1959); Swarn et al. (2017). The essence of this method is that the value of the desired function at a given point may be thought of as the value of some linear functional on the initial data. An expression for this functional was first found by Riemann (Riemann, 1948; Marchenko, 1986). The Riemann function plays a fundamental role in the theory of linear equations of hyperbolic type and with its help it is usually possible to write solutions to the Cauchy and Goursat problems in quadratures.

It is known that with a practical point of view, the numerical solutions of hyperbolic equations are of particular interest. In this regard, the question arises of the implementation of the Riemann method for discrete analogues of hyperbolic equations. In this work, the question of

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the possibility of using the Riemann method for discrete analogues of hyperbolic equations is studied. As far as we know, this issue was not considered before.

Let the real coefficients $a(n) > 0, b(n)$ satisfy the conditions $a(n+N) = a(n)$, $b(n+N) = b(n)$ where, $N \geq 2$ is a positive integer. Suppose $f(n)$, $g(n)$ be given compactly supported sequences. Under these assumptions about $a(n), b(n)$ and $f(n)$, $g(n)$, consider the following Cauchy problem

$$\begin{aligned} &a(n)U(n+1,m) + b(n)U(n,m) + a(n-1)U(n-1,m) = \\ &= a(m)U(n,m+1) + b(m)U(n,m) + a(m-1)U(n,m-1), \\ &n = 0, \pm1, \pm2, ..., \ m > m_0 \end{aligned} \tag{1}$$

$$U(n,m_0) = f(n), \ U(n,m_0+1) = g(n). \tag{2}$$

The consideration of equation of the form (1) is connected with the fact that it and the associated discrete Hill equation have important applications in crystallography Toda (1980), Khanmamedov (2005).

We will call the function $R(n,m,n_0, m_0)$ the Riemann function of equation (4) if it satisfies equation (1) and taking the value 1 on the characteristics $n - n_0 = \pm(m - m_0 + 1)$ of this equation. In the present paper the Riemann function of equation (1) is investigated. An explicit form of the Riemann function is found in terms of Floquet solutions of the discrete Hill equation. A representation of the solution to the Cauchy problem through Riemann functions is found. The last result also extends to the case of the Goursat problem. In addition, the results of this work can be used to construct transformation operators for difference equations (see, for example, Khanmamedov (2005)).

2 A representation of the solution by means of Riemann function

We need some preliminaries concerning the discrete Hill equation. Consider the equation

$$a(n-1)y(n-1) + b(n)y(n) + a(n)y(n+1) = \lambda y(n), n = 0, \pm1, \pm2, ..., \tag{3}$$

where λ is a complex parameter. Let

$$\chi(1,0;\lambda) = \chi(N, N-1;\lambda) \equiv 1,$$

$$\chi(m,s;\lambda) = (-1)^{s+m+1} \times$$

$$\times \begin{vmatrix} b(m)-\lambda & a(m) & 0 & ... & 0 & 0 & 0 \\ a(m) & b(m+1)-\lambda & 1 & ... & 0 & 0 & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & 0 & 0 & ... & 1 & b(s-1)-\lambda & a(s-1) \\ 0 & 0 & 0 & ... & 0 & a(s-1) & b(s)-\lambda \end{vmatrix}, \ 1 \leq m \leq s \leq N-1.$$

$$\Delta_{\pm}(\lambda) = \begin{vmatrix} b(0)-\lambda & a(0) & 0 & ... & 0 & 0 & \pm a(N-1) \\ a(0) & b(1)-\lambda & a(1) & ... & 0 & 0 & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & 0 & 0 & ... & a(N-3) & b(N-2)-\lambda & a(N-2) \\ \pm a(N-1) & 0 & 0 & ... & 0 & a(N-2) & b(N-1)-\lambda \end{vmatrix},$$

$$\mu = \Delta_{\pm}(\lambda) \pm (-1)^N 2a(1)a(2)...a(N).$$

Denote the roots of the polynomials $\Delta_+(\lambda)$ and $\Delta_-(\lambda)$- by $\lambda_1^* < \lambda_2^* \leq ... \leq \lambda_N^*$ and $\lambda_1^- < \lambda_2^- \leq ... \leq \lambda_N^-$. As is well known (Khanmamedov, 2005), these roots alternate, and we can arrange them as follows:

$$\lambda_1^* < \lambda_1^- \leq .\lambda_2^-. < \lambda_2^*. \leq \lambda_3^* < ... < \lambda_{N-1}^* \leq \lambda_N^- < \lambda_N^*, \text{ if } N \text{ is even},$$

$$\lambda_1^- < \lambda_1^+ \leq .\lambda_2^+. < \lambda_2^-. \leq \lambda_3^* < ... < \lambda_{N-1}^* \leq \lambda_N^- < \lambda_N^* \text{ , if } N \text{ is odd}.$$

Let Γ be the complex λ-plane with cuts along the segments $I_1, I_2, ..., I_N$, where I_j has end-points $\lambda_j^\pm, j = 1, 2, ..., N$. We introduce the function

$$z = z(\lambda) = \frac{(-1)^N \mu}{2a(1)\,a(2)\,...a(N)} + \sqrt{\left(\frac{(-1)^N \mu}{2a(1)\,a(2)\,...a(N)}\right)^2 - 1},$$

taking a regular branch of $z(\lambda)$ in Γ such that $z(\infty) = 0$. Using the definition of $z(\lambda)$, we can prove as in Khanmamedov (2005) that

$$z(\lambda) = \frac{a(1)\,a(2)\,...a(N)}{\lambda^N} + O\left(\frac{1}{\lambda^{N+1}}\right) \text{ , } \lambda \to \infty.$$

As shown in Khanmamedov (2005), equation (6) has solutions$\psi^\pm(n, \lambda)$, which can be represented as

$$\psi^\pm(nN + j - 1, \lambda) = E_j^\pm(\lambda)\, z^{\pm n}, \quad j = 1, ..., N, \quad |n| = 0, 1, 2, ..., \tag{4}$$

where

$$E_1^\pm(\lambda) \equiv 1, \quad E_j^\pm(\lambda) = \chi^{-1}(1, N-1; \lambda)\left\{\chi(1, j-2; \lambda)\, z^{\pm 1} + \chi(j, N-1; \lambda)\right\}, \ j = 2, .., N.$$

As is known (Khanmamedov (2005)), the Wronskian $W(\lambda) = a(n)\,(\psi^+(n, \lambda), \psi^-(n+1, \lambda) - -\psi^+(n+1, \lambda), \psi^-(n, \lambda))$ is defined by the formula

$$W(\lambda) = a(1)\,a(2)\,...a(N)\ \chi^{-1}(1, N-1; \lambda)\left(z^{-1} - z\right).$$

We set

$$\rho(\lambda) = \frac{a(1)\,a(2)\,...a(N)\,\chi(1, N-1; \lambda)}{z - z^{-1}}. \tag{5}$$

We have the formula

$$\frac{1}{2\pi i}\int_{\partial\Gamma} \rho(\lambda)\,\psi^+(n, \lambda)\,\psi^-(m, \lambda) = \delta_{nm}, \ n, m = 0, \pm 1, ..., \tag{6}$$

where i is the imaginary unit, and δ_{nm} is the Kronecker delta.

Now, introduce the function

$$F(n, k, m, r) =$$

$$= \frac{1}{2\pi i}\int_{\partial\Gamma} \rho^2(\lambda)\left[\psi^+(n, \lambda)\,\psi^-(k, \lambda) - \psi^+(k, \lambda)\,\psi^-(n, \lambda)\right]\psi^+(m, \lambda)\,\psi^-(r, \lambda)\,d\lambda. \tag{7}$$

Note that $F(n, k, m, r) = 0$ if $k = n$ and $F(n, k, m, r) = -F(k, n, m, r)$. Therefore, without loss of generality, we can assume that $k > n$.

Theorem 1. *The following equalities hold:*

$$F(n, k, m, r) = 0, \pm r \geq \pm(m \pm (k - n)),$$

$$F(n, k, m, r) = -\frac{1}{a(n)}, \pm r = \pm(m \pm (k - n) \mp 1),$$

Proof. It follows from the definition of $F(n,k,m,r)$ and the residue theorem that

$$F(n,k,m,r) = \underset{\lambda=\infty}{res}\, \rho^2(\lambda) \left[\psi^+(n,\lambda)\psi^-(k,\lambda) - \psi^+(k,\lambda)\psi^-(n,\lambda)\right]\psi^+(m,\lambda)\psi^-(r,\lambda) =$$

$$= \underset{\lambda=\infty}{res}\, \rho^2(\lambda)\psi^+(n,\lambda)\psi^-(k,\lambda)\psi^+(m,\lambda)\psi^-(r,\lambda) -$$

$$- \underset{\lambda=\infty}{res}\, \rho^2(\lambda)\psi^+(k,\lambda)\psi^-(n,\lambda)\psi^+(m,\lambda)\psi^-(r,\lambda) .$$

Let $r \le m-(k-n)$ and $k > n$. Suppose that $n = n_1N + n_2 - 1$, $k = k_1N + k_2 - 1$, $m = m_1N + m_2 - 1$, $r = r_1N + r_2 - 1$, where the numbers n_2, k_2, m_2 and r_2 vary from 1 to N. By (4), we find up to a constant factor that

$$\psi^+(n,\lambda)\psi^-(k,\lambda)\psi^+(m,\lambda)\psi^-(r,\lambda) = \lambda^{k-n+r-m}(1+o(1)),\ \lambda\to\infty.$$

From the definition of $\rho(\lambda)$ we now obtain

$$\rho(\lambda) \sim \frac{1}{\lambda},\ \lambda\to\infty.$$

Recalling that $k-n+r-m \le 0$, from the last two relations we conclude that

$$\underset{\lambda=\infty}{res}\, \rho^2(\lambda)\psi^+(n,\lambda)\psi^-(k,\lambda)\psi^+(m,\lambda)\psi^-(r,\lambda) = 0.$$

On other hand, if $k-n+r-m = 1$, then

$$\underset{\lambda=\infty}{res}\, \rho^2(\lambda)\psi^+(n,\lambda)\psi^-(k,\lambda)\psi^+(m,\lambda)\psi^-(r,\lambda) = -1.$$

Since the condition $k-n+r-m \le 1$, implies$n-k+r-m \le 0$, we similarly find that

$$\underset{\lambda=\infty}{res}\, \rho^2(\lambda)\psi^+(k,\lambda)\psi^-(n,\lambda)\psi^+(m,\lambda)\psi^-(r,\lambda) = 0.$$

Thus,

$$F(n,k,m,r) = 0$$

for $r \le m-(k-n)$ and

$$F(n,k,m,r) = -\frac{1}{a(n)}$$

for $r = m-(k-n)+1$. Moreover, since the function $\rho^2(\lambda)$ takes real values for $\lambda\in\partial\Gamma$ and $\psi^+(k,\lambda)$ while $\psi^-(k,\lambda)$ are complex conjugates, passing to the conjugate expressions in the last formula, we arrive at

$$F(n,k,r,m) = 0$$

for $m \ge r+(k-n)$ and

$$F(n,k,r,m) = -\frac{1}{a(n)}$$

for $m = r+(k-n)-1$. □

The theorem is proved.

Now we introduce the function

$$R(n,m,n_0,m_0) = -a(m)F(m,m_0,n_0,n). \tag{8}$$

We will call the function $R(n,m,n_0,m_0)$ the Riemann function of equation (1). As follows from Theorem 1, the Riemann function vanishes when $n \ge n_0+|m-m_0|$ or $n \le n_0-|m-m_0|$. In addition, this function takes the value 1 on the "characteristics" $n = n_0+|m-m_0|-1$ or $n = n_0-|m-m_0|+1$.

Theorem 2. *The solution of the problem* (1), *(2) can be represented as*

$$U(n,m)=\sum_{j=n-(m-m_0)}^{n+(m-m_0)} f(j)\,\frac{a(m_0)}{a(m)}R(j,m,n,m_0+1)+\sum_{j=n-(m-m_0)+1}^{n+(m-m_0)-1} g(j)\,R(j,m_0,n,m) \quad (9)$$

Proof. It is easy to check that, Eq. (1) has a solution of the form

$$U(n,m)=\frac{1}{2\pi i}\int_{\partial\Gamma}\left\{f_1(\lambda)\,\psi^+(m,\lambda)+f_2(\lambda)\,\psi^-(m,\lambda)\right\}\psi^+(n,\lambda)\,d\lambda$$

where $f_1(\lambda)$ and $f_2(\lambda)$ are functions to be determined. Moreover, from formula (6) it follows that the decomposition formula for $h(n)\in\ell_2(-\infty,+\infty)$ is valid:

$$h(n)=\frac{1}{2\pi i}\int_{\partial\Gamma}\psi^-(n,\lambda)\sum_{m=-\infty}^{+\infty}h(m)\,\psi^+(m,\lambda)\,d\lambda.$$

On this basis, we set

$$\tilde f(\lambda)=\sum_{n=-\infty}^{+\infty}f(n)\,\psi^-(n,\lambda)\,,\, f(n)=\frac{1}{2\pi i}\int_{\partial\Gamma}\tilde f(\lambda)\,\psi^+(n,\lambda)\,d\lambda$$

$$\tilde g(\lambda)=\sum_{n=-\infty}^{+\infty}g(n)\,\psi^-(n,\lambda)\,,\, g(n)=\frac{1}{2\pi i}\int_{\partial\Gamma}\tilde g(\lambda)\,\psi^+(n,\lambda)\,d\lambda$$

and require the solution $U(n,m)$ to satisfy the initial conditions (2). These conditions surely hold if

$$\begin{cases} f_1(\lambda)\,\psi^+(m_0,\lambda)+f_2(\lambda)\,\psi^-(m_0,\lambda)=\tilde f(\lambda)\,, \\ f_1(k)\,\psi^+(m_0+1,\lambda)+f_2(k)\,\psi^-(m_0+1,\lambda)=\tilde g(\lambda)\,. \end{cases}$$

Solving this system of equations with respect to$f_1(k)$ and $f_2(k)$ and using (5), (6), we obtain

$$f_1(\lambda)=a(m_0)\,\tfrac{\tilde f(\lambda)\psi^-(m_0+1,\lambda)-\tilde g(\lambda)\psi^-(m_0,\lambda)}{W(\lambda)},$$

$$f_2(\lambda)=-a(m_0)\,\tfrac{\tilde f(\lambda)\psi^+(m_0+1,\lambda)-\tilde g(\lambda)\psi^+(m_0,\lambda)}{W(\lambda)}$$

Therefore,

$$U(n,m)=\frac{1}{2\pi i}\int\limits_{\partial\Gamma}\frac{\tilde f(\lambda)}{W(\lambda)}a(m_0)\left\{\psi^-(m_0+1,\lambda)\,\psi^+(m,\lambda)-\psi^+(m_0+1,\lambda)\,\psi^-(m,\lambda)\right\}\times$$

$$\times\psi^+(n,\lambda)\,d\lambda+\frac{1}{2\pi i}\int\limits_{\partial\Gamma}\frac{\tilde g(\lambda)}{W(\lambda)}a(m_0)\left\{\psi^+(m_0,\lambda)\,\psi^-(m,\lambda)-\psi^-(m_0,\lambda)\,\psi^+(m,\lambda)\right\}\psi^+(n,\lambda)\,d\lambda.$$

Since $\tilde f(\lambda)=\sum\limits_{j=-\infty}^{+\infty}f(j)\,\psi^-(j,\lambda)$, $\tilde g(\lambda)=\sum\limits_{j=-\infty}^{+\infty}g(j)\,\psi^-(j,\lambda)$, it follows that

$$U(n,m)=+\sum_{j=-\infty}^{+\infty}g(j)\,\frac{1}{2\pi i}\int_{\partial\Gamma}\frac{a(m_0)}{W(\lambda)}\left\{\psi^+(m_0,\lambda)\,\psi^-(m,\lambda)-\right.$$

$$\left.-\psi^+(m,\lambda)\,\psi^-(m_0,\lambda)\right\}\psi^+(n,\lambda)\,\psi^-(j,\lambda)\,d\lambda-$$

$$-\sum_{j=-\infty}^{+\infty} f(j) \frac{1}{2\pi i} \int_{\partial\Gamma} \frac{a(m_0)}{W(\lambda)} \left\{ \psi^+(m_0+1,\lambda)\, \psi^-(m,\lambda) - \right.$$

$$\left. - -\psi^+(m,\lambda)\, \psi^-(m_0+1,\lambda) \right\} \psi^+(n,\lambda)\, \psi^-(j,\lambda)\, d\lambda.$$

Using then formula (7) and theorem 1, we obtain

$$U(n,m) = \sum_{j=-\infty}^{\infty} g(j)\, a(m_0)\, F(m_0,m,n,j) - \sum_{j=-\infty}^{\infty} f(j)\, a(m_0)\, F(m_0+1,m,\, n,j) =$$

$$= -\sum_{j=n-(m-m_0)}^{n+(m-m_0)} f(j)\, a(m_0)\, F(m,m_0+1,n,j) - \sum_{j=n-(m-m_0)+1}^{n+(m-m_0)-1} g(j)\, a(m_0)\, F(m_0,m,n,j)$$

From the last formula and equality (8) follows the validity of representation (3). □

The theorem is proved.

3 Conclusions

In this paper the Cauchy problem is considered for a discrete analogue of a second -order hyperbolic equation with periodic coefficients. The value of the solution at each point is considered as the value of some linear functionality on the initial data. A clear appearance of this functionality has been found. Thus, it was established that the Riemann method is also applicable to discrete hyperbolic equations.

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