Advanced Mathematical Models & Applications
Vol.10, No.1, 2025, pp.88-99
https://doi.org/10.62476/amma.10188

SOLUTE TRANSPORT IN AN ELEMENT OF A FRACTURED-POROUS MEDIUM TAKING INTO ACCOUNT ANOMALIES IN POROUS BLOCK

Bakhtiyor Khuzhayorov[1,2], **Jamol Makhmudov**[1], **Tursunpulat Dzhiyanov**[1,3], **Zarifjon Eshdavlatov**[1*]

[1]Department of Mathematical Modelling, Samarkand State University named after Sharaf Rashidov, 140104, Samarkand, Uzbekistan
[2]V.I. Romanovski Institute of Mathematics, Academy of Sciences, Tashkent, Uzbekistan
[3]Kimyo International University in Tashkent branch Samarkand, Samarkand, Uzbekistan

Abstract. A problem of anomalous solute transport in an element of a fractured-porous medium (FPM), consisting of a single fracture and an adjacent porous block (matrix), is considered. The solute transport in the fracture, which is considered as a one-dimensional object, is carried out on the basis of convective and diffusion phenomena, and in the porous block - on the basis of only diffusion phenomena. In the porous block, the diffusion process occurs anomalously, which means that Fick's law is violated. The solute mass transfer through the common boundary of the fracture and the porous block is also accounted. The mathematical model consists of two differential equations of fractional order for the fracture and the porous block, respectively. A problem of injection of a liquid, containing solute, into a fracture is considered. The problem is numerically solved and the fields of solute concentration and relative mass transfer through the common boundary of the fracture and the porous block are determined. The effect of anomalies on the distribution of solute concentration and relative mass transfer is estimated.

Keywords: Diffusion, fractional differential equations, fracture, fractured-porous medium, porous block, solute transport.

AMS Subject Classification: 76R50.

Corresponding author: Zarifjon Eshdavlatov, Department of Mathematical modelling, Samarkand State University, 140104, Samarkand, Uzbekistan, Tel.: +99889303093, e-mail: *zarifjon1911@gmail.com*

Received: 17 July 2024; Revised: 8 January 2025; Accepted: 13 January 2025; Published: 10 April 2025.

1 Introduction

Oil and gas are the main energy resources in the world and it is necessary to increase in their production which is in need of humanity. The geological structure of oil and gas layers are complex. Analysis of mass transfer in these media shows that the classical diffusion equation based on Fick's law which cannot model adequately the anomalous nature of the solute transport due to the diffusion, observed in field and laboratory experiments. Due to the complexity of the processes of solute transport in fractal media, nonlocal advection-diffusion equations with fractional derivatives with respect to time and coordinates used to compose a mathematical model (Fomin et al., 2011; Zhou, 2023; Yang, 2019; Podlubny, 1998). In this paper, we improved

How to cite (APA): Khuzhayorov, B., Makhmudov, J., Dzhiyanov, T. & Eshdavlatov, Z. (2025). Solute transport in an element of a fractured-porous medium taking into account anomalies in porous block. *Advanced Mathematical Models & Applications*, *10*(1), 88-99 https://doi.org/10.62476/amma.10188

the traditional derivatives of the mathematical model of the considerable assumption with a fractional derivative.

If a porous medium is considered as a multiphase material body in which a representative elementary volume (REV) can be distinguished, it always contain a solid phase, i.e. the skeleton of the porous medium and void space. The size of the REV is such that the physical parameters that represent the distributions of the void space and the solid matrix in it are statistically significant (Bear et al., 2012). In particular, a fractured porous medium can be considered as a composition of two distinct formations, namely individual fractures, fracture networks, and a solid medium or porous matrix that exists between the fractures (also referred to as the porous matrix). A detailed description of FPM and fracture networks can be found in Adler et al. (1999); Sahimi (2011). Whereas, individual fractures look like flat discontinuities or like two solid surfaces, which are surrounded by a porous medium - a matrix. Fractures are considered as porous media with usually higher permeability than the adjacent porous matrix (Rezaei et al., 2022; Nelson et al., 1977; Gale et al., 1980; Grisak et al., 1981).

Numerous experiments and field data showed that the true nature of the solute transport in layered geological structures has anomalous behavior (Li et al., 2021; Sahimi, 1993; Edery, 2014) such as anomalous non-Fickian velocities and mineral concentrations, asymmetry, a sharp leading edge, and elongated "tails" in concentration profiles (Hatano et al., 1993; Haggerty et al., 2002; Radilla et al., 2012; Cardenas et al., 2007). It also confirms the presence of anomalous non-Fickian diffusion in media with a complex structure (Kärger et al., 2016). The anomalous nature of the pollutant transport can also be confirmed by the analysis of the corresponding characteristics of nuclear waste storage facilities (Hodgkinson et al., 2009). These effects cannot be predicted by traditional transport equations even in the case of their various modifications (Chen et al., 2017; Hyman et al., 2021; Haggerty et al., 2000; Sudicky et al., 1982). Considering the effects of spatial heterogeneity of fractured rocks and the effects of flow dispersion by introducing spatially and temporally variable dispersion coefficients and assuming multivelocity solute transport (Fiori et al., 2015; Becker et al., 2000; Bodin et al., 2003; Grathwohl, 2012). The stochastic approach to the derivation of the solute transport equations led to fractional derivatives in these equations (Berkowitz et al., 2006; Benson, 2000; Morales-Casique et al., 2006). A comparison of the results obtained which was based on the equations with fractional derivatives showed a good agreement between these results and the real concentrations of substances measured in a porous medium (Schumer et al., 2009; Liu et al., 2003). For this purpose, in Fomin et al. (2005), a fractured-porous aquifer was considered and the effects of the interaction of a solution with porous blocks were modeled by fractional derivatives with respect to time. Numerical experiments showed that varying the parameters representing the orders of fractional derivatives with respect to the time, makes it possible to obtain various types of time distributions of the concentration that characterize the experimentally observed data.

In Chugunov et al. (2020), fractional differential equations were used to model the motion of radioactive materials in a fracture surrounded by a porous matrix of a fractal structure. A new form of a fractional differential equation was proposed for modeling the migration of radioactive contamination inside a fracture and particular boundary value problems and it was solved using the Laplace transform. The inclusion of fractional derivatives in the model makes by considering the exchange of contaminants between the fracture and the surrounding porous matrix with fractal geometry. Exact solutions were obtained for the concentration of dissolved substances in the fracture and the surrounding porous medium for the case of a non-stationary source of radioactive contamination located at the entrance to the fracture.

In this paper, the solute transport in an element of the FPM is considered, taking into account the anomalous transport. The problem of solute transport in the element of fractured porous medium which consists of a single fracture and a porous block adjacent to it was also considered in works such as Grisak et al. (1981), Khuzhayorov et al. (2020), unlike these works, current study improved the advective dispersion equation model using fractional derivatives.

The solute transport in the fracture was modeled by the anomalous diffusion equation derived from the corresponding anomalous Fickian law. This form of the diffusion equation through a fractal-type porous medium is based on the concept of memory. These equations with initial and boundary conditions are solved numerically. Using the fractional derivative model we consider the mass exchange between the fracture and the neighboring porous matrix of fractal geometry. The problem of solute transport, when a fluid is with substances injected from one end of the fracture, is posed. A numerical solution to the problem is obtained and analyzed. The concentration fields of the solute in the fracture and the adjacent porous medium have been determined. The influence of anomalous phenomena on transport characteristics is studied.

2 Formulation of the Problem

Let's consider an element of FPM consisting of one fracture and a porous block (matrix) adjacent to it (Figure 1). A fracture is a semi-infinite one-dimensional object (Khuzhayorov et al., 2020) and the distribution of the solute and the fluid flow over its cross-section is considered to be uniform. In this formulation, the second dimension of the fracture, i.e. its thickness is not considered here. The porous block occupies the first quadrant of the plane. Thus, the area is $R\{0 \le x < \infty,\, 0 \le y < \infty\}$. Let the fluid with solute concentration c_0 is injected to the fracture with a given constant velocity v from the point $x = 0$ of the fracture. Initially, a fracture and a porous block are considered to be filled with a pure (without substance) liquid. Convective-diffusion solute transport occurs in a fracture, and only diffusion occurs in a porous block. We consider the transfer process in a porous block to be anomalous while in a fracture the process occurs without the manifestation of anomalous phenomena.

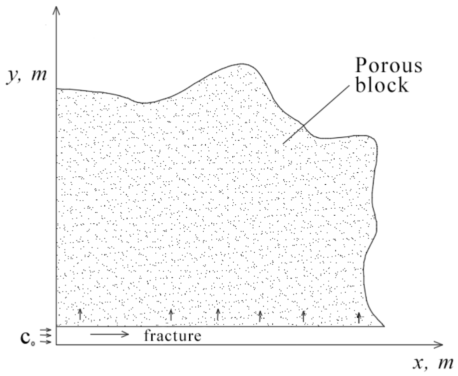

Figure 1: An element of fractured-porous medium

The equations for the solute transport and the fluid flow in the element of FPM are given as

$$\frac{\partial c_f}{\partial t} + v\frac{\partial c_f}{\partial x} = D_f\frac{\partial^2 c_f}{\partial x^2} + m_0 D_m \frac{\partial^{1-\gamma}}{\partial t^{1-\gamma}}\left(\frac{\partial^\delta c_m}{\partial y^\delta}\right)\Bigg|_{y=0}, \tag{1}$$

$$\frac{\partial^\gamma c_m}{\partial t^\gamma} = D_m\frac{\partial^{1+\delta} c_m}{\partial y^{1+\delta}}, \tag{2}$$

where $c_f = c_f(t, x)$ is the solute concentration in the fracture, $c_m = c_m(t, x, y)$ is the solute concentration in the matrix, D_f is the diffusion coefficient in the fracture, v is the velocity of the fluid, $\gamma$and δ are the orders of derivatives ($0 < \gamma,\, \delta \le 1$), m_0 is the coefficient of matrix porosity, t is time, s, x, y are the coordinates. Due to anomality of the solute transport in the matrix, diffusion belongs to non-Fickian law. As a consequence, the diffusion coefficient D_m is the anomalous diffusion coefficient with the dimension $m^{1+\delta}/s^\gamma$.

The initial and boundary conditions for the above formulation of the problem are as follows:

$$c_f(0,x) = 0, \quad c_m(0,x,y) = 0, \tag{3}$$

$$c_f(t,0) = c_0, \quad c_f(t,\infty) = 0, \tag{4}$$

$$c_m(t,x,0) = c_f(t,x), \quad c_m(t,x,\infty) = 0. \tag{5}$$

3 Numerical Solution of the Problem

To solve problem (1) - (5) we use the finite difference method (Samarskii, 2001; Khuzhayorov et al., 2022a;b;c; 2019; 2020). In the area $D = \{0 \le t \le T,\ 0 \le x < \infty,\ 0 \le y < \infty\}$, a net is introduced, where T is the maximum value of the time during the process is studied. To do this, we divide the interval $x \in [0,\infty)$ with the step size h_1, and the interval $y \in [0,\infty)$ with the step size h_2, and the interval $[0,T]$ is divided into J parts with a step τ. As a result, we have a net:

$$\begin{gathered}\bar{\omega}_{h_1 h_2 \tau} = \{(x_i, y_k, t_j),\ i = 0,\ 1,\ 2, ...;\ k = 0,\ 1,\ 2,\ ...;\\ j = 0,\ 1,\ ...,\ J;\ \ x_i = ih_1\ ;\ y_k = kh_2\ \ t_j = j\tau;\ \ \tau = T/J\}\ .\end{gathered} \tag{6}$$

Equations (1) and (2) are approximated on the net $\omega_{h_1 h_2 \tau}$. To do this, we use an explicit difference scheme, and we define fractional derivatives in the sense of Caputo. Therefore, the approximations have the form

$$\begin{gathered}\frac{(c_f)_i^{j+1} - (c_f)_i^j}{\tau} + v\frac{(c_f)_{i+1}^j - (c_f)_i^j}{h_1} = \\ = D_f \frac{(c_f)_{i-1}^j - 2(c_f)_i^j + (c_f)_{i+1}^j}{h_1^2} + \frac{\tau^{\gamma-1}\cdot m_0 \cdot D_m}{\Gamma(1+\gamma)\Gamma(2-\delta)h_2^\delta}\times \\ \times\Big[\sum_{l=0}^{j-1}((c_m)_{i,0}^{l+1} - (c_m)_{i,0}^l - (c_m)_{i,1}^{l+1} + (c_m)_{i,1}^l)((j-l)^{1-\gamma} - (j-l-1)^{1-\gamma})\Big],\end{gathered} \tag{7}$$

$$\begin{gathered}\frac{1}{\Gamma(2-\gamma)\tau^\gamma}[\sum_{l=0}^{j-2}((c_m)_{i,k}^{l+1} - (c_m)_{i,k}^l)\times((j-l+1)^\gamma - (j-l)^\gamma) + (c_m)_{i,k}^{j+1} - (c_m)_{i,k}^j] = \\ = D_m \frac{1}{(2-\delta)h_2^{1+\delta}}\cdot\sum_{r=0}^{k-1}((c_m)_{i,k-(r-1)}^j - 2(c_m)_{i,k-r}^j + (c_m)_{i,k-(r+1)}^j)((r+1)^{1-\delta} - r^{1-\delta}),\end{gathered} \tag{8}$$

where $(c_f)_i^j, (c_m)_{i,k}^j$ are the net values of concentrations $c_f(t,x)$ and $c_m(t,x,y)$ at net points (t_j, x_i) and (t_j, x_i, y_k), respectively, $\Gamma(\cdot)$ is the gamma function. The initial and boundary conditions are discretized as:

$$(c_f)_i^0 = 0, \tag{9}$$

$$(c_m)_{i,k}^0 = 0, \tag{10}$$

$$(c_f)_0^j = c_0, \tag{11}$$

$$(c_m)_{i,0}^j = (c_f)_i^j, \tag{12}$$

Table 1: Values of y at which the mode switching occurs

x	From $\gamma = 1$ to $\gamma = 0.9$	From $\gamma = 1$ to $\gamma = 0.8$	From $\gamma = 1$ to $\gamma = 0.7$
0.1	0.00395300	0.0027371	0.00196010
0.3	0.00937385	0.0068135	0.00511980
0.5	0.01280000	0.0097042	0.00759160
0.7	0.01520000	0.1190000	0.00964800

$$(c_f)_I^j = 0, \tag{13}$$

$$(c_m)_{i,K}^j = 0. \tag{14}$$

Net equations (7) and (8) are reduced in the form as:

$$\begin{aligned}&(c_f)_i^{j+1} = (c_f)_i^j + D_f \frac{(c_f)_{i-1}^j - 2(c_f)_i^j + (c_f)_{i+1}^j}{h_1^2} + \frac{m_0 D_m}{\Gamma(1+\gamma)\tau^{1-\gamma}\Gamma(2-\delta)h_2^\delta}\times \\ &\times \left[\sum_{l=0}^{j-1}((c_m)_{i,0}^{l+1} - (c_m)_{i,0}^l - (c_m)_{i,1}^{l+1} + (c_m)_{i,1}^l)((j-l)^{1-\gamma} - (j-l-1)^{1-\gamma})\right] - \\ &\tau\, v \frac{(c_f)_{i+1}^j - (c_f)_i^j}{h_1}\end{aligned} \tag{15}$$

$$\begin{aligned}&(c_m)_{i\,k}^{j+1} = (c_m)_{i\,k}^j + \frac{D_m \Gamma(2-\gamma)\tau^\gamma}{h_2^{\delta+1}\Gamma(2-\delta)} \sum_{r=0}^{k-1}((c_m)_{i,k-(r-1)}^j - \\ &-2(c_m)_{i,k-r}^j + (c_m)_{i,k-(r+1)}^i) \times ((r+1)^{1-\delta} - r^{1-\delta}) \\ &- \sum_{l=0}^{j-2}((c_m)_{i,k}^{l+1} - (c_m)_{i,k}^l) \times ((j-l+1)^\gamma - (j-l)^\gamma), \\ &i = \overline{0, I-1},\ j = \overline{0, J-1},\ \ k = \overline{0, K-1}.\end{aligned} \tag{16}$$

4 Results and Discussion

Computational work carried out to calculate the fields $(c_f)_i^j, (c_m)_{i,k}^j$ from the (15) and (16). The following values of the initial parameters were used in the calculations: $c_0 = 0.01, D_m = 5 \cdot 10^{-6}$, $D_f = 2 \cdot 10^{-5}$, $v = 1 \cdot 10^{-5}$, $m_0 = 0.35$, $T = 3600$ and various γ, δ (Crank, 1979; Bear, 2013).

Figure 2 shows the sections of the concentration surface c_m for the classical case $\gamma = 1$ and $\delta = 1$. The diffusion transport in a porous block occurs without manifestation of anomalous effects at three points $x = 0.1$, $x = 0.3$ and $x = 0.5$ for different time. In this case, for small values of x relatively large concentrations c_m are obtained. Obviously, this occurs due to the formation of relatively high concentrations of the substance in the fracture and large concentration gradients $\partial c_m/\partial y$ at $y = 0$. The concentration field increases with time in both the direction x and y.

In Figure 3, it is seen that the sections of the concentration surface c_m on three points $x = 0.1\ m$, $x = 0.3\ m$, $x = 0.5\ m$ for the anomalous case, when the values of γ decrease from one and with decreasing γ from 1 the concentrations profiles are generally understated. However, near the fracture, the profiles on decreasing the value of γ takes higher values. Consequently, with a decrease in γ in the concentration profiles, c_m the mode switching from high concentrations to low ones in the direction of y increase from the fracture. Table 1. Shows the values of y in various sections for x when γ decreases from 1 by a certain amount.

Figure 4 shows the sections of the concentration surface c_m by x when decreasing δ from 1. This leads to an increase in the diffusion of the solute transport in the medium. An analysis of the nature of changes in concentration profiles shows that with a decrease in δ from 1 in some

Table 2: Values of y at which a mode switching occurs

x	From $\delta = 1$ to $\delta = 0.9$	From $\delta = 1$ to $\delta = 0.8$	From $\delta = 1$ to $\delta = 0.7$
0.1	0.2050	0.1935	0.1818
0.3	0.1858	0.1765	0.1676
0.5	0.1690	0.1621	0.1548
0.7	0.1545	0.1485	0.1436

Table 3: Values of y at which the mode change occurs

x,m	From $\gamma = 0.9$, $\delta = 0.7$ to $\gamma = 0.7, \delta = 0.9$	From $\gamma = 0.9$, $\delta = 0.7$ to $\gamma = 0.8, \delta = 0.8$	From $\gamma = 0.7$, $\delta = 0.9$ to $\gamma = 0.8, \delta = 0.8$
0.1	0.0005238	0.0005583	0.0004910
0.3	0.0016046	0.0016825	0.0015280
0.5	0.0027422	0.0028401	0.0026430
0.7	0.0039600	0.0040600	0.0038550

zones, a decrease is observed in c_m, and in the rest of the environment it is increases. To get more accuracy in the nature of the change in c_m, it's profiles are constructed. As in the previous case, there is a mode switching in the change of c_m. However, in contrast to the previous case, due to the enhancement of diffusion transport in the porous block near the fracture in the direction of y, a zone with reduced values of c_m is formed. After switching the mode with decrease in δ ,a zone with elevated values of c_m is formed. To evaluate mode switching points in the direction of y on various sections along x shown in Table 2 , which shows the values at y the mode switching occurs.

It should be noted that, in contrast to the previous case, the mode switching occurs at relatively large values of y.

Various values of $\gamma\,(< 1)$ and $\delta\,(< 1)$ are also analyzed. Considering that the decrease in γ leads to the slow diffusion, and a decrease in δ to the fast diffusion. For various combinations of γ and δ depending on their values one can get both slow and fast diffusion. Figure 5 shows the cross sections of the surface c_m along the x direction for three pairs of values of γ and δ. In case $\gamma = 0.9$ and $\delta = 0.7$received fast diffusion and in the case of $\gamma = 0.7,\ \ \delta = 0.9$, it is slow diffusion, and intermediate variant of diffusion for $\gamma = 0.8,\ \ \delta = 0.8$. As in the previous cases, there is a mode switching in the change of c_m according to y. The values of y at which the mode switching occurs are shown in Table 3. The data in Table 3 shows that as the distance increases, the mode switching x points shift in the direction of growth of y.

Figures 6-8 depicts the changes in the relative flow rates of solute across the common boundary of the medium.

The current relative flow rate of the solute through $y = 0$ is defined as

$$Q = -m_0 D_m \frac{\partial^{1-\gamma}}{\partial t^{1-\gamma}} \left(\frac{\partial^{\delta} c_m}{\partial y^{\delta}} \right) \Bigg|_{y=0} . \tag{17}$$

The total Q_{total} and summing Q_{sum} relative flow rates across the border $y = 0$ are also determined.

$$Q_{total} = \int_0^{\infty} Q\, dx, \tag{18}$$

$$Q_{sum} = \int_0^t Q_{total} dt = \int_0^t \int_0^\infty Q\, dxdt. \tag{19}$$

It can be seen from Figure 6(a), with a decrease of γ the current flow rates through $y = 0$ increases. This is explained by the fact that due to slow diffusion in the porous block and relatively large concentration gradients that are characterized by $\frac{\partial^{1-\gamma}}{\partial t^{1-\gamma}}(\frac{\partial^\delta c_m}{\partial y^\delta})$, are formed across the common boundary $y = 0$. In the case of decreasing δ, fast diffusion occurs and at $y = 0$ relatively smaller concentration gradients are formed. This leads to a decrease in flow rates through $y = 0$ (Figure 6(b)). With a simultaneous decrease in values of γ and δ, different flow rate modes can be obtained, and there can be both increase and decrease in its values (Figure 6(c)).

In turn, the total and summing flow rates across the boundary increases as the current flow rates increases and, conversely, decreases as it decreases (Figure 7, and 8).

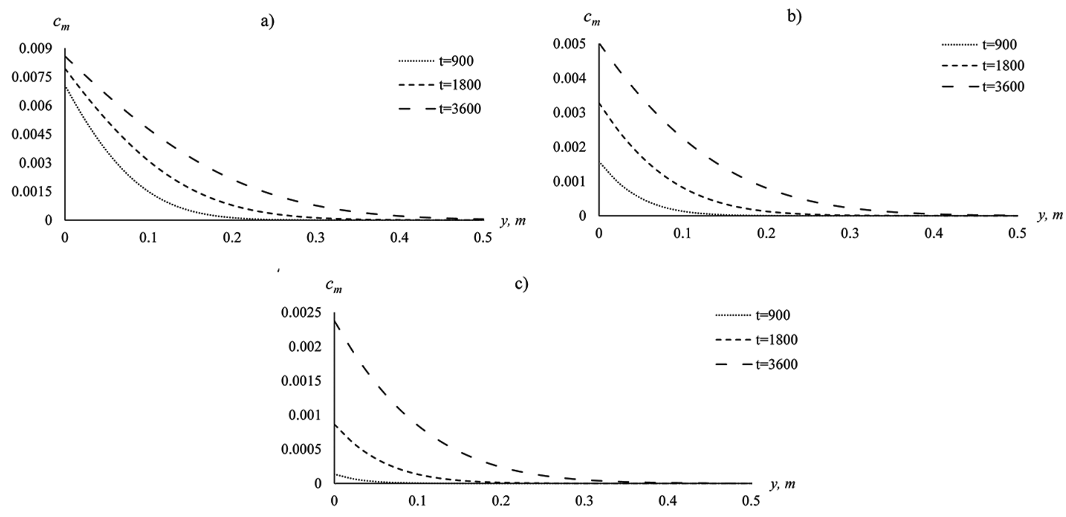

Figure 2: Concentration profiles c_m at $\gamma = 1$, $\delta = 1$ $t = 3600\,\mathrm{s}$ and different time values, $x = 0.1\ m$(a), $0.3\ m$ (b), $0.5\ m$ (c)

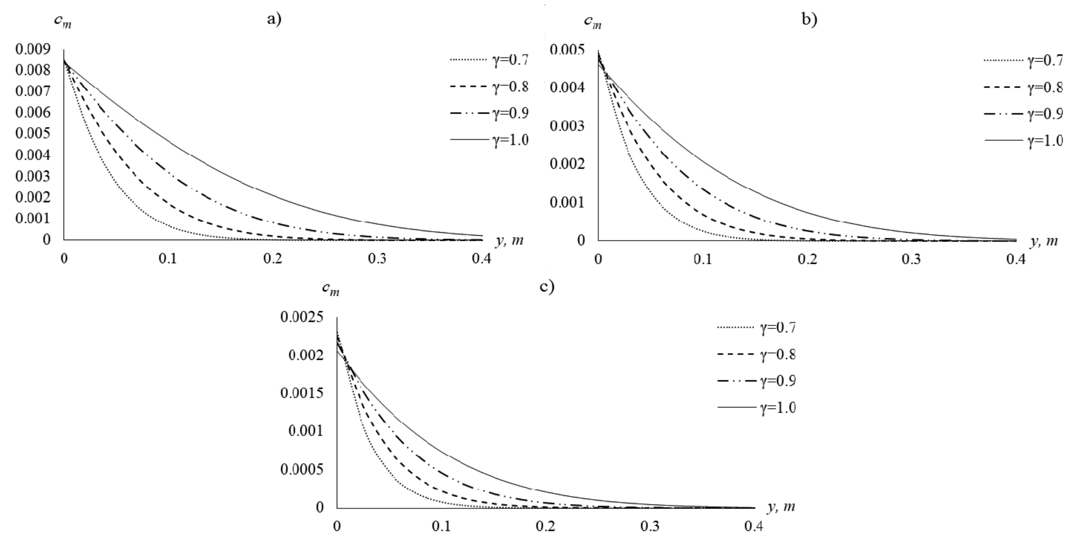

Figure 3: Concentration profiles c_m at $\delta = 1$, $t = 3600\,\mathrm{s}$ and different values of γ, $x = 0.1\ m$(a), $0.3\ m$ (b), $0.5\ m$ (c)

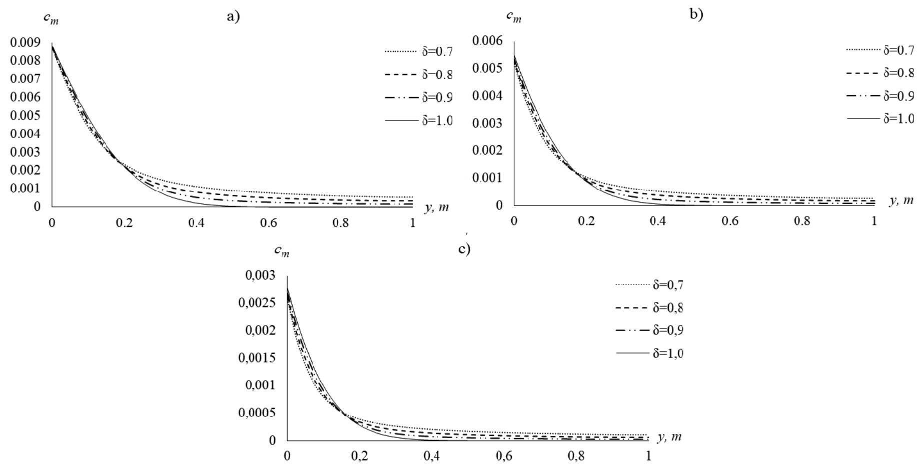

Figure 4: Concentration profiles c_m at $\gamma = 1$, $t = 3600\,\mathrm{s}$ and different values of δ, $x = 0.1\ m$ (a), $0.3\ m$ (b), $0.5\ m$ (c)

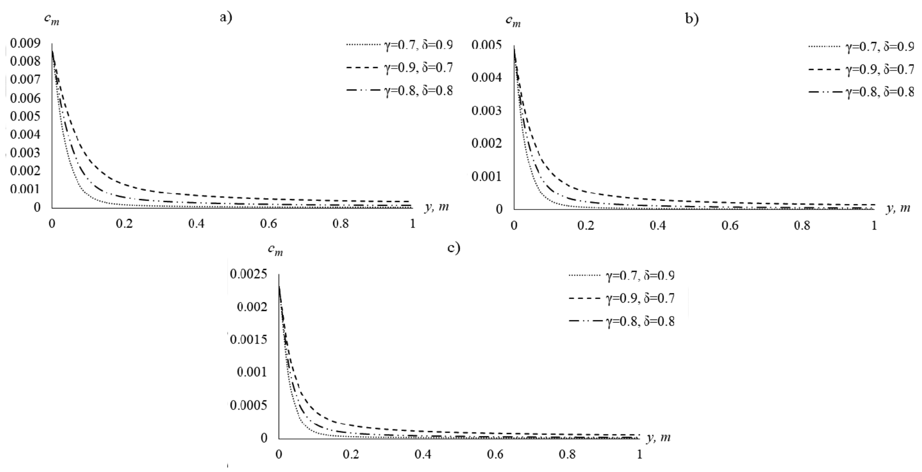

Figure 5: Concentration profiles of c_m at $t = 3600\,\mathrm{s}$ and different values of γ, δ, $x = 0.1\ m$ (a), $0.3\ m$ (b), $0.5\ m$ (c)

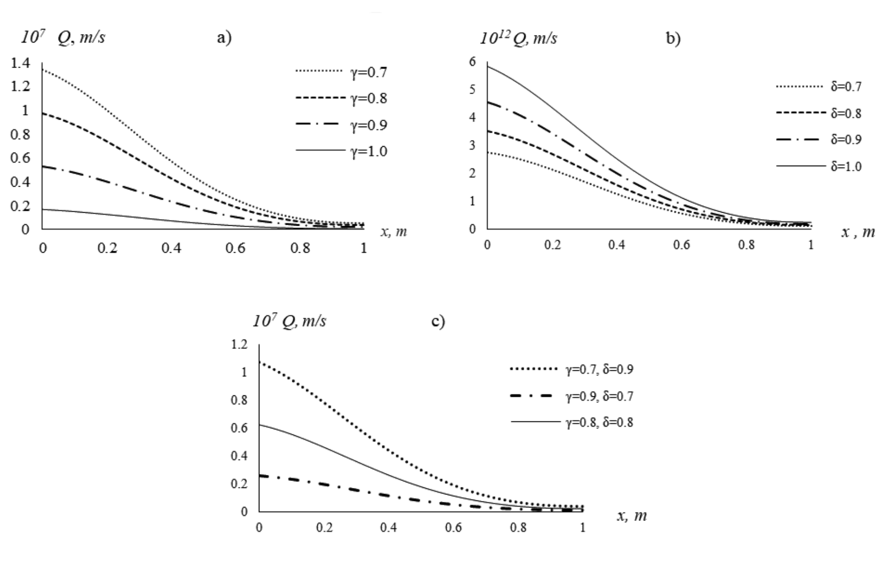

Figure 6: Current flow rates profiles Q at $t = 3600\,s$ $\delta = 1$ (a), $\gamma = 1$ (b), different values of γ, δ (c)

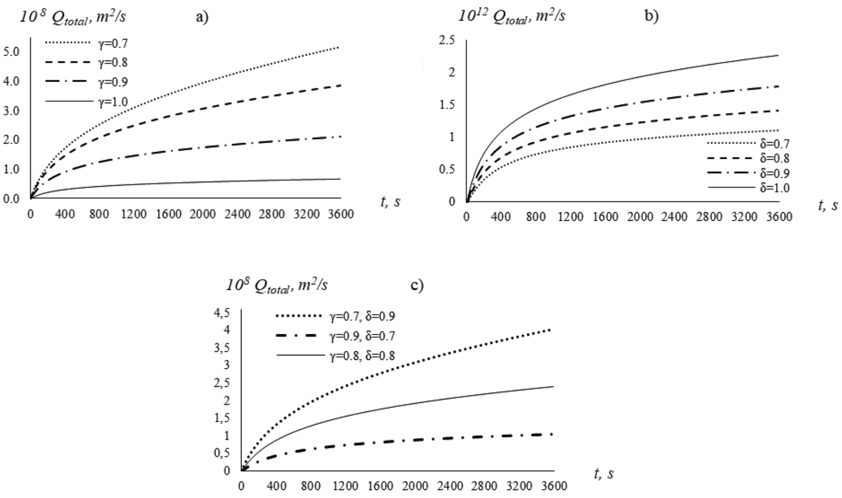

Figure 7: Total flow rates dynamics Q_{total} at $\delta = 1$ (a), $\gamma = 1$ (b), different values of γ, δ(c)

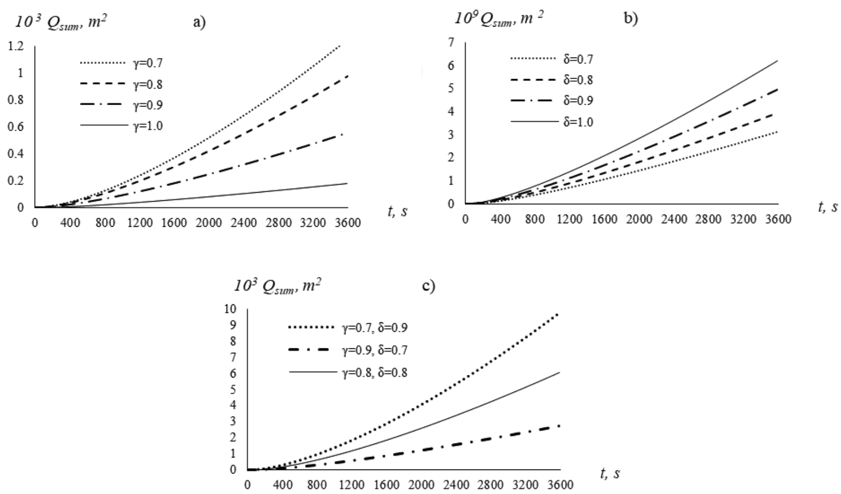

Figure 8: Summing flow rates dynamics Q_{sum} at $\delta = 1$ (a), $\gamma = 1$ (b), different values of γ, δ(c)

5 Conclusion

The problem of anomalous solute transport in FPM element consisting of a single fracture and a porous block attached to it. Fractional differential equations (FDE) are significantly more successful in understanding more in the complex transport phenomena than classical models designed for mass transport. Unlike traditional Fickian diffusion, which assumes that dissolved particles transport independently and randomly, FDE takes into account non-local and non-Markovian behavior. This means they can model phenomena such as long-range memory and anomalous diffusion, which are often observed in porous media. It is shown that the anomalous nature of the solute transport in the porous block of the FPM element has a different effect on the distribution of its concentration both in the fracture and in the porous block. With slow diffusion in a porous block, the process of transport in a fracture is intensified, on the contrary, with fast diffusion in a porous block. When the orders of derivatives in the diffusion equations are changed, the mode switching‘s profiles of the solute concentration in the porous block changes from higher to lower or vice versa. Flow rates through the common boundary of a fracture and a porous block tends to increase or decrease depending on the manifestation of the transfer anomaly. The obtained results reflect the features of solute transport process in FPM. Although here the problem is considered only in a highlighted element of the FPM, the general dynamics of the solute transport characteristics can be used to explain the solute transport in the FPM as a whole. These results are very important in the analysis and design of water flooding processes of oil reservoirs with fractured-porous rocks, where various pollutants enter the reservoir together with water. As a further development of the work, it is possible to consider a heterogeneous porous block, where the solute transport process, in addition to anomalous effects, occurs in different zones of the block in different ways. In addition, it is necessary to consider the anomalous effects of solute transport not only in the porous block, but also in the fracture.

Acknowledgement Authors greatly appreciated anonymous reviewers for their extremely detailed and useful comments.

References

Adler, P.M., Thovert, J.F. (1999). *Theory and applications of transport in porous media: fractures and fracture networks*. Kluwer Academic Publishers: Dordrecht. https://doi.org/10.1007/978-94-017-1599-7

Bear, J. (2013). *Dynamics of fluids in porous media*. Courier Corporation. https://doi.org/10.1097/00010694-197508000-00022

Bear, J., Tsang, C.F, & De Marsily, G. (2012). *Flow and contaminant transport in fractured rock*. Academic Press, 561. https://doi.org/10.1016/B978-0-12-083980-3.50005-X

Becker, M.W., Shapiro, A.M. (2000). Tracer transport in fractured crystalline rock: evidence of nondiffusive breakthrough tailing. *Water Resources Research 36*(7), 1677–1686. https://doi.org/10.1029/2000WR900080

Benson, D.A., Wheatcraft, S.W., & Meerschaert, M.M. (2000). Application of a fractional advectiondispersion equation. *Water Resources Research. 36*(6), 1403-1412. https://doi.org/10.1029/2005RG000178

Berkowitz, B., Cortis, A., Dentz, M., & Scher, H. (2006). Modeling non-Fickian transport in geological formations as a continuous time random walk. *Reviews of Geophysics. 44*. https://doi.org/10.1029/2005RG000178

Bodin, J., Delay, F., & De Marsily, G. (2003). Solute transport in a single fracture with negligible matrix permeability: 2. mathematical formalism. *Hydrogeology Journal*. 11, 434-454. https://doi.org/10.1007/s10040-003-0269-1

Cardenas, M.B., Slottke, D.T., Ketcham, R.A., & Sharp, Jr J.M. (2007). Navier -Stokes flow and transport simulations using real fractures shows heavy tailing due to eddies. *Geophysical Research Letters 34*(14). https://doi.org/10.1029/2007GL030545

Chen, K., Zhan, H., & Yang, Q. (2017). Fractional models simulating non- Fickian behavior in four-stage single-well push-pull tests. *Water Resources Research 53*(11), 9528-9545. https://doi.org/10.1002/2017WR021411

Chugunov, V., Fomin, S. (2020). Effect of adsorption, radioactive decay and fractal structure of matrix on solute transport in fracture. *Philosophical Transactions of the Royal Society A. 378*(2172), 20190283. https://doi.org/10.1098/rsta.2019.0283

Crank, J. (1979). *The mathematics of diffusion.* Oxford University Press.

Edery, Y., Guadagnini, A., Scher, H., & Berkowitz, B. (2014). Origins of anomalous transport in heterogeneous media: structural and dynamic controls. *Water resources research 50*(2), 1490–1505. https://doi.org/10.1002/2013WR015111

Fiori, A., Becker, M.W. (2015). Power law breakthrough curve tailing in a fracture: The role of advection. *Journal of Hydrology* 525, 706-710. https://doi.org/10.1016/j.jhydrol.2015.04.029

Fomin, S., Chugunov, V., & Hashida, T. (2011). Mathematical modeling of anomalous diffusion in porous media. *Fractional Differential Calculus. 1*(1), 1-28. https://dx.doi.org/10.7153/fdc-01-01

Fomin, S., Chugunov, V., & Hashida, T. (2005). The effect of non-Fickian diffusion into surrounding rocks on contaminant transport in a fractured porous aquifer. *Proceedings of the Royal Society a Mathematical, Physical and Engineering Sciences. 461*(2061), 2923-2939. https://doi.org/10.1098/rspa.2005.1487

Gale, J.E., Quinn, O., Wilson, C., Forster, C., Witherspoon, P.A., & Jacobson, L. (1980). Hydrogeologic Characteristics of Fractured Rocks for Waste Isolation—The Stripa Experience. *Scientific Basis for Nuclear Waste Management.* 507-518. https://doi.org/10.1007/978-1-4684-3839-0-61

Grathwohl, P. (2012). Diffusion in natural porous media: contaminant transport, sorption/desorption and dissolution kinetics. *Springer Science & Business Media. 1.* https://doi.org/10.1007/978-1-4615-5683-1

Grisak, G.E., Pickens, J.F. (1981). An analytical solution for solute transport through fractured media with matrix diffusion. *Journal of Hydrology 52*(1-2), 47-57. https://doi.org/10.1016/0022-1694(81)90095-0

Haggerty, R., McKenna, S.A., & Meigs, L.C. (2000). On the late-time behavior of tracer test breakthrough curves. *Water Resources Research. 36*(12), 3467–3479. https://doi.org/10.1029/2000WR900214

Haggerty, R., Wondzell, S.M., & Johnson, M.A. (2002). Power-law residence time distribution in the hyporheic zone of a 2nd-order mountain stream. *Geophysical Research Letters 29*(13), 18-1-18-4. https://doi.org/10.1029/2002GL014743

Hatano, Y., Hatano, N. (1998). Dispersive transport of ions in column experiments: an explanation of long-tailed profiles. *Water resources research. 34*(5), 1027–1033. https://doi.org/10.1029/98WR00214

Hodgkinson, D., Benabderrahmane, H., Elert, M., Hautojärvi, A., Selroos, J.O., Tanaka, Y., & Uchida, M. (2009). An overview of Task 6 of the Aspö Task Force: modeling groundwater and solute transport: improved understanding of radionuclide transport in fractured rock. *Hydrogeology Journal 17*(5), 1035–1049. https://doi.org/10.1007/s10040-008-0416-9

Hyman, J.D., Dentz, M. (2021). Transport upscaling under flow heterogeneity and matrix-diffusion in three-dimensional discrete fracture networks. *Advances in Water Resources. 155*, 103994. https://doi.org/10.1016/j.advwatres.2021.103994

Kärger, J., Ruthven, D.M. (2016). Diffusion in nanoporous materials: fundamental principles, insights and challenges. *New Journal of Chemistry. 40*(5), 4027–4048. https://doi.org/10.1039/C5NJ02836A

Khuzhayorov, B.Kh., Mustafokulov, J.A., Dzhiyanov, T.O., & Zokirov, M.S. (2022a). Solute Transport with Non-Equilibrium Adsorption in a NonHomogeneous Porous Medium. *WSEAS transactions on fluid mechanics. 17*, 181-188. https://doi.org/10.37394/232013.2022.17.18

Khuzhayorov, B.Kh., Dzhiyanov, T.O., Mamatov, Sh.S., & Shukurov, V.S. (2022b). Mathematical Model of Substance Transport in Two-Zone Porous Media. *AIP Conference Proceedings*, *2637*(1), 040011. (htFPM://doi.org/10.1063/5.0118450).

Khuzhayorov, B.Kh ., Dzhiyanov, T.O., Eshdavlatov, Z.Z. (2022c). Numerical Investigation of Solute transport in a non-homogeneous porous medium using nonlinear kinetics. *International Journal of Mechanical Engineering and Robotics Research. 11*(2), 79–85. https://doi.org/10.18178/ijmerr.11.2.79-85

Khuzhayorov, B.Kh., Djiyanov, T.O., & Yuldashev, T.R. (2019). Anomalous Nonisothermal Transfer of a Substance in an Inhomogeneous Porous Medium. *Journal of Engineering Physics and Thermophysics this link is disabled. 92*(1), 104–113. https://doi.org/10.1007/s10891-019-01912-y

Khuzhayorov, B., Saydullaev, U., & Fayziev, B. (2020). Relaxation equations of consolidating cake filtration. *Journal of Advanced Research in Fluid Mechanics and Thermal Sciences*, *74*(2), 168-182. https://doi.Org/10.37934/arfmts.74.2.168182