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SOLUTION OF THE MIXED PROBLEM FOR THE NON-HOMOGENEOUS PARABOLIC EQUATION WITH TIME DERIVATIVE IN THE BOUNDARY CONDITION

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Abstract. In the paper a mixed problem with time derivative in the boundary conditions for the second order inhomogeneous parabolic equation with complex coefficients is considered. It is impossible to apply the Fourier decomposition method to find a solution to problems of this type. Despite the fact that the boundary conditions of the considered problem seem to be local, in fact they do not contain the property of locality, since the boundary conditions involve a derivative with respect to time. The spectral problem in the complex plane corresponding to this problem is formulated and studied. Then the solution of the corresponding Cauchy problem is constructed. The solution of the mixed problem is found in the form of an infinite sequence of deductions using the residue method proposed by M.L. Rasulov.

Keywords: Parabolic type equation, mixed problem, residue method, Cauchy problem, Green's function, asymptotics, eigenvalues.

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1 Introduction

The work is devoted to finding a solution to the mixed problem for the parabolic equation (Stepanov, 2004; Tikhonov & Samarsky, 2004; Vladimirov & Zharinov, 2008). The derivatives included in the boundary conditions in the problem are partial derivatives of the sought function with respect to both the time and the space variables (Abbasova & Ahmedov, 2020). It is impossible to apply the Fourier decomposition method to find a solution to problems of this type. Despite the fact that the boundary conditions of the considered problem seems to be local, in fact they do not contain the property of locality, since the boundary conditions involve a derivative with respect to time (Ahmadov & Allahyarova, 2023; Khankishiev, 2012; 2012). Therefore, the eigenfunctions of the corresponding problem do not form a complete system. When finding a solution to this problem, the residue method is applied (Asadova & Aliyev, 2012; Ahmadov, 2023; Iqbal et al., 2024). According to the general scheme of the residue method, two problems are posed corresponding to the initial mixed problem. One of these problems is the boundary value problem for the ordinary differential equation containing a complex parameter (i.e., the spectral

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problem) (An & Cao, 2023; Konechnaja et al., 2023) and the other is the Cauchy problem (Filippov, 1970; Mamedov et al., 2014; Salazar & Mendioroz, 2012; Guseynov & Mamedov, 2024). Note that the derivatives involved in the spectral problem are with respect to the space variable (with respect to x), while the derivatives involved in the Cauchy problem are with respect to time (with respect to t). In the paper, the Laplace integral transform is applied to the initial problem to construct the spectral problem (Knyazev, 1969). The boundary conditions of the obtained spectral problem involve a complex parameter (Ahmadov & Alasgarova, 2014; Ahmadov & Alasgarova, 2012). The Green function of the obtained spectral problem was constructed and the analyticity property of this function is studied (Fedoryuk, 2022; Markushevich, 1996). The asymptotics of the zeros of the characteristic determinant of the Green function are found and it is shown that these are simple poles of the Green function (Fedoryuk, 1987). Estimates for the Green's function and its derivatives with respect to x outside the δ neighborhood of the found poles are obtained (Asadova & Aliyev, 2012). Using the solution of the corresponding Cauchy problem and the Green's function, the solution of the mixed problem is found in the form of a sequence of residues.

In order to find the asymptotics of eigenvalues and the Green's function, the separation theorem for functions of a certain class is used. Then, the solution of the corresponding Cauchy problem is found and the solution found was evaluated for large values of the parameter. Finally, using the Green's function and the solution of the corresponding Cauchy problem, the formal solution of the original problem is derived in the form of an infinite sequence of residues. In order to justify this solution, certain algebraic conditions are imposed on the initially given functions, along with the smoothness conditions.

Let us consider the following mixed problem for the second order non-homogenous parabolic type equation i.e. it needs to find a solution of the equation

$$\frac{\partial \upsilon}{\partial t} - a\frac{\partial^2 \upsilon}{\partial x^2} + b\upsilon = f(x,t), x \in (0,1), t > 0 \tag{1}$$

with boundary conditions

$$\left.\begin{array}{l} \left[\frac{\partial \upsilon}{\partial t} + \alpha_0 \frac{\partial \upsilon}{\partial x} + \alpha_1 \upsilon\right]_{x=0} = 0 \\ \left[\frac{\partial \upsilon}{\partial t} + \beta_0 \frac{\partial \upsilon}{\partial x} + \beta_1 \upsilon\right]_{x=1} = 0 \end{array}\right\} \tag{2}$$

and initial condition

$$\upsilon(x,0) = \Phi(x), \ x \in [0,1] \tag{3}$$

Here $\Phi(x)$ is a differentiable function satisfying the condition$\Phi(0) = 0$; the coefficients a and b are complex constants; $Rea > 0$; the coefficients α_k, $\beta_k (k \in 0,1)$ in the boundary conditions are complex constants; $f(x;t)$ is a smooth enough function in the set $D = \{(x,t) : 0 \leq x \leq 1, t \geq 0\}$; $f(0,0) = 0$.

Following the procedures of the residual method first a boundary value and Cauchy problem have to be constructed corresponding to problem (1)-(3). Since the variables in the equation and boundary conditions are not dependent, the non-trivial solution to the homogeneous equation corresponding to (1)

$$\frac{\partial \upsilon}{\partial t} - a\frac{\partial^2 \upsilon}{\partial x^2} + b\upsilon = 0 \tag{4}$$

that satisfies condition (2) we search in the form

$$\upsilon(x,t) = y(x)z(t). \tag{5}$$

Considering (5) in boundary condition (2) and equation (4) the following spectral problem can be obtained

$$ay'' - \lambda^2 y = h(x), \tag{6}$$

$$\left.\begin{array}{l} \left[\alpha_0 y' + (\lambda^2 + \alpha_1 - b)y\right]\big|_{x=0} = 0 \\ \left[\beta_0 y' + (\lambda^2 + \beta_1 - b)y\right]\big|_{x=1} = 0 \end{array}\right\}, \tag{7}$$

where $h(x)$ is any continuous function.
Problem (6)-(7) is called to be a spectral problem corresponding to mixed problem (1)-(3).

According to the general scheme of the residue theory, we obtain the following Cauchy problem corresponding to mixed problem (1)-(3) (Abbasova & Ahmedov, 2020)

$$\frac{\partial z}{\partial t} + bz + \lambda^2 z = f(x,t), \tag{8}$$

$$z(x,0) = \Phi(x). \tag{9}$$

2 Construction of the solution to the spectral problem

Now let us give the procedure for constructing the solution to the spectral problem using the Green's function. The solution that satisfies the spectral problem (6), (7) is constructed using the Green's function as follows

$$y(x,\lambda) = \int_0^1 G(x,\xi,\lambda)h(\xi)d\xi, \tag{10}$$

where $G(x,\xi,\lambda)$ is the Green's function that is defined as

$$G(x,\xi,\lambda) = \frac{\Delta(x,\xi,\lambda)}{\Delta(\lambda)}. \tag{11}$$

$\Delta(\lambda)$ is called the characteristic determinant of the Green's function of problem (6), (7) that is defined by the following determinant

$$\Delta(\lambda) = \begin{vmatrix} \alpha_0 \frac{\lambda}{\sqrt{a}} + \lambda^2 + \alpha_1 - b & -\alpha_0 \frac{\lambda}{\sqrt{a}} + \lambda^2 + \alpha_1 - b \\ \left(\beta_0 \frac{\lambda}{\sqrt{a}} + \lambda^2 + \beta_1 - b\right) e^{\frac{\lambda}{\sqrt{a}}} & \left(-\beta_0 \frac{\lambda}{\sqrt{a}} + \lambda^2 + \beta_1 - b\right) e^{-\frac{\lambda}{\sqrt{a}}} \end{vmatrix}. \tag{12}$$

Finally, the numerator of the fraction in the right hand side of identity (11) is defined by the formula

$$\Delta(x,\xi,\lambda) = \begin{vmatrix} g(x,\xi,\lambda) & y_1(x,\lambda) & y_2(x,\lambda) \\ \left(\alpha_0 g'_x + \left(\lambda^2 + \alpha_1 - b\right) g\right)_{x=0} & & \Delta(\lambda) \\ \left(\beta_0 g'_x + \left(\lambda^2 + \beta_1 - b\right) g\right)_{x=1} & & \end{vmatrix}. \tag{13}$$

The Cauchy function $g(x,\xi,\lambda)$ here is defined as

$$g(x,\xi,\lambda) = \begin{cases} \frac{\sqrt{a}}{4\lambda}\left\{e^{\frac{\lambda}{\sqrt{a}}(x-\xi)} - e^{-\frac{\lambda}{\sqrt{a}}(x-\xi)}\right\}, & 0 \le \xi \le x \le 1 \\ -\frac{\sqrt{a}}{4\lambda}\left\{e^{\frac{\lambda}{\sqrt{a}}(x-\xi)} - e^{-\frac{\lambda}{\sqrt{a}}(x-\xi)}\right\}, & 0 \le x \le \xi \le 1. \end{cases}$$

The functions $y_1(x,\lambda)$ and $y_2(x,\lambda)$ are linearly independent partial solutions of the homogeneous equation corresponding to (6).

3 Finding the asymptotics of the zeros of the characteristic determinant

Here we construct the asymptotics of the determinant $\Delta(\lambda)$ for large enough values of $|\lambda|$. To do this let us write determinant (12) in the form

$$\begin{aligned} \Delta(\lambda) = \left(\frac{\alpha_0\lambda}{\sqrt{a}} + \lambda^2 + \alpha_1 - b\right)\left(-\beta_0 \frac{\lambda}{\sqrt{a}} + \lambda^2 + \beta_1 - b\right) e^{-\frac{\lambda}{\sqrt{a}}} - \\ - \left(-\alpha_0 \frac{\lambda}{\sqrt{a}} + \lambda^2 + \alpha_1 - b\right)\left(\beta_0 \frac{\lambda}{\sqrt{a}} + \lambda^2 + \beta_1 - b\right) e^{\frac{\lambda}{\sqrt{a}}}. \end{aligned} \tag{14}$$

Denote $\delta_k = (-1)^{k-1}\frac{1}{\sqrt{a}}$ $(k = 1,\ 2)$. To estimate the Green's function we divide the λ complex plane by the sections in each of which the inequality $Re\lambda\delta_1 < 0 < Re\lambda\delta_2$ is satisfied. Satisfying this inequality provides a comparison of the modulus of the exponential function. For this, let's first calculate the expressions $Re\lambda\delta_k,\ (k = 1,\ 2)$

$$Re\lambda\delta_1 = Re\,(\lambda_1 + i\lambda_2)\,(\delta_{11} + i\delta_{12}) = \lambda_1\delta_{11} - \lambda_2\delta_{12},$$

$$Re\lambda\delta_2 = Re\,(\lambda_1 + i\lambda_2)\,(\delta_{21} + i\delta_{22}) = \lambda_1\delta_{21} - \lambda_2\delta_{22}.$$

As we see from the expression of δ_k, if $Re\lambda\delta_1 < 0$,then $Re\lambda\delta_2 > 0$. In this case for the large enough values of $|\lambda|$ in the expression of $\Delta(\lambda)$ the expression $e^{\frac{\lambda}{\sqrt{a}}}$ will be a decreasing function, while $e^{-\frac{\lambda}{\sqrt{a}}}$ will be an increasing function, like an exponent function.

Therefore, if we move the multiplication $\lambda^4 e^{-\frac{\lambda}{\sqrt{a}}}$ outside the brackets on the right hand side of (14), then we get the following expression for $\Delta(\lambda)$

$$\begin{gathered}\Delta(\lambda) = \lambda^4 e^{-\frac{\lambda}{\sqrt{a}}}\left\{\left(\tfrac{\alpha_0}{\lambda\sqrt{a}} + 1 + \tfrac{\alpha_1}{\lambda^2} - \tfrac{b}{\lambda^2}\right)\left(-\tfrac{\beta_0}{\lambda\sqrt{a}} + 1 + \tfrac{\beta_1}{\lambda^2} - \tfrac{b}{\lambda^2}\right) - \right.\\ \left. - \left(-\tfrac{\alpha_0}{\lambda\sqrt{a}} + 1 + \tfrac{\alpha_1}{\lambda^2} - \tfrac{b}{\lambda^2}\right)\left(\tfrac{\beta_0}{\lambda\sqrt{a}} + 1 + \tfrac{\beta_1}{\lambda^2} - \tfrac{b}{\lambda^2}\right) e^{\frac{2\lambda}{\sqrt{a}}}\right\}.\end{gathered} \tag{15}$$

The last equality may be written as

$$\Delta\left(\lambda\right) = \lambda^4 e^{-\frac{\lambda}{\sqrt{a}}}\left\{\Delta_0(\lambda) + H(\lambda)\right\}. \tag{16}$$

Here

$$\Delta_0(\lambda) = 1 - e^{\frac{2\lambda}{\sqrt{a}}}, \tag{17}$$

$$\begin{gathered}H(\lambda) = \tfrac{1}{\lambda\sqrt{a}}\left(\alpha_0 - \beta_0\right)\left(1 + e^{\frac{2\lambda}{\sqrt{a}}}\right) + \tfrac{1}{\lambda^2}\left(\alpha_1 + \beta_1 - \tfrac{\alpha_0\beta_0}{a} - 2b\right) \times \\ \times\left(1 - e^{\frac{2\lambda}{\sqrt{a}}}\right) + \tfrac{1}{\lambda^3\sqrt{a}}\left(\alpha_0\beta_1 - \alpha_1\beta_0 - b\left(\alpha_0 - \beta_0\right)\left(1 + e^{\frac{2\lambda}{\sqrt{a}}}\right)\right) + \\ + \tfrac{1}{\lambda^4}\left(\alpha_1\beta_1 + \alpha_1 b - \beta_1 b + b^2\right)\left(1 - e^{\frac{2\lambda}{\sqrt{a}}}\right).\end{gathered} \tag{18}$$

As is seen from expression (18) of $H(\lambda)$ for the large enough values of $|\lambda|$ and for $\lambda \in E_2, E_2 = \{\lambda : Re\lambda\delta_1 > 0\}$ the following estimate is valid for $H(\lambda)$

$$H(\lambda) = O\left(\frac{1}{|\lambda|}\right) \tag{19}$$

or

$$|H(\lambda)| \leq \frac{c}{|\lambda|},\ c > 0. \tag{20}$$

From expression (16) of the characteristic determinant and from (20) we obtain that $\Delta_0(\lambda)$ is the main part of the determinant $\Delta(\lambda)$. Therefore by the Roche theorem (Rasulov, 1989) the number of zeros of $\Delta(\lambda)$ and $\Delta_0(\lambda)$ in the set E_2 is same and the zeros of $\Delta(\lambda)$ differs from the zeros of $\Delta_0(\lambda)$ by infinitely small quantity.

Solving the equation $\Delta_0(\lambda)$=0 instead of the equation $\Delta(\lambda)$=0 we get

$$\Delta_0(\lambda) = 0 \Leftrightarrow 1 - e^{\frac{2\lambda}{\sqrt{a}}} = 0 \tag{21}$$

$$\lambda_\nu = i\pi\sqrt{a}\nu, \quad \nu = 1, 2, 3... \tag{22}$$

Then by the Roche theorem we get the following asymptotic formulas for the roots of the equation $\Delta(\lambda)$=0

$$\lambda_\nu = i\pi\sqrt{|a|}\nu\left\{1 + O\left(\frac{1}{\nu}\right)\right\}, \qquad \nu \to \infty. \tag{23}$$

Now we define the order of the roots obtained by formula (22). For this purpose calculate $\Delta'(\lambda)$

$$\Delta'(\lambda) = \left(\lambda^4 e^{-\frac{\lambda}{\sqrt{a}}}\right) \{\Delta_0(\lambda) + H(\lambda)\}' =$$

$$= \left(4\lambda^3 - \frac{\lambda^5}{\sqrt{a}}\right) e^{-\frac{\lambda}{\sqrt{a}}} \{\Delta_0(\lambda) + H(\lambda)\} + +\lambda^4 e^{-\frac{\lambda}{\sqrt{a}}} \left\{\Delta'_0(\lambda) + H'(\lambda)\right\}.$$

From relations (16), (19) and (21) and from Roche theorem we get fulfilment of the identity $\Delta_0(\lambda) + H(\lambda) = 0$. It gives

$$\Delta'(\lambda_\nu) = \lambda_\nu^4 e^{-\frac{\lambda_\nu}{\sqrt{a}}} \left\{\Delta'_0(\lambda_\nu) + H'(\lambda_\nu)\right\}. \tag{24}$$

From (24) we have $\Delta'(\lambda_\nu) \neq 0$.

As we see from (24) $\lambda_\nu (\nu = 1, 2, 3...)$are the simple roots for (21).

From other hand

$$|\lambda_{\nu+1} - \lambda_\nu| = \pi\sqrt{|a|}(\nu = 1, 2, 3...).$$

It means that the roots $\lambda_\nu (\nu = 1, 2, 3...)$ are not densing. It in its turn means that each root λ_ν has some δ neighbourhood and these roots are placed in the band with finite width containing the line $Re\frac{\lambda}{\sqrt{a}} = 0$.

Particularly, when $a > 0$ is a real number, the roots are placed in the band containing the line $\lambda_1 = 0$. Therefore omitting the δ neighbohood of the roots $\lambda_\nu (\nu = 1, 2, 3...)$ from the sector E_2 it is easy to prove the validity of the relation

$$\left|\lambda^{-4} e^{\frac{\lambda}{\sqrt{a}}} \Delta(\lambda)\right| \geq N_\delta > 0. \tag{25}$$

for the characteristic determinant $\Delta(\lambda)$ in the remind part. Here N_δ is a positive number depending only on δ.

Therefore the following theorem is proved.

Theorem 1. *If the coefficient a is non-zero number then the characteristic determinant of the Green function of spectral problem* (6)-(7)

1. *Has infinite number of roots, asymptotics* (23) *is valid for these roots for the large enough values of $|\lambda|$ and they are placed on the line $Re\frac{\lambda}{\sqrt{a}} = 0$ in the band of finite width.*

2. *Inequality* (25) *is satisfied in the λ plane after omitting the δneighborhood of the zeros of $\Delta(\lambda)$.*

3. *The zeros of $\Delta(\lambda)$ are the simple poles of the Green function.*

4 Estimation of the Green's function

To estimate the Green's function $G(x, \xi, \lambda)$ we give the upper estimation for the determinant $\Delta(x, \xi, \lambda)$. For this purpose we make some transformation in (13). For the sake of simplicity we denote the boundary conditions as $e_\nu(y), (\nu = 1, 2)$ and write the determinant $\Delta(x, \xi, \lambda)$in the form

$$\Delta(x, \xi, \lambda) = \begin{vmatrix} g(x, \xi, \lambda) & y_1(x, \lambda) & y_2(x, \lambda) \\ l_1(g)_x & l_1(y_1) & l_1(y_2) \\ l_2(g)_x & l_2(y_1) & l_2(y_2) \end{vmatrix}. \tag{26}$$

In the expression of $\Delta(x, \xi, \lambda)$ there are increasing and decreasing exponents as well as exponents for sufficiently large values of $|\lambda|$ and for$\lambda \in E_2$.

To eliminate the increasing terms in the first column elements of the determinant $\Delta(x,\xi,\lambda)$ we multiply the second and third columns of the determinant (26) by $\frac{\sqrt{a}}{4\lambda}e^{-\frac{\lambda\xi}{\sqrt{a}}}$ and $\frac{\sqrt{a}}{4\lambda}e^{\frac{\lambda\xi}{\sqrt{a}}}$ correspondingly, and add them to the first column elements.

The determinant obtained after this transformation becomes as follows

$$\Delta(x,\xi,\lambda)=\Delta_0(x,\xi,\lambda)=\begin{vmatrix} g_0(x,\xi,\lambda) & y_1(x,\lambda) & y_2(x,\lambda) \\ l_1(g_0)_x & l_1(y_1) & l_1(y_2) \\ l_2(g_0)_x & l_2(y_1) & l_2(y_2) \end{vmatrix}. \tag{27}$$

Thus

$$g_0(x,\xi,\lambda)=\begin{cases} \frac{\sqrt{a}}{2\lambda}e^{\frac{\lambda}{\sqrt{a}}(x-\xi)}; & 0\le\xi\le x\le 1 \\ -\frac{\sqrt{a}}{2\lambda}e^{-\frac{\lambda}{\sqrt{a}}(x-\xi)}; & 0\le x\le\xi\le 1 \end{cases} \tag{28}$$

$$\begin{aligned} l_1(g_0)_x &= \left(\frac{\alpha_0\lambda}{\sqrt{a}}-\lambda^2-\alpha_1+b\right)\frac{\sqrt{a}}{2\lambda}e^{\frac{\lambda}{\sqrt{a}}\xi}; \quad 0\le\xi\le x\le 1 \\ l_2(g_0)_x &= \frac{\sqrt{a}}{2\lambda}\left(\beta_0\frac{\lambda}{\sqrt{a}}+\lambda^2+\beta_1-b\right)e^{\frac{\lambda}{\sqrt{a}}(1-\xi)}; \quad 0\le x\le\xi\le 1. \end{aligned} \tag{29}$$

After some simple transformations on $\Delta(x,\xi,\lambda)$ we get

$$\begin{gathered} \Delta(x,\xi,\lambda)=g_0(x,\xi,\lambda)\cdot\Delta(\lambda)+l_1(g_0)_x\left[l_2(y_1)y_2-l_2(y_2)y_1\right]+ \\ +e_2(g_0)_x\left[l_2(y_2)y_1-l_1(y_1)y_2\right]=g_0(x,\xi,\lambda)\cdot\Delta(\lambda)-\frac{\sqrt{a}}{2\lambda}\left(-\frac{\alpha_0\lambda}{\sqrt{a}}+\lambda^2+\alpha_1-b\right)e^{\frac{\lambda}{\sqrt{a}}\xi}\times \\ \times\left\{\left(-\frac{\alpha_0\lambda}{\sqrt{a}}+\lambda^2+\alpha_1-b\right)e^{\frac{\lambda}{\sqrt{a}}x}-\left(\frac{\alpha_0\lambda}{\sqrt{a}}+\lambda^2+\alpha_1-b\right)e^{-\frac{\lambda}{\sqrt{a}}x}\right\}+\frac{\sqrt{a}}{2\lambda}\left(\frac{\beta_0\lambda}{\sqrt{a}}+\lambda^2+\beta_1-b\right)e^{\frac{\lambda}{\sqrt{a}}(1-\xi)}\times \\ \times\left\{\left(\frac{\beta_0\lambda}{\sqrt{a}}+\lambda^2+\beta_1-b\right)e^{\frac{\lambda}{\sqrt{a}}(1-x)}-\left(-\frac{\beta_0\lambda}{\sqrt{a}}+\lambda^2+\beta_1-b\right)e^{-\frac{\lambda}{\sqrt{a}}(1-x)}\right\}. \end{gathered} \tag{30}$$

Assume that $\lambda\in E_2$. Then $Re\frac{\lambda}{\sqrt{a}}<0<Re(-\frac{\lambda}{\sqrt{a}})$ and from (30) we obtain

$$\begin{gathered} \Delta(x,\xi,\lambda)=g_0(x,\xi,\lambda)\cdot\Delta(\lambda)-e^{\frac{\lambda}{\sqrt{a}}(\xi-x)}\frac{\sqrt{a}}{2\lambda}\left(-\frac{\alpha_0\lambda}{\sqrt{a}}+\lambda^2+\alpha_1-b\right)\times \\ \times\left\{\left(-\frac{\alpha_0\lambda}{\sqrt{a}}+\lambda^2+\alpha_1-b\right)e^{\frac{2\lambda}{\sqrt{a}}}-\left(\frac{\alpha_0\lambda}{\sqrt{a}}+\lambda^2+\alpha_1-b\right)\right\}+\frac{\sqrt{a}}{2\lambda}e^{\frac{\lambda}{\sqrt{a}}(x-\xi)}\left(\frac{\beta_0\lambda}{\sqrt{a}}+\lambda^2+\beta_1-b\right)\times \\ \times\left\{\left(\frac{\beta_0\lambda}{\sqrt{a}}+\lambda^2+\beta_1-b\right)e^{\frac{2\lambda}{\sqrt{a}}(1-x)}-\left(-\frac{\beta_0\lambda}{\sqrt{a}}+\lambda^2+\beta_1-b\right)\right\}. \end{gathered} \tag{31}$$

Dividing both sides of (31) by the function $\Delta(\lambda)$ we get

$$\begin{gathered} G(x,\xi,\lambda)=\frac{\Delta(x,\xi,\lambda)}{\Delta(\lambda)}=g_0(x,\xi,\lambda)+\frac{A_1(\lambda)}{\Delta(\lambda)}e^{\frac{\lambda}{\sqrt{a}}(x+\xi)}+\frac{A_2(\lambda)}{\Delta(\lambda)}e^{\frac{\lambda}{\sqrt{a}}(\xi-x)}+ \\ +\frac{A_3(\lambda)}{\Delta(\lambda)}e^{\frac{\lambda}{\sqrt{a}}(2-\xi-x)}+\frac{A_4(\lambda)}{\Delta(\lambda)}e^{\frac{\lambda}{\sqrt{a}}(x-\xi)}. \end{gathered} \tag{32}$$

Here

$$\left.\begin{aligned} A_1(\lambda) &= -\frac{\sqrt{a}}{2\lambda}\left(-\frac{\alpha_0\lambda}{\sqrt{a}}+\lambda^2+\alpha_1-b\right)^2 \\ A_2(\lambda) &= \frac{\sqrt{a}}{2\lambda}\left(\left(\lambda^2+\alpha_1-b\right)^2-\frac{\alpha_0^2\lambda^2}{a}\right) \\ A_3(\lambda) &= \frac{\sqrt{a}}{2\lambda}\left(\frac{\beta_0\lambda}{\sqrt{a}}+\lambda^2+\beta_1-b\right)^2 \\ A_4(\lambda) &= \frac{\sqrt{a}}{2\lambda}\left(\left(\lambda^2+\beta_1-b\right)^2-\frac{\beta_0^2\lambda^2}{a}\right) \end{aligned}\right\}. \tag{33}$$

It can be shown by direct calculation that the following estimates are true for all $\lambda\in E_2$.

$$|g_0(x,\xi,\lambda)|\le\frac{C\times exp\left[-\varepsilon\left|\lambda\right|\left|x-\xi\right|\right]}{|\lambda|}, \tag{34}$$

$$|A_k(\lambda)|\le M|\lambda|^3(k=\overline{1,4}), \tag{35}$$

where $\varepsilon>0$ and M is a positive number.

Now, if we consider estimates (10), (34), (35) in (32), we can easily prove that the following estimate is true for the Green's function for all $x \in (0,1),\ \lambda \in E_2$.

$$|G(x,\xi,\lambda)| \leq \frac{C \times exp\left[-\varepsilon\left|\lambda\right|\left|x-\xi\right|\right]}{|\lambda|}. \tag{36}$$

Based on this estimate, we can say that the spectral problem is regular. In the same way, for $x \in (0,1), \lambda \in E_1, E_1 = \{\lambda : Re\lambda\delta_1 < 0\}$ inequality (36) can be proved.

Now write the determinant $\Delta(x,\xi,\lambda)$ in the form

$$\varDelta(x,\xi,\lambda) = g_0(x,\xi,\lambda)\varDelta(\lambda) + B(x,\lambda)e^{\lambda\xi} + C(x,\lambda)e^{-\lambda\xi}. \tag{37}$$

As we see from (36) here

$$B(x,\lambda) = A_1(\lambda)e^{\frac{\lambda}{\sqrt{a}}x} + A_2(\lambda)e^{-\frac{\lambda}{\sqrt{a}}x}, \tag{38}$$

$$C(x,\lambda) = A_3(\lambda)e^{\frac{\lambda}{\sqrt{a}}(2-x)} + A_4(\lambda)e^{\frac{\lambda}{\sqrt{a}}x} \tag{39}$$

and $A_k(\lambda)$, $(k = \overline{1,4})$ are found by formula (33).

5 Solution of the Cauchy problem

Problem (8), (9) is the Cauchy problem for a first-order ordinary differential equation with a complex parameter λ. It is known from the literature that the solution to this problem is found by the following formula (Arnold, 1992; Yegorov, 2005).

$$Z(x,t) = \Phi(x)e^{-(\lambda^2+b)t} + \int_0^t f(x,\tau)e^{-(\lambda^2+b)(t-\tau)}d\tau. \tag{40}$$

6 Solution of the mixed problem

According to the scheme of the residue method, referring to the works of M. Rasulov, it can be directly shown that the solution of the mixed problem (1)-(3) is expressed as a complete residue of the solutions of the spectral and Cauchy problems (Rasulov, 2014; Salazar & Mendioroz, 2012).

$$\upsilon(x,t) = \frac{1}{2\pi\sqrt{-1}}\sum_{\nu}\int_{C_\nu}\lambda d\lambda\int_0^1 G(x,\xi,\lambda)Z(\xi,t,\lambda)d\xi. \tag{41}$$

Here $\sum_v$ is a sum over all poles, C_v is a simple closed counter containing one poles of the Green's function. The Green's function $G(x,\xi,\lambda)$ is a solution of the homogeneous boundary problem, and $Z(x,t,\lambda)$ is a solution of the Cauchy problem Cauchy problem.

If to consider the expression (11) of the Green's function in (41) take into account that the zeros of $\Delta(\lambda)$ in fact are the simple poles of the Green's function, then the integral in the right hand side of (41) may be rewritten as

$$\upsilon(x,t) = -\sum_{\nu}\int_0^1\frac{\Delta(x,\xi,\lambda_\nu)}{\Delta'(\lambda_\nu)}\lambda_\nu Z(\xi,t,\lambda_\nu)d\xi. \tag{42}$$

Here $\Delta(x,\xi,\lambda)$ is defined by (37), $\Delta'(\lambda)$ by (24) and $Z(x,t,\lambda)$ by (40), λ_v are zeros of the characteristic determinant $\Delta(\lambda)$ with the asymptotic representations (23).

Let us put expression (37) of $\Delta(x,\xi,\lambda)$ into (42). It gives

$$\upsilon(x,t) = -\sum\left\{\int_0^1\frac{B(x,\lambda_\nu)}{\Delta'(\lambda_\nu)}\,\lambda_\nu Z(\xi,t,\lambda_\nu)e^{\frac{\lambda_\nu\xi}{\sqrt{a}}}d\xi + \int_0^1\frac{C(x,\lambda_\nu)}{\Delta'(\lambda_\nu)}\lambda_\nu Z(\xi,t,\lambda_\nu)e^{-\frac{\lambda_\nu\xi}{\sqrt{a}}}d\xi\right\}. \tag{43}$$

Indeed, if to consider expressions (38), (39) for $B(x,\lambda)$ and $C(x,\lambda)$ in (43) then for $\upsilon(x,t)$ we obtain the formula

$$\upsilon(x,t) = -\sum \left\{ \frac{\lambda_\nu A_1(\lambda_\nu)}{\Delta'(\lambda_\nu)} e^{\frac{\lambda_\nu x}{\sqrt{a}}} \int_0^1 Z(\xi,t,\lambda_0) e^{\frac{\lambda_\nu \xi}{\sqrt{a}}} d\xi + \frac{\lambda_\nu A_2(\lambda_\nu)}{\Delta'(\lambda_\nu)} e^{-\frac{\lambda_\nu}{\sqrt{a}} x} \int_0^1 Z(\xi,t,\lambda_\nu) e^{\frac{\lambda_\nu \xi}{\sqrt{a}}} d\xi + \right.$$
$$\left. + \frac{\lambda_\nu A_3(\lambda_\nu)}{\Delta'(\lambda_\nu)} e^{\frac{\lambda_\nu}{\sqrt{a}}(2-x)} \int_0^1 Z(\xi,t,\lambda_\nu) e^{-\frac{\lambda_\nu \xi}{\sqrt{a}}} d\xi + \frac{\lambda_\nu A_4(\lambda_\nu)}{\Delta'(\lambda_\nu)} e^{\frac{\lambda_\nu}{\sqrt{a}} x} \int_0^1 Z(\xi,t,\lambda_\nu) e^{-\frac{\lambda_\nu \xi}{\sqrt{a}}} d\xi \right\}.$$

Finally if to denote

$$\begin{array}{c} \int_0^1 Z(\xi,t,\lambda_\nu) e^{\frac{\lambda_\nu \xi}{\sqrt{a}}} d\xi = Z_1(t,\lambda_\nu) \\ \int_0^1 Z(\xi,t,\lambda_\nu) e^{-\frac{\lambda_\nu \xi}{\sqrt{a}}} d\xi = Z_2(t,\lambda_\nu) \end{array}.$$

For the solution of mixed problem (1)-(3) we obtain the following expression

$$\upsilon(x,t) = -\sum_\nu \left\{ \frac{\lambda_\nu A_1(\lambda_\nu) e^{\frac{\lambda_\nu x}{\sqrt{a}}} + \lambda_\nu A_2(\lambda_\nu) e^{-\frac{\lambda_\nu}{\sqrt{a}} x}}{\Delta'(\lambda_\nu)} Z_1(t,\lambda_\nu) + \right.$$
$$\left. + \frac{\lambda_\nu A_3(\lambda_\nu) e^{\frac{\lambda_\nu (2-x)}{\sqrt{a}}} + \lambda_\nu A_4(\lambda_\nu) e^{\frac{\lambda_\nu}{\sqrt{a}} x}}{\Delta'(\lambda_\nu)} Z_2(t,\lambda_\nu) \right\}.$$

Using the formula we obtained for solving the mixed problem, one can find a numerical solution to the problem.

7 Conclusion

The study of the solution of the mixed problem for the one-dimensional heat transfer equation by numerical methods has been investigated by many famous scientists. In the two-dimensional case, the problem is non-local with respect to the time variable, and local with respect to the space variable. Unlike such cases, the presence of a time derivative in the boundary conditions differs from the considered works. The practical significance of the considered problem is of many importance, such as the study of the thermal properties of materials, the simulation of heat dissipation for the thermal control of microprocessors, the analysis of heat flow in heat exchangers, and the modeling of temperature distribution in the casting process of molten metals.

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