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SUFFICIENT CONDITIONS FOR IDENTICAL SYNCHRONIZATION IN A FULL NETWORK OF n ORDINARY DIFFERENTIAL SYSTEMS OF FITZHUGH-NAGUMO TYPE WITH LINEAR COUPLING AND TIME-DEPENDENT EXTERNAL ELECTRICAL STIMULATION

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Abstract. This paper explores the sufficient conditions necessary for achieving identical synchronization in a full network of n interacting ordinary differential systems of Fitz-Hugh-Nagumo type. These systems are characterized by linear coupling and time-dependent external electrical stimulation. The sufficient conditions are established through the construction of an appropriate Lyapunov function. Additionally, this study utilizes R programming to present numerical results that validate the theoretical findings. In this study, we focus exclusively on identical synchronization. Identical synchronization occurs when the dynamic systems within a network exhibit the same properties and patterns at a specific moment in time. In simpler terms, if a network consists of two dynamic equations, synchronization means that one system will replicate the properties and shape of the other system starting from a particular point. When this happens, the network is considered to be in synchronization.

Keywords: FitzHugh-Nagumo model, identical synchronization, Lyapunov function, sufficient condition.
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1 Introduction

Synchronization is an important natural phenomenon that has significant implications in both practical applications and nonlinear sciences. This is especially true for networks composed of interconnected dynamic systems (Aziz-Alaoui, 2006; Corson, 2009; Pecora & Carroll, 1998, 1990; Yanchuk et al., 2000). There are various types of synchronization, such as identical synchronization, generalized synchronization, and phase synchronization (see Aziz-Alaoui (2006)). In this study, we focus exclusively on identical synchronization. Identical synchronization occurs when the dynamic systems within a network exhibit the same properties and patterns at a specific moment in time. In simpler terms, if a network consists of two dynamic equations, synchronization means that one system will replicate the properties and shape of the other system starting from a particular point. When this happens, the network is considered to be in synchronization.

In recent years, synchronization has been extensively studied across various fields, revealing that many natural phenomena exhibit this concept. Examples include the formation of flocks of birds in the sky, schools of fish swimming in unison in a lake, the coordinated movements of a parade, and the way groups of cells communicate and share information (Aziz-Alaoui, 2006; Aguirre et al., 2004; Belykh et al., 2005; Braun et al., 1994; Izhikevich, 2007). The human brain, composed of numerous interconnected cells, functions as a network. This network operates as a system of cells that are physiologically linked. The interactions among these cells primarily depend on electrochemical processes. Individual neurons can display irregular behavior, while groups of neurons often synchronize to convey biological information or produce regular rhythmic activities. Understanding the synchronization of populations of interacting neurons is crucial for addressing key issues in neuroscience.

Many research studies have investigated the synchronization of cell networks, predominantly using models based on the FitzHugh-Nagumo equations (Ambrosio et al., 2012; Ambrosio & Aziz-Alaoui, 2013) or the Hindmarsh-Rose differential equations (Corson, 2009). However, these studies generally assume that the externally applied current remains constant. In reality, the current influencing the cell membrane voltage typically varies in both amplitude and frequency over time (Keener & Sneyd, 2009; Heagy et al., 1995; Murray, 2010). This variability can lead to fluctuating cell membrane voltages, potentially disrupting the transmission process (Ermentrout & Terman, 2009).

This paper explores the identical synchronization of a network consisting of n linearly connected cells, each described by a FitzHugh-Nagumo model, with external electrical stimulation that varies with time. The time-dependent nature of the stimulation introduces chaotic behavior into the FitzHugh-Nagumo model (see Figure 2 and Figure 4) and reveals rich dynamics that resemble the activation patterns of real biological cells.

In 1952, Hodgkin and Huxley introduced a four-dimensional mathematical model to approximate the properties of cell voltage (Badola et al., 1992; FitzHugh, 1960; Hodgkin & Huxley, 1952; Khalil, 2002). This model has since inspired the development of numerous simpler models to describe cell voltage dynamics. In 1962, FitzHugh R. and Nagumo J. published a new model known as the FitzHugh-Nagumo model (FitzHugh, 1960; Nagumo et al., 1962; Pecora & Carroll, 1998, 1990). This model is a simplified two-dimensional representation derived from Hodgkin-Huxley's well-known system of equations (Hodgkin & Huxley, 1952). Despite its simplicity, the FHN model yields many significant analytical results and maintains its biological relevance. It captures the equilibrium, activity, and bursting behavior of cell voltage when external electrical stimulation is applied in the first equation (see Figure 2 and Figure 4). The FHN model is commonly expressed through two equations involving two variables, u and v. The first variable, $u = u(t)$, is a fast excitatory variable that represents the transmembrane voltage. The second variable, $v = v(t)$, is a slower recovery variable that describes various physical quantities, including the electrical conductivity of ion currents across the membrane. The FitzHugh-Nagumo equations (FHN) are formulated as follows (see Ambrosio et al. (2012); Ambrosio & Aziz-Alaoui (2013); Phan (2018, 2019)):

$$\begin{cases} \dfrac{du}{dt} = u(u-1)(1-bu) - v + I(t), \\ \\ \dfrac{dv}{dt} = cu, \end{cases} \tag{1}$$

where b and c are positive constants; t presents the time; $I(t)$ presents the time-dependent external electrical stimulation and given by:

$$I(t) = \frac{a}{2\pi f}\cos 2\pi f t, \tag{2}$$

where a and f are dimensionless parameters presenting, respectively, the amplitude and the frequency of the applied electrical stimulus Phan (2019).

In 2019, a paper was published that discussed the identical synchronization of two ordinary differential systems of FitzHugh-Nagumo type with linear coupling, as the frequency of the externally applied current varied Phan (2019). The results presented in this paper are relatively straightforward to prove, as there are only two nodes in the network. Furthermore, the numerical simulations were conducted using C++. This can pose challenges for many researchers, particularly mathematics students who are new to coding and its practical applications.

In this study, we will perform the simulations using R software, an open-source program that simplifies the coding of dynamic equation systems compared to C++. Currently, there is limited research on the full network of n ordinary differential systems of FitzHugh-Nagumo type with linear coupling and external electrical stimulation that varies over time. Exploring this topic is therefore valuable and offers significant practical applications in the field of applied mathematics.

2 Sufficient conditions for identical synchronization in a full network of n ordinary differential systems of FitzHugh-Nagumo type with linear coupling and time-dependent external electrical stimulation

This study examines identical synchronization within a fully connected network, where each node is connected to every other node in the network Ambrosio et al. (2012); Ambrosio & Aziz-Alaoui (2013); Corson (2009). For instance, Figure 1 illustrates full networks (graphs) containing between 3 and 10 nodes. In this context, each node represents a neuron, modeled by a system of ordinary differential equations of the FHN type, while each edge symbolizes a synaptic connection, modeled by a coupling function.

Specifically, we construct a network of n systems (1) as follows:

$$\begin{cases} \dfrac{du_i}{dt} = u_i(u_i - 1)(1 - bu_i) - v_i - h(u_i, u_j) + I(t), \\ \\ \dfrac{dv_i}{dt} = cu_i, \\ \\ i, j = 1, ..., n, i \neq j, \end{cases} \tag{3}$$

where $(u_i, v_i), i = 1, 2, ..., n$ is defined by (1).

The function h describes the coupling mechanism that dictates the nature of the connection between the i-th neuron and the j-th neuron. Neuronal connections can be classified into two main types: chemical and electrical connections, with chemical connections being more prevalent than electrical ones. For the purposes of this research, the focus will be solely on electrical connections. In this context, the coupling function is linear Badola et al. (1992); Yanchuk et al. (2000); Wolf et al. (1985) and is defined by the following formula:

$$h(u_i, u_j) = g_{syn} \sum_{j=1, j\neq i}^{n} c_{ij}(u_i - u_j), \quad i = 1, 2, ..., n, \tag{4}$$

where g_{syn} is a positive constant and represents the coupling strength. The coefficients c_{ij} are the elements of the connectivity matrix $C_n = (c_{ij})_{n\times n}$, defined by: $c_{ij} = 1$ if u_i and u_j are coupled, $c_{ij} = 0$ if u_i and u_j are not coupled, where $i, j = 1, 2, ..., n, i \neq j$. Therefore, a full network of n "neurons" (1) coupled by the electrical connection is given as follows:

$$\begin{cases} \dfrac{du_i}{dt} = u_i(u_i - 1)(1 - bu_i) - v_i - \displaystyle\sum_{j=1, j\neq i}^{n} g_{syn}(u_i - u_j) + I(t), \\ \\ \dfrac{dv_i}{dt} = cu_i, \end{cases} \tag{5}$$

where $(u_i, v_i), i = 1, 2, ..., n$ are the state variables of the i-th neurons and g_{syn} is the coupling strength.

Remark 1. To simplify the study, we keep $I(t)$ and the constants a, b, c, and g_{syn} the same in any subsystem of system (5).

Definition 1. *(see Aziz-Alaoui (2006); Ambrosio & Aziz-Alaoui (2013)) Let* $\sum_{i=2}^{n} |u_i - u_1| + |v_i - v_1|$ *be the identical synchronization error. We say that the network* (5) *identically synchronizes if the identical synchronization error reaches zero as t approaches infinity.*

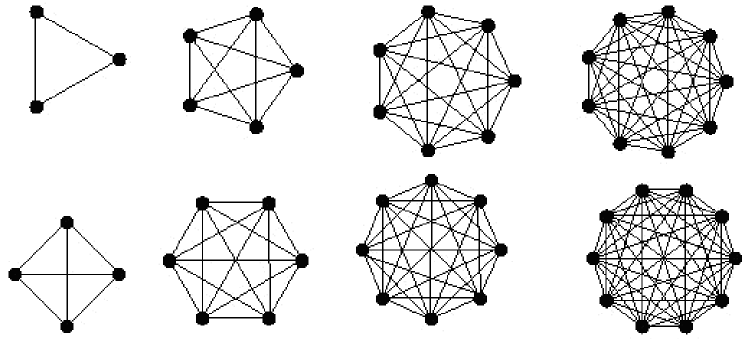


Figure 1: Full graphs (networks) from 3 to 10 nodes

The primary outcome of this paper is presented in the following theorem. In this theorem, we construct an appropriate Lyapunov function and, utilizing Lyapunov stability theory along with LaSalle's invariance principle (Aeyels, 1995), we identify the sufficient conditions regarding the coupling strength necessary to achieve the desired synchronization.

Theorem 1. *Suppose that the coupling strength g_{syn} satisfies the following condition:*

$$g_{syn} > \frac{M\left[(2(1+b)+3Mb\right]-1}{n-1}, \tag{6}$$

where M is a positive constant such that $|u_i(t)| \leq M$, for all $i = 1, 2, \ldots, n$, and $t \geq 0$. Then, for every initial condition $u_i(0), v_i(0), i = 1, 2, ..., n$, the network (5) *synchronizes in the sense of Definition 1.*

Remark 2. In the paper Ambrosio et al. (2018), the authors demonstrated the existence of a global attractor for networks comprising n systems of the FitzHugh-Nagumo type, regardless of their structure or initial conditions. This implies that there exists a positive constant M such that $|u_i(t)| \leq M$, for all $i = 1, 2, \ldots, n$, and $t \geq 0$, regardless of the initial conditions.

Proof. To prove this theorem , we construct the Lyapunov function as follows:

$$\Phi(t) = \frac{1}{2}\left[\sum_{i=2}^{n}\left(c(u_i-u_1)^2+(v_i-v_1)^2\right)\right]. \tag{7}$$

The derivative of the function Φ with respect to t is given by:

$$\begin{aligned}\frac{d\Phi(t)}{dt} &= \sum_{i=2}^{n}\left[c(u_i-u_1)(\frac{du_i}{dt}-\frac{du_1}{dt})+(v_i-v_1)(\frac{dv_i}{dt}-\frac{dv_1}{dt})\right]\\ &= \sum_{i=2}^{n}\left[c(u_i-u_1)\left(u_i(u_i-1)(1-bu_i)-v_i-g_{syn}\sum_{k=1,k\neq i}^{n}(u_i-u_k)-u_1(u_1-1)(1-bu_1)\right.\right.\\ &\qquad\left.\left.+v_1+g_{syn}\sum_{l=2}^{n}(u_1-u_l)\right)+c(u_i-u_1)(v_i-v_1)\right]\\ &\le \sum_{i=2}^{n}c\left(-1-(n-1)g_{syn}+(1+b)(u_i+u_1)-b(u_i^2+u_iu_1+u_1^2)\right)(u_i-u_1)^2.\end{aligned} \tag{8}$$

Since the solution of the system (5) is bounded (see Ambrosio et al. (2018)), let the bound be M (i.e., $|u_i|\le M, i=1,2,...,n$). Then,

$$\frac{d\Phi(t)}{dt} \le -c\left[1+(n-1)g_{syn}-M(2(1+b)+3Mb)\right](u_i-u_1)^2. \tag{9}$$

Hence, if

$$1+(n-1)g_{syn}-M(2(1+b)+3Mb)\ge 0 \Leftrightarrow g_{syn}\ge \frac{M(2(1+b)+3Mb)-1}{n-1}, \tag{10}$$

then

$$\frac{d\Phi(t)}{dt}\le 0. \tag{11}$$

From (11), it can be seen that $0\le\Phi(t)\le\Phi(0)$, this together with (7) implies that $\Phi(t)$ is bounded. It is based on Lyapunov stability theory and LaSalle's invariance principle Aeyels (1995), we have:

$$\lim_{t\to+\infty}\sum_{i=2}^{n}|u_i-u_1|+|v_i-v_1|=0.$$

Therefore, it implies that the network (5) achieves identical synchronization in the sense of Definition 1. The theorem is proved.

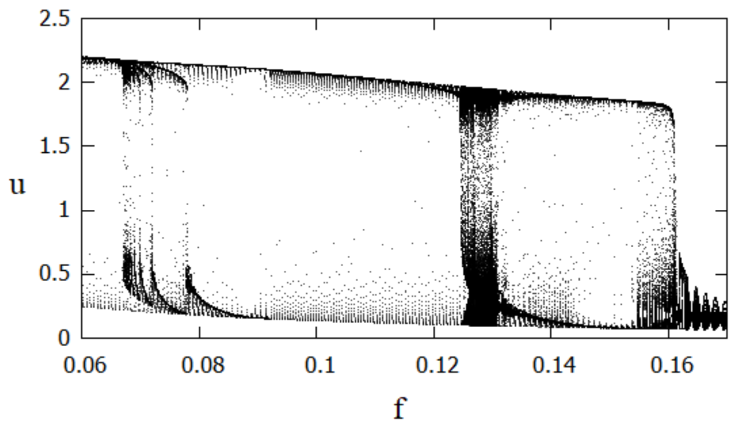
□

Remark 3. The theorem above provides a sufficient condition for the identical synchronization of the network described in system (5). We observe that as the number of cells in the full network increases, the coupling strength required for synchronization decreases. This is evident from inequality (6), where the denominator is variable (specifically, $n-1$) and the numerator is constant $(M\,(2(1+b)+3Mb)-1)$. Therefore, if n increases, the value of the fraction on the right side of (6) decreases. This indicates that achieving identical synchronization becomes easier as the number of cells in the full network increases.

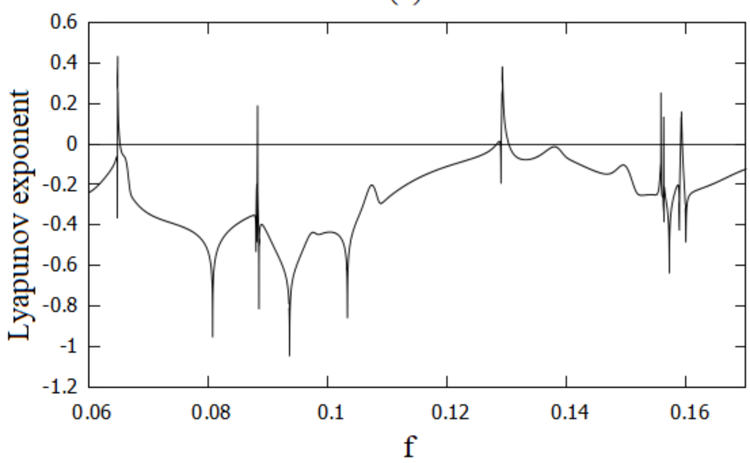
Remark 4. If the right-hand side in (10) is negative, it will be better since then $1-M(2(1+b)+3Mb)>0$. It implies that $1+(n-1)g_{syn}-M(2(1+b)+3Mb)>0$ in (9). And then we also get (11).

3 Numerical results and discussion

In this section, we aim to verify the theoretical results discussed earlier (see Theorem 1 and Remark 3) by programming in R. First, we will demonstrate why the FitzHugh-Nagumo model with time-dependent external electrical stimulation is more intriguing than one with a constant external electrical stimulus. The former can better represent the membrane voltage of a cell. In 2018, a paper was published that investigated the fixed points and Hopf bifurcation of the FitzHugh-Nagumo model with a constant external electrical current Phan (2018). The author of that paper showed that the model exhibits a Hopf bifurcation, meaning that its solutions can either be a stable focus or a stable limit cycle, depending on fluctuations in the external electrical current. In other words, this model does not exhibit chaotic behavior. Specifically, Figure 3 illustrates the state variables of the system described by (1), with parameters set to $I(t) = 0.5$, $b = 10$, $c = 1$, and $a = 0.1$, over the time interval $t \in [0, 50000]$. The initial conditions are $(u(0), v(0)) = (0.5, 0)$. In Figure 3(a), the state variable $u(t)$ is plotted against time t, while Figure 3(b) shows the state variable $v(t)$ against time t. The results indicate that when $I(t)$ is constant, the solution to system (1) behaves as a stable focus, and no chaotic behavior is observed for other constant values of $I(t)$ Phan (2018). In contrast to the chaotic oscillations and routes to chaos typically observed in real biological neurons, which are effectively described by the Hodgkin–Huxley model Aihara et al. (1985), this study focuses on the FitzHugh-Nagumo model under time-dependent external electrical stimulation, as it exhibits chaotic characteristics. To investigate the chaotic behavior of system (1), we analyze bifurcation diagrams and the variation of the largest Lyapunov exponents Aguirre et al. (2004); Wolf et al. (1985). We have fixed the parameters at $b = 10$, $c = 1$, and $a = 0.1$, using the frequency f of the external electrical stimulation as a bifurcation parameter.



(a)



(b)

Figure 2: The bifurcation diagram (upper) and the variation of the largest Lyapunov exponent (lower) of the system described in (1) over the frequency range of $0.06 < f < 0.17$, with parameters $a = 0.1$, $b = 10$, and $c = 1$.

Figure 2 presents the bifurcation diagram (upper part) and the corresponding variation of the largest Lyapunov exponent (lower part) of the periodically stimulated FitzHugh-Nagumo neuron over the frequency range $0.06 < f < 0.17$. As shown in Figure 2, a rich dynamic behavior is observed, with certain regions exhibiting chaos. This is confirmed by the positive values of the largest Lyapunov exponents noted in the lower part of the figure.

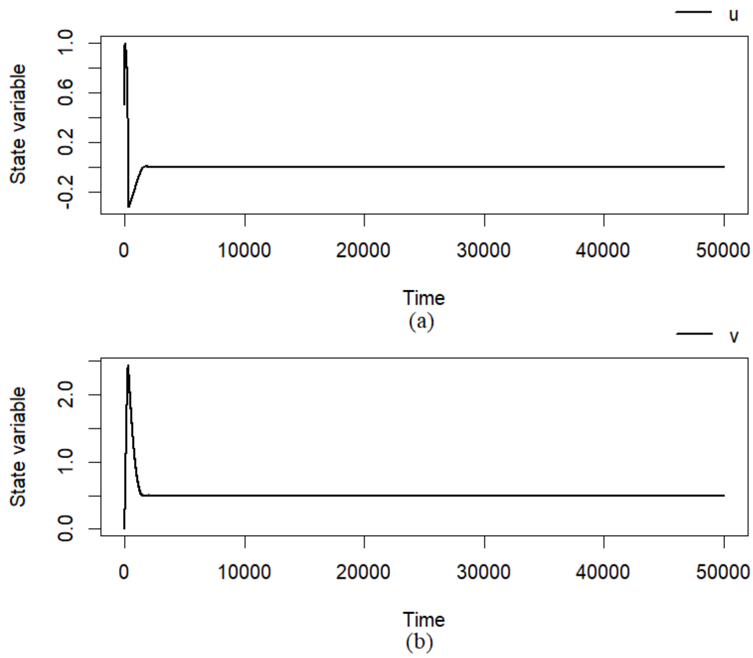


Figure 3: This figure illustrates the state variables of the system defined by (1) with parameters $I(t) = 0.5$, $b = 10$, $c = 1$, $a = 0.1$, for $t \in [0, 50000]$, and initial conditions $(u(0), v(0)) = (0.5, 0)$. It consists of two parts: (a) the time series of the state variable $u(t)$ as a function of time t; (b) the time series of the state variable $v(t)$ as a function of time t. These graphs demonstrate that when $I(t)$ is constant, the solution of the system (1) exhibits characteristics of a stable focus.

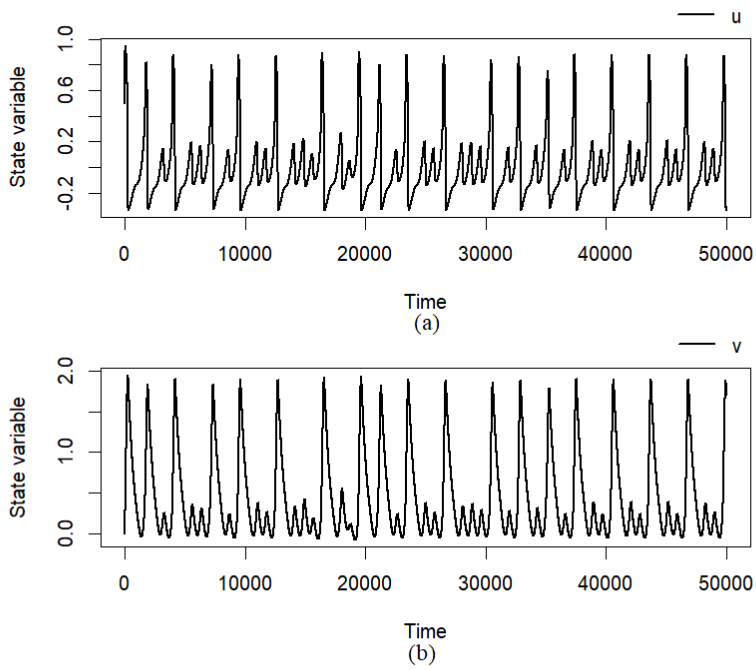


Figure 4: This figure illustrates the state variables of the system described by equation (1), with the input current defined as $I(t) = \dfrac{a}{2\pi f}\cos(2\pi f t)$, where $b = 10$, $c = 1$, $a = 0.1$, $f = 0.129$, and $t \in [0, 50000]$. The initial conditions are set as $(u(0), v(0)) = (0.5, 0)$. The figure includes: (a) the time series of state variable $u(t)$ plotted against time t; (b) the time series of state variable $v(t)$ plotted against time t. These results indicate that the solution to the system described by (1), with $I(t)$ varying over time, exhibits chaotic behavior.

Specifically, Figure 4 illustrates the state variables of system (1) with $I(t) = \dfrac{a}{2\pi f}\cos(2\pi f t)$, where $b = 10$, $c = 1$, $a = 0.1$, $f = 0.129$, and $t \in [0, 50000]$, with initial conditions $(u(0), v(0)) = (0.5, 0)$. Figure 4(a) displays the state variable $u(t)$ as a function of time t, while Figure 4(b) shows the state variable $v(t)$ over the same time period. Both figures demonstrate that the solution of system (1), with $I(t)$ depending on time, is chaotic. This time-dependent nature of $I(t)$ makes the model particularly interesting, which is why we have chosen it for this study.

Next, we will verify the theoretical results presented in Section 2 by examining various values of n ranging from 2 to 20. For each value of n, we will determine the minimal coupling strength required to achieve identical synchronization and assess whether the required coupling strength aligns with the law described in Theorem 1. To simplify our analysis, we will first consider a fully connected network of two nodes in Example 1, followed by a fully connected network of three nodes in Example 2 to observe the synchronization phenomenon. Subsequently, we will conduct a similar analysis for fully connected networks with n nodes, where n ranges from 4 to 20.

In the simulation results below, the parameter values of network (5) are specified as follows:

$$b = 10, c = 1, a = 0.1, f = 0.129.$$

Example 1. In this example, we examine identical synchronization in a fully connected network of two nodes ($n = 2$) with linear coupling and time-dependent external electrical stimulation. Our focus is on determining the minimal coupling strength required to achieve the desired synchronization within this network.

The system describing a fully connected network of two nodes with linear coupling and time-dependent external electrical stimulation is given by the following equations:

$$\begin{cases} \dfrac{du_1}{dt} = u_1(u_1 - 1)(1 - bu_1) - v_1 - g_{syn}(u_1 - u_2) + s(t), \\[2ex] \dfrac{dv_1}{dt} = cu_1, \\[2ex] \dfrac{du_2}{dt} = u_2(u_2 - 1)(1 - bu_2) - v_2 - g_{syn}(u_2 - u_1) + s(t), \\[2ex] \dfrac{dv_2}{dt} = cu_2, \end{cases} \tag{12}$$

where $s(t) = \dfrac{a}{2\pi f}\cos 2\pi f t$.

Let $|u_2 - u_1| + |v_2 - v_1|$ represent the identical synchronization error of this network. According to Definition 1, we say that the network achieves identical synchronization if the identical synchronization error approaches zero as time t approaches infinity. To demonstrate this, we can integrate the system described in (12) using the following initial condition:

$$(u_1(0), v_1(0), u_2(0), v_2(0)) = (0.5, 0, -0.5, 0.5),$$

and $t \in [0, 50000]$.

Figure 5 illustrates the synchronization errors of the network described by system (12) for different values of coupling strength. This analysis aims to determine which coupling strength facilitates achieving the desired synchronization. Specifically, Figures 5(a) and 5(b) present simulations of the network for coupling strengths of $g_{syn} = 0.005$ and $g_{syn} = 0.08$, respectively. The simulations indicate that the synchronization errors do not reach zero, suggesting that identical synchronization does not occur in the network under consideration.

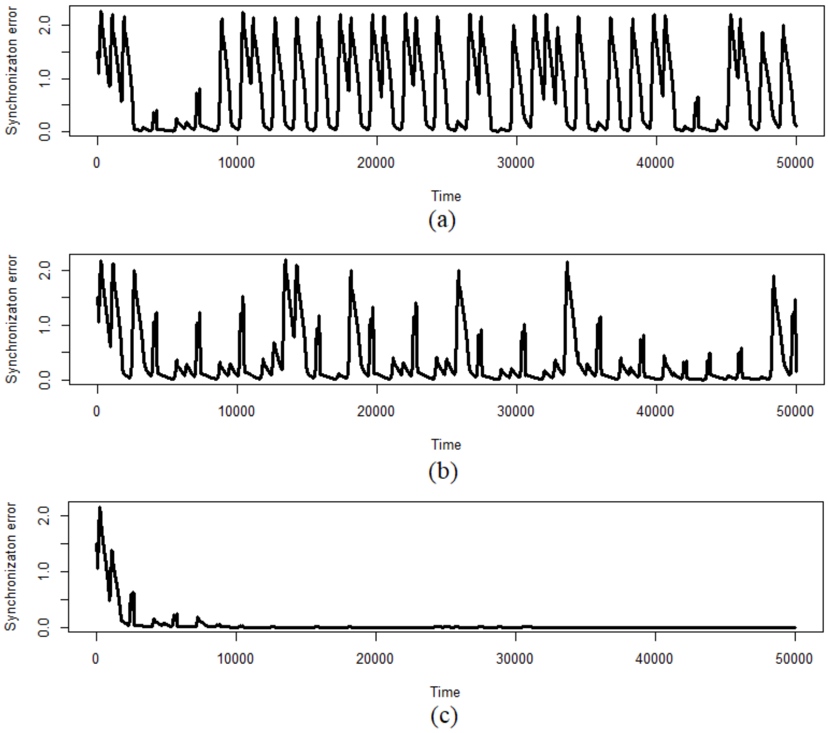


Figure 5: Identical synchronization errors in the network are analyzed with respect to different values of coupling strength: (a) $g_{syn} = 0.005$, (b) $g_{syn} = 0.08$, and (c) $g_{syn} = 0.1$. In Figures 5(a) and 5(b), the identical synchronization error remains above zero, indicating that synchronization has not been achieved. However, in Figure 5(c), the identical synchronization error reaches zero, signifying that identical synchronization is successful.

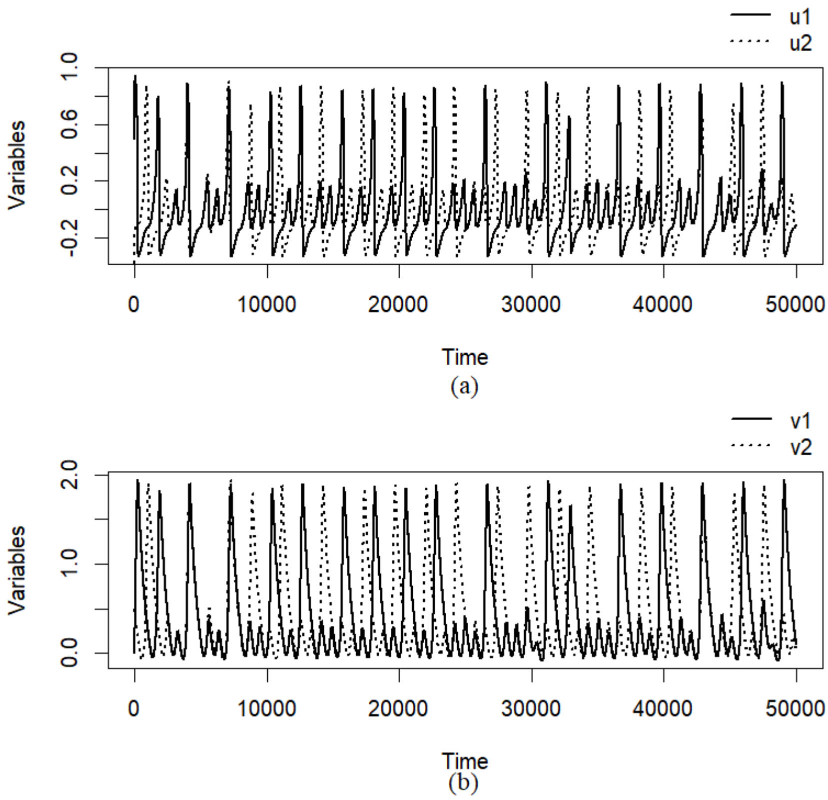
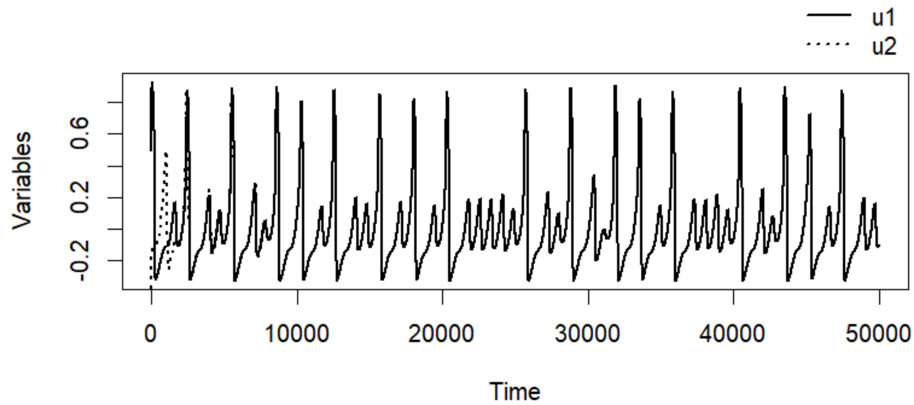
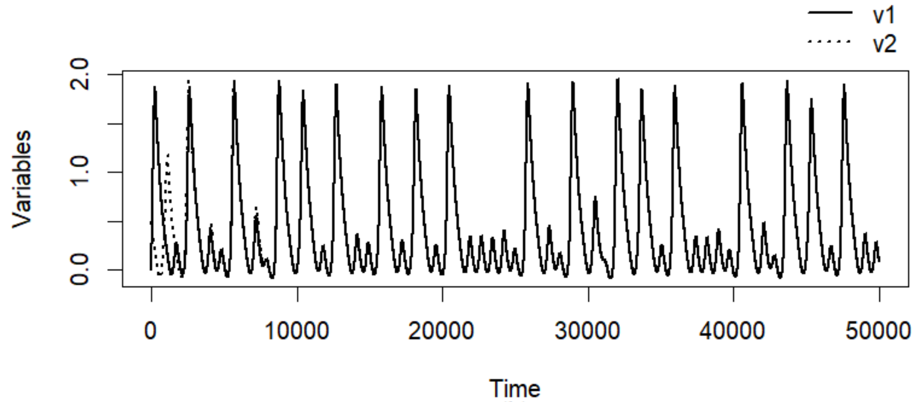


Figure 6: The time series for all state variables of the system described in (12) with $g_{syn} = 0.005$ is presented. In Figure 6(a), the variable u_1 is shown with a solid line, while u_2 is represented by a dotted line. Similarly, v_1 and v_2 are illustrated in Figure 6(b). The dotted lines do not mirror the behavior of the solid lines, indicating that identical synchronization does not occur.



(a)



(b)

Figure 7: The time series of all state variables in the system described by system (12) with $g_{syn} = 0.1$ is presented. In Figure 7(a), the variable u_1 is represented by a solid line, while u_2 is depicted with a dotted line. Similarly, Figure 7(b) shows v_1 and v_2 with the same line styles. The behavior of the dotted lines closely follows that of the solid lines, indicating that a synchronization phenomenon is occurring between these variables.

Additionally, Figure 6 displays the time series of all state variables in the system (12) for $g_{syn} = 0.005$. In Figure 6(a), the solid line represents the variable u_1, while the dotted line corresponds to u_2. Similarly, Figure 6(b) shows v_1 as a solid line and v_2 as a dotted line. The results demonstrate that the behavior of the solid lines does not match that of the dotted lines, further indicating that identical synchronization does not occur in this scenario.

However, in Figure 5(b), we simulate the network (12) with $g_{syn} = 0.1$. The simulation shows that the identical synchronization error reaches zero. Clearly, Figure 7 represents the time series of all state variables of the system (12) with $g_{syn} = 0.1$. In Figure 7(a), the variable u_1 is presented by the solid line, and the dotted line for u_2 (respectively, v_1 and v_2 in Figure 7(b)). We can see that the solid lines copy the behaviour of the dotted ones. In other words, the identical synchronization phenomenon occurs in this case.

Example 2. In this example, we will examine identical synchronization in a fully connected network of three nodes ($n = 3$) that employs linear coupling and external electrical stimulation that varies with time. Our goal is to determine the minimum coupling strength required to achieve the desired synchronization within this network.

The system in this case, which consists of three nodes with linear coupling and time-dependent external electrical stimulation, is described by the following equations:

$$\begin{cases} \dfrac{du_1}{dt} = u_1(u_1-1)(1-bu_1) - v_1 - g_{syn}(u_1-u_2) - g_{syn}(u_1-u_3) + s(t), \\ \dfrac{dv_1}{dt} = cu_1, \\ \dfrac{du_2}{dt} = u_2(u_2-1)(1-bu_2) - v_2 - g_{syn}(u_2-u_1) - g_{syn}(u_2-u_3) + s(t), \\ \dfrac{dv_2}{dt} = cu_2, \\ \dfrac{du_3}{dt} = u_3(u_3-1)(1-bu_3) - v_3 - g_{syn}(u_3-u_1) - g_{syn}(u_3-u_2) + s(t), \\ \dfrac{dv_3}{dt} = cu_3, \end{cases} \tag{13}$$

where $s(t) = \dfrac{a}{2\pi f}\cos 2\pi f t$.

Let $|u_2-u_1|+|v_2-v_1|+|u_3-u_1|+|v_3-v_1|$ represent the identical synchronization error of this network. According to Definition 1, we say that the network achieves identical synchronization if the identical synchronization error approaches zero as t approaches infinity. To demonstrate this, we will integrate the system described in (13) using the following initial condition:

$$(u_1(0), v_1(0), u_2(0), v_2(0), u_3(0), v_3(0)) = (0.5, 0, -0.5, 0.5, 1, -0.5),$$

and $t \in [0, 50000]$.

Figure 8 illustrates the identical synchronization errors of the network (13) for different values of coupling strength to determine which value facilitates the desired synchronization. Specifically, Figures 8(a) and 8(b) show the simulations for the network (13) with coupling strengths of $g_{syn} = 0.005$ and $g_{syn} = 0.05$, respectively. The simulations indicate that the identical synchronization errors do not reach zero, suggesting that the identical synchronization phenomenon does not occur in this network.

Figure 9 displays the time series of all state variables for the system (12) with a coupling strength of $g_{syn} = 0.005$. In Figure 9(a), the variable u_1 is represented by a solid line, u_2 by a dotted line, and u_3 by a dashed line. Correspondingly, v_1, v_2, and v_3 are shown in Figure 9(b) using the same line styles. It is evident that the solid lines do not mimic the behavior of the dotted lines, further confirming that the identical synchronization phenomenon does not occur in this case.

In Figure 8(b), we simulate the network described in (12) with a coupling strength of $g_{syn} = 0.07$. The simulation indicates that the identical synchronization error converges to zero. Figure 10 illustrates the time series of all state variables from the system (12) under the same coupling strength. In Figure 10(a), the variable u_1 is represented by a solid line, u_2 by a dotted line, and u_3 by a dashed line. Similarly, in Figure 10(b), the variables v_1, v_2, and v_3 are presented in a corresponding manner. It is evident that the solid lines closely follow the behavior of the dotted lines, demonstrating that identical synchronization occurs in this scenario.

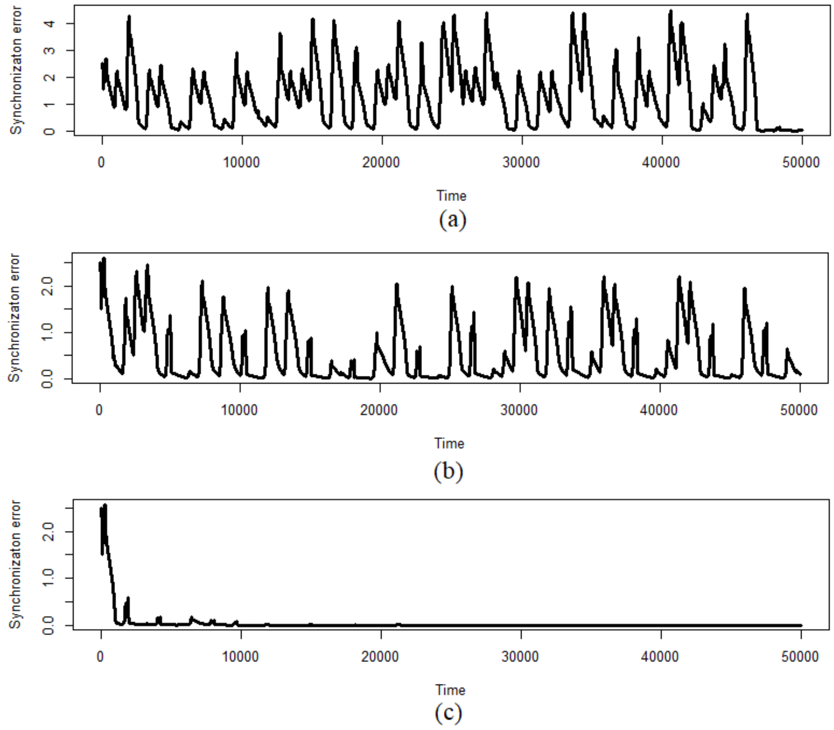


Figure 8: The identical synchronization errors of the network (13) are examined for different values of coupling strength: (a) $g_{syn} = 0.005$; (b) $g_{syn} = 0.05$; (c) $g_{syn} = 0.07$. In Figures 8(a) and 8(b), the synchronization errors do not reach zero, indicating that synchronization has not been achieved. In contrast, in Figure 8(c), the synchronization error reaches zero, signifying that identical synchronization has been successfully attained.

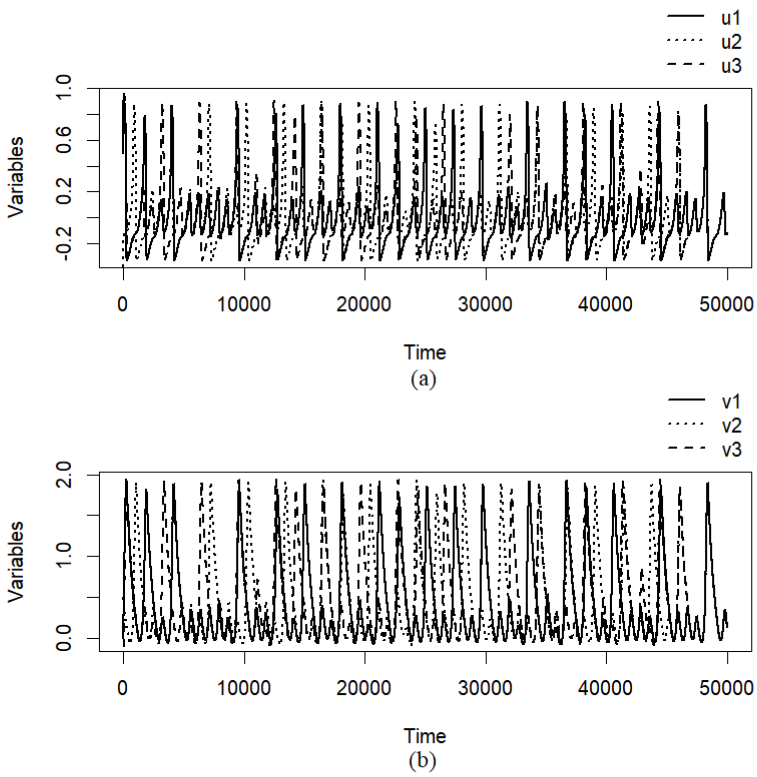


Figure 9: The time series of all state variables for the system described in equation (13) with $g_{syn} = 0.005$ is shown. In Figure 9(a), the variable u_1 is represented by a solid line, u_2 by a dotted line, and u_3 by a dashed line. Similarly, in Figure 9(b), the variables are represented as v_1 with a solid line, v_2 with a dotted line, and v_3 with a dashed line. It is evident that the solid lines do not follow the same patterns as the dotted lines. In other words, identical synchronization is not observed in this scenario.

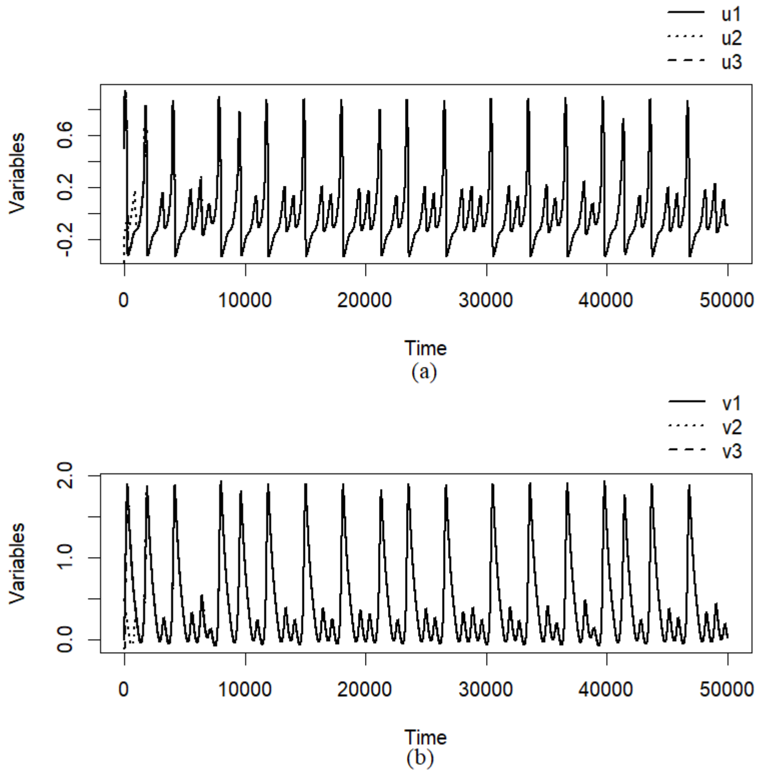


Figure 10: The time series of all state variables for the system described in equation (13) with $g_{syn} = 0.1$ is presented. In Figure 10(a), the variable u_1 is represented by the solid line, u_2 by the dotted line, and u_3 by the dashed line. Similarly, in Figure 10(b), the variables are represented as v_1, v_2, and v_3. Notably, the solid lines display behavior that mirrors that of the dotted lines. This indicates that an identical synchronization phenomenon occurs.

To assess the relationship between the number of neurons and coupling strength in full networks, we examined systems with neuron counts ranging from 2 to 20. The results are summarized in Table 1. In this table, for each value of n, we identify a critical coupling strength required to achieve synchronization in full networks as the number of neurons varies from 2 to 20.

Table 1: Minimal coupling strength is essential for achieving identical synchronization in full networks with linear coupling and external electrical stimulation over time.

n	2	3	4	5	6
g_{syn}	0.1	0.07	0.06	0.055	0.052
n	7	8	9	10	11
g_{syn}	0.05	0.049	0.048	0.047	0.046
n		12	13	14	15
g_{syn}		0.0454	0.0452	0.045	0.0442
n	16	17	18	19	20
g_{syn}	0.0441	0.044	0.0435	0.0433	0.043

Based on the numerical experiments conducted, it is evident that the coupling strength required for synchronization among n neurons is dependent on the number of neurons in the network. The data points illustrated in Figure 11 show the relationship between the coupling strength necessary for synchronization and the number of neurons in the complete networks listed in Table 1. Our aim is to establish a relationship between the number of neurons, n, and the corresponding coupling strength outlined in Table 1. This relationship can be represented by the following function:

$$g_{syn} = \frac{0.06}{n-1} + 0.04. \tag{14}$$

In Figure 11, the function represented by (14) is depicted as a curve, with points corresponding to the coupling strengths nearly aligned. This observation indicates that the coupling strength required for achieving the desired synchronization in fully connected networks adheres to the law defined by (14). The simulations demonstrate that as the number of neurons increases, the necessary coupling strength decreases. This suggests that achieving identical synchronization becomes easier when there are more neurons in fully connected networks.

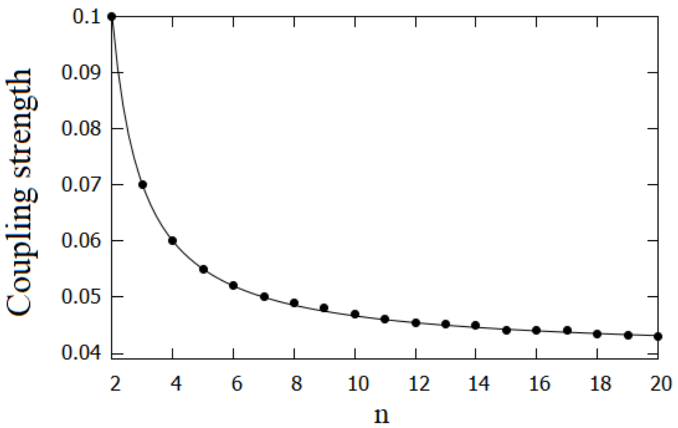


Figure 11: The relationship between coupling strength and the number of nodes in fully connected networks with linear coupling, as well as the effect of external electrical stimulation over time.

4 Conclusion

This study provides a sufficient condition for the coupling strength needed to achieve identical synchronization in full networks of n linearly coupled FitzHugh-Nagumo systems, with external electrical stimulation that varies over time. The findings indicate that the model can effectively represent the cell membrane voltage, as changes in the frequency of the external electrical stimulation can induce chaos in the FitzHugh-Nagumo model. Notably, Theorem 1 demonstrates that as the value of n increases, the required coupling strength g_{syn} decreases. This implies that a greater number of neurons makes synchronization easier to achieve. Additionally, numerical results support this conclusion. Further research is needed to explore different synchronization regimes in networks coupled through chemical synapses.

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