Advanced Mathematical Models & Applications
Vol.10, No.1, 2025, pp.26-42
https://doi.org/10.62476/amma.10126

SYNCHRONOUS CONTROLLER OF DRIVE-RESPONSE NEURAL NETWORKS OF n REACTION-DIFFUSION SYSTEMS OF HINDMARSH-ROSE TYPE

Phan Van Long Em*

An Giang University, Vietnam National University, Ho Chi Minh City, Vietnam

Abstract. In the paper we study the identical synchronization of drive-response neural networks of n reaction diffusion systems of Hindmarsh-Rose type with arbitrary topological structures. First the definition of the identical synchronization between the drive-response complex networks of n reaction-diffusion systems and some preliminary knowledge is given. By using the Lyapunov theory and LaSalle's invariance principle, we design some schemes to construct the response networks to achieve the desired synchronization. We propose a nonlinear adaptive controller in order to achieve the desired synchronization of these two networks of the same node dynamics. At the end of the paper two numerical examples are given to verify the simulation results and to demonstrate the effectiveness of the proposed method.

Keywords: drive-response neural networks, identical synchronization, synchronous controller, reaction-diffusion system of Hindmarsh-Rose.

AMS Subject Classification: Primary 00A69, 35Q68, 35Q90.

Corresponding author: Phan Van Long Em, An Giang University, Vietnam National University, Ho Chi Minh City, Vietnam, e-mail: *pvlem@agu.edu.vn*

Received: 25 July 2024; Revised: 17 October 2024; Accepted: 18 October 2024; Published: 10 April 2025.

1 Introduction

Synchronization is known as one of the most important topics of dynamical systems that have been investigated in recent years. Especially, it has been interesting in complex networks. As everyone knows, complex networks are various in nature and human society such as transportation networks, biological networks, social relationship networks, neuronal networks, etc (Aziz-Alaoui, 2006; Izhikevich, 2007; Jordan et al., 2007; Keener et al., 2009; Murray, 2010; Pikovsky et al., 2001), and have been studied in many domains, especially in neuroscience, and applied mathematics, (Jordan et al., 2007; Murray, 2010). There are many researchs reflecting the synchronization of complex networks, and their application in the real world (Ambrosio et al., 2012, 2013; Corson, 2009; Phan, 2022, 2023a, 2017, 2021a,b). The meaning of *synchronization* is to the same behavior at the same time (Aziz-Alaoui, 2006). Its principle can be seen in many different applications, also in social production and human activities (Jordan et al., 2007; Murray, 2010). Many studies on the synchronization of complex networks have been achieved. Especially, internal synchronization in a complex network is studied in many papers, for example, (Ambrosio et al., 2012, 2013; Corson, 2009; Phan, 2022, 2023a, 2017, 2021a,b). In reality, synchronization can also be seen between two networks presenting two different groups of cells with

How to cite (APA): Em, P.V.L. (2025). Synchronous controller of drive-response neural networks of n reaction-diffusion systems of Hindmarsh-Rose type. *Advanced Mathematical Models & Applications*, *10*(1), 26-42 https://doi.org/10.62476/

arbitrary network structures such as real-world systems or large-scale neural networks. Lately, some scientists have been researching partition dimension problems, which involve dividing a graph's vertex set into a minimum number of disjoint sets so that each vertex is distinct with respect to the representation from each disjoint set (Nithya et al., 2024; Raj et al., 2024). This method's development has led to numerous applications in fields like drug design, robot navigation, pattern recognition, and image processing. They have calculated the partition dimension of oxide and zigzag benzenoid networks, subdivision of benzenoid hydrocarbons and triangular benzene networks (Raj et al., 2024), as well as the partition dimension of the line graph of the honeycomb network, the Aztec diamond network, and the extended Aztec diamond network (Nithya et al., 2024). In essence, there is increasing work related to real-life networks, which is why our research also focuses on studying the drive-response networks with arbitrary topological structures. Clearly, in this paper, the identical synchronization of two complex networks of the same reaction-diffusion systems is studied. Specifically, each node is presented by one reaction-diffusion system of Hindmarsh-Rose type (see Figure 1). In Figure 1, the left graph is called the drive network, and the right one is called the response network. They may have different structures and connections which means it is difficult to synchronize them. That is a reason why in this paper we would like to design a nonlinear controller so that the identical synchronization of two complex networks with the same nodes and different topological structures is realized.

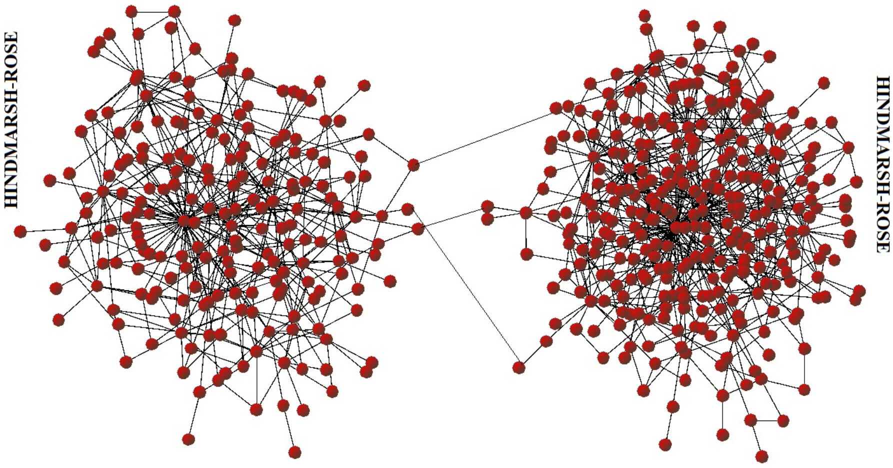

Figure 1: An example of the drive-response networks of reaction-diffusion systems of Hindmarsh-Rose type.

As everyone knows, there are many kinds of synchronization (Aziz-Alaoui, 2006), and the identical one is studied the most. Specifically, the identical synchronization of a network of n reaction-diffusion systems of Hindmarsh-Rose type is studied in Phan (2022, 2023a). There are also a lot of works studying the identical synchronization of n dynamical systems in complete networks such as (Corson, 2009; Phan, 2017, 2021a,b). However, there are not any studies on the identical synchronization of two neural networks of n reaction-diffusion systems of Hindmarsh-Rose model. In other words, this paper studies the problem of identical synchronization for drive-response neural networks of n reaction-diffusion systems of Hindmarsh-Rose type. In this work, the author considers the drive-response neural networks with arbitrary topological structures and designs an appropriate adaptive controller in order to get the identical synchronization of these two networks. The proof is based on Lyapunov stability theory and LaSalle's invariance principle. This paper also shows the numerical results to verify if the proposed method is effective.

The author organizes this paper as follows. In Section 2, the definition of the identical synchronization between the drive-response complex networks of n reaction-diffusion systems and some preliminary knowledge is given. By using the Lyapunov theory and LaSalle's invariance principle, we design some schemes to construct the response networks to achieve the desired synchronization. In Section 3, the author displays two numerical examples to see if the proposed method in Section 2 is effective. Finally, conclusions are displayed in Section 4.

2 Identical synchronization between two complex networks of n reaction-diffusion systems of Hindmarsh - Rose type

A community of cells that are physiologically connected is called a neuronal network. Especially, it is considered a kind of complex network, and the exchange between neurons is principle based on electrochemical processes (Corson, 2009). To express some dynamical properties of neural membrane potential, Hodgkin and Huxley published their research about a mathematical model in 1952 that could help present many interesting properties of neuronal membrane potential (Corson, 2009; Ermentrout et al., 2009; Hodgkin et al., 1952). However, it is a four-dimensional model, so it is not easy to investigate. Based on this system, many scientists published their simpler models to describe the voltage dynamics of neurons. Among them, Hindmarsh J. L. and Rose R. M. presented a mathematical model of two ordinary differential equations called Hindmarsh-Rose model in 1982 (Hindmarsh et al., 1982; Izhikevich, 2007). Although it is simpler compared to Hodgkin-Huxley model, it retains the dynamical properties and biological significance of cells, especially many extraordinary analytical results such as equilibrium state, activity, and bursting of the neuron voltage. Specifically, the Hindmarsh-Rose model is a system of two ordinary differential equations in two variables X and Y given by Corson (2009):

$$\begin{cases} \dfrac{dX}{dt} = X_t = g(X) + Y + I, \\[2ex] \dfrac{dY}{dt} = Y_t = 1 - bX^2 - Y, \end{cases} \tag{1}$$

where $g(X) = -X^3 + aX^2$; a, b are positive constants determined by practical experience; I presents the external current; t presents the time. The first variable is fast and excitatory. It introduces the transmembrane voltage of neurons. The second one is slow and presents some physical quantities, for example, the electrical conductivity of ion currents across the membrane (Phan, 2022, 2023a).

The model (1) is a system of two ordinary differential equations, so it can not help describe the distribution of action potential along the axon of the cell. To solve this problem, the Laplace operator could be added to the first equation of the model (1) in order to obtain a model that can describe the propagation of action potential. It is called the reaction-diffusion system of Hindmarsh-Rose type (HR) which is written as follows:

$$\begin{cases} X_t = g(X) + Y + I + d\Delta X, \\[2ex] Y_t = 1 - bX^2 - Y, \end{cases} \tag{2}$$

where $X = X(x,t),\ Y = Y(x,t),\ (x,t) \in \Omega \times \mathbb{R}^+,\ d$ is a positive constant, ΔX is the Laplace operator of X, $\Omega \subset \mathbb{R}^N$ is a regular bounded open set and with Neumann zero flux boundary conditions, and N is a positive integer. The model (2) consists of two nonlinear partial differential equations. Specifically, the first one of (2) is similar to the cable equation that could help describe the propagation of the membrane potential along the axon of a single neuron Ermentrout et al. (2009); Izhikevich (2007). Hereafter, we consider the system (2) as a neuronal model, and a network of n coupled reaction-diffusion systems (2) is constructed as follows:

$$\begin{cases} X_{it} = g(X_i) + Y_i + I_i + d_i\Delta X_i + \sum\limits_{j=1}^{n} c_{ij} h(X_i, X_j), \\ Y_{it} = 1 - bX_i{}^2 - Y_i, \end{cases} \quad i = 1, 2, ..., n. \tag{3}$$

where $(X_i, Y_i), I_i, d_i, i = 1, 2, ..., n$ is defined as (2). The coefficients c_{ij} are the elements of the connectivity matrix $C_n = (c_{ij})_{n\times n}$, defined by: $c_{ij} > 0$ if neuron ith and jth are coupled; $c_{ij} = 0$ if neuron ith and jth are not coupled. This matrix also presents the network topology. The function h presents the coupling function that could help describe the type of connection between cell ith and jth. As everyone knows, neurons connect through synapses, and there are chemical connections and electrical ones between cells.

If the cells connect through the chemical synapse, then the function h is nonlinear (Corson, 2009; Phan, 2023a) and its formula is given as follows:

$$h(X_i, X_j) = -(X_i - V_{syn})g_{syn}\frac{1}{1+\exp(-\lambda(X_j - \theta_{syn}))}, j \neq i, \tag{4}$$

where g_{syn} represents a positive number that is called the coupling strength; V_{syn} introduces the reversal potential, and its value must be larger than $X_i(x,t)$, for all $i = 1, 2, ..., n$, $x \in \Omega, t \geq 0$ since synapses are supposed excitatory; θ_{syn} represents the threshold value that is reached by every action potential; λ is a positive number that could be big enough to approach the Heaviside function (Belykh et al., 2005; Corson, 2009).

If the cells connect through the electrical synapse, then the function h is linear (Corson, 2009; Phan, 2022) and its formula is given as follows:

$$h(X_i, X_j) = -g_{syn}.(X_i - X_j), j \neq i. \tag{5}$$

The network (3) is considered a drive network of n reaction-diffusion systems of HR with arbitrary structures, and the following one corresponds to the response network:

$$\begin{cases} \overline{X}_{it} = g(\overline{X}_i) + \overline{Y}_i + I_i + d_i\Delta\overline{X}_i + \sum\limits_{j=1}^{n}\overline{c}_{ij}\overline{h}(\overline{X}_i, \overline{X}_j) + w_i, \\ \overline{Y}_{it} = 1 - b\overline{X}_i^2 - \overline{Y}_i, \end{cases} \quad i = 1, 2, ..., n. \tag{6}$$

where $(\overline{X}_i, \overline{Y}_i), i = 1, 2, ..., n$ is defined as (2). The coefficients $\overline{c}_{ij}$ $(i, j = 1, 2, ..., n)$ and the function $\overline{h}$ are defined as c_{ij} and h in network (3), respectively; $w_i, i = 1, 2, ..., n$ is the synchronous controller that we need to design in this paper.

We can see from the model of drive-response neuronal networks of reaction-diffusion systems (3) and (6) that the structures of these two networks can be different. For example, all cells in (3) are coupled by chemical synapses, and all cells in (6) are coupled by electrical synapses. Then, it is difficult to synchronize them because of their different structures. Therefore, we need to design the adaptive controllers to observe our desired synchronization.

Before going to the main results, we need to see some following remarks that help to prove our desired results.

Remark 1. *The function g satisfies the following condition:*

$$|g(X_i) - g(X_j)| \leq \alpha\,|X_i - X_j|\,,\ i, j = 1, 2, ..., n, \tag{7}$$

where X_i, X_j present the transmembrane voltages, and α is a positive number.

Proof. For all $X_i, X_j, i, j = 1, 2, ..., n$, we have:

$$\begin{aligned} g(X_i) - g(X_j) &= -X_i^3 + aX_i^2 + X_j^3 - aX_j^2 \\ &= (X_i - X_j)\left[a(X_i + X_j) - (X_i^2 + X_iX_j + X_j^2)\right]. \end{aligned}$$

Since $X_i, X_j, i, j = 1, 2, ..., n$ are bounded in Phan (2023b), then we can find a positive constant α such that:

$$|g(X_i) - g(X_j)| \leq \alpha\,|X_i - X_j|\,,\ i, j = 1, 2, ..., n.$$

□

Remark 2. *The function* h *defined by* (4) *satisfies the following condition:*

$$|h(X_i, X_k) - h(X_j, X_l)| \leq \beta \left| X_i - X_j \right|, \ i, j, k, l = 1, 2, ..., n, \tag{8}$$

where X_i, X_j, X_k, X_l *present the transmembrane voltages, and* β *is a positive number.*

Proof. For all $X_i, X_j, i, j = 1, 2, ..., n$, we have:

$$\begin{aligned}|h(X_i, X_k) - h(X_j, X_l)| = \Bigg| & g_{syn}(X_i - V_{syn}) \frac{1}{1 + \exp(-\lambda(X_k - \theta_{syn}))} \\ & - g_{syn}(X_j - V_{syn}) \frac{1}{1 + \exp(-\lambda(X_l - \theta_{syn}))} \Bigg|,\end{aligned}$$

where $k \neq i, l \neq j$.

Since $X_k, X_l, k, l = 1, 2, ..., n$ are bounded in Phan (2023b), then we can find a positive constant K such that:

$$-K \leq \frac{1}{1 + \exp(-\lambda(X_k - \theta_{syn}))} \leq K,$$

and

$$-K \leq \frac{1}{1 + \exp(-\lambda(X_l - \theta_{syn}))} \leq K.$$

Besides that $X_i - V_{syn} < 0$; $X_j - V_{syn} < 0$, for all $X_i, X_j, i, j = 1, 2, ..., n$, since we only consider the rapid chemical excitatory synapses (Belykh et al., 2005; Corson, 2009).

Thus

$$\begin{aligned}|h(X_i, X_k) - h(X_j, X_l)| &\leq |K g_{syn}(X_i - V_{syn}) + K g_{syn}(X_j - V_{syn})| \\ &\leq K g_{syn} \left| X_i + X_j - 2V_{syn} \right|.\end{aligned}$$

Moreover, $X_i, X_j, i, j = 1, 2, ..., n$ are bounded in Phan (2023b), then we can find a positive constant β such that:

$$|h(X_i, X_k) - h(X_j, X_l)| \leq \beta \left| X_i - X_j \right|.$$

□

Remark 3. *It is easy to see that the function* h *defined by* (5) *satisfies the following condition:*

$$|h(X_i, X_k) - h(X_j, X_l)| \leq \beta \left| X_i - X_j \right|, \ i, j, k, l = 1, 2, ..., n, \tag{9}$$

where X_i, X_j, X_k, X_l *present the transmembrane voltages, and* β *is a positive number.*

Remark 4. *The function* $\overline{h}$ *in network* (6) *is defined as* h *in network* (3), *then it also verifies the conditions in Remark 2 and Remark 3.*

Definition 1 (see Ambrosio et al. (2012))**.** *Let* $S_i = (X_i, Y_i), \ i = 1, 2, ..., n$ *and* $S = (S_1, S_2, ..., S_n)$ *be a network. We say that* S *is identically synchronous if*

$$\lim_{t \to +\infty} \sum_{i=1}^{n-1} \left(\|X_i - X_{i+1}\|_{L^2(\Omega)} + \|Y_i - Y_{i+1}\|_{L^2(\Omega)} \right) = 0,$$

where $L^2(\Omega)$ *is function space on* Ω *defined using a natural generalization of the 2-norm for finite-dimensional vector spaces.*

By applying Definition 1 to this study, we let the node error of the identical synchronization between two systems (3) and (6) be $e_i = \overline{X}_i - X_i, \overline{e}_i = \overline{Y}_i - Y_i$, $i = 1, 2, ..., n$.

If we can design a controller w_i such that the Definition 1 satisfies, it means:

$$\lim_{t \to +\infty} \sum_{i=1}^{n} \left(\|e_i\|_{L^2(\Omega)} + \|\overline{e}_i\|_{L^2(\Omega)} \right) = 0,$$

then the drive-response networks (3) and (6) are said to be identical synchronization.

To get the identical synchronization of drive-response networks (3) and (6), the controller w_i can be designed as follows:

$$w_i = X_{it} - g(X_i) - Y_i - I_i - d_i \Delta X_i - \sum_{j=1}^{n} \overline{c}_{ij} \overline{h}(X_i, X_j) - k_i e_i, \tag{10}$$

with the updated rules defined as follows:

$$k_{it} = r_i e_i^2, \tag{11}$$

where r_i is an arbitrary positive constant, for $i = 1, 2, ..., n$.

Based on the action of the controller, we can write the error dynamic equations of the system as follows:

$$\begin{aligned} e_{it} &= (\overline{X}_{it} - X_{it}) \\ &= g(\overline{X}_i) + \overline{Y}_i + I_i + d_i \Delta \overline{X}_i + \sum_{j=1}^{n} \overline{c}_{ij} \overline{h}(\overline{X}_i, \overline{X}_j) \\ &\qquad - g(X_i) - Y_i - I_i - d_i \Delta X_i - \sum_{j=1}^{n} \overline{c}_{ij} \overline{h}(X_i, X_j) - k_i e_i \\ &= g(\overline{X}_i) - g(X_i) + (\overline{Y}_i - Y_i) + d_i \Delta (\overline{X}_i - X_i) \\ &\qquad + \sum_{j=1}^{n} \overline{c}_{ij} (\overline{h}(\overline{X}_i, \overline{X}_j) - \overline{h}(X_i, X_j)) - k_i e_i, \end{aligned} \tag{12}$$

and

$$\overline{e}_{it} = \overline{Y}_{it} - Y_{it} = -b(\overline{X}_i^2 - X_i^2) - (\overline{Y}_i - Y_i), \tag{13}$$

for $i = 1, 2, ..., n$.

Next, we investigate the identical synchronization problem of drive-response networks (3) and (6), and the theorem below is the main result.

Theorem 1. *The drive-response neural networks* (3) *and* (6) *achieve identical synchronization under the adaptive controller* (10) *and updated rule* (11).

Proof. To prove this theorem, we need to construct the Lyapunov function as follows:

$$V(x,t) = \frac{1}{2} \sum_{i=1}^{n} \int_{\Omega} \left[e_i^2 + \overline{e}_i^2 + \frac{1}{r_i} (k_i - k)^2 \right] dx, \tag{14}$$

where k is a positive constant that we need to determine.

We calculate the derivative of $V(x,t)$ with respect to the time t and along the error systems (12) and (13), we get:

$$\begin{aligned} \frac{dV(x,t)}{dt} &= \sum_{i=1}^{n} \int_{\Omega} \left[e_i e_{it} + \overline{e}_i \overline{e}_{it} + \frac{1}{r_i} (k_i - k) k_{it} \right] dx \\ &= \sum_{i=1}^{n} \int_{\Omega} \Big[e_i \Big(g(\overline{X}_i) - g(X_i) + (\overline{Y}_i - Y_i) + d_i \Delta (\overline{X}_i - X_i) \\ &\qquad + \sum_{j=1}^{n} \overline{c}_{ij} (\overline{h}(\overline{X}_i, \overline{X}_j) - \overline{h}(X_i, X_j)) - k_i e_i \Big) \\ &\qquad + \overline{e}_i \left(-b(\overline{X}^2{}_i - X_i^2) - (\overline{Y}_i - Y_i) \right) + \frac{1}{r_i} (k_i - k) k_{it} \Big] dx. \end{aligned} \tag{15}$$

By using the Green formula and Neumann zero flux boundary conditions, (15) becomes:

$$
\begin{aligned}
\frac{dV(x,t)}{dt} \leq & \sum_{i=1}^{n} \int_{\Omega} \left[e_i \left(g(\overline{X}_i) - g(X_i) \right) + e_i \overline{e}_i \right. \\
& \quad + e_i \sum_{j=1}^{n} \overline{c}_{ij} (\overline{h}(\overline{X}_i, \overline{X}_j) - \overline{h}(X_i, X_j)) \\
& \quad \left. - k_i e_i^2 - b e_i \overline{e}_i (\overline{X}_i + X_i) - \overline{e}_i^2 + k_i e_i^2 - k e_i^2 \right] dx \\
\leq & \sum_{i=1}^{n} \int_{\Omega} \Bigg[e_i \left(g(\overline{X}_i) - g(X_i) \right) + e_i \sum_{j=1}^{n} \overline{c}_{ij} (\overline{h}(\overline{X}_i, \overline{X}_j) - \overline{h}(X_i, X_j)) \\
& \quad \left. e_i \overline{e}_i - b e_i \overline{e}_i (\overline{X}_i + X_i) - \overline{e}_i^2 - k e_i^2 \right] dx.
\end{aligned} \tag{16}
$$

By using Remarks 1-4, it is easy to obtain:

$$
\begin{aligned}
\frac{dV(x,t)}{dt} \leq & \sum_{i=1}^{n} \int_{\Omega} \left[\alpha e_i^2 - \bar{e}_i^2 - k e_i^2 \right. \\
& \left. + |e_i| \, |\overline{e}_i| \left(1 + b(|\overline{X}_i| + |X_i|) \right) + \sum_{j=1}^{n} \beta \, |\bar{c}_{ij}| \, |e_i| \, |e_j| \right] dx.
\end{aligned} \tag{17}
$$

By using the Young's inequality for every $\delta > 0$, we can see:

$$
\begin{aligned}
|e_i| \, |\bar{e}_i| \left(1 + b(|\overline{X}_i| + |X_i|) \right) & \leq (1 + b(|\overline{X}_i| + |X_i|)) \left(\frac{1}{2\delta} e_i^2 + \frac{\delta}{2} \bar{e}_i^2 \right) \\
& \leq \frac{M}{2\delta} e_i^2 + \frac{M\delta}{2} \bar{e}_i^2,
\end{aligned} \tag{18}
$$

where M is a positive constant, since $X_i, \overline{X}_i, i = 1, 2, ..., n$ are bounded (see Phan (2023b)).

Besides that, we have:

$$
\begin{aligned}
\sum_{i=1}^{n} \sum_{j=1}^{n} \beta \, |\bar{c}_{ij}| \, |e_i| \, |e_j| & = \beta \sum_{i=1}^{n} \left(|e_i| \sum_{j=1}^{n} |\bar{c}_{ij}| \, |e_j| \right) \\
& \leq \beta \left(\sum_{i=1}^{n} e_i^2 . \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |\bar{c}_{ij}| \, |e_j| \right)^2 \right)^{\frac{1}{2}} \\
& \leq \beta \left(\sum_{i=1}^{n} e_i^2 . \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \bar{c}_{ij}{}^2 . \sum_{j=1}^{n} e_j^2 \right) \right)^{\frac{1}{2}} \\
& \leq \beta \left(\left(\sum_{i=1}^{n} e_i^2 \right)^2 . \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \bar{c}_{ij}{}^2 \right) \right)^{\frac{1}{2}} \\
& \leq \beta \left(\left(\sum_{i=1}^{n} e_i^2 \right)^2 . n^2 \max_{1 \leq i,j \leq n} \bar{c}_{ij}{}^2 \right)^{\frac{1}{2}} \\
& \leq \beta n \max_{1 \leq i,j \leq n} |\bar{c}_{ij}| \sum_{i=1}^{n} e_i^2.
\end{aligned} \tag{19}
$$

Combining (17), (18) and (19) yields:

$$\begin{aligned}\frac{dV(x,t)}{dt} &\leq \sum_{i=1}^{n}\int_{\Omega}[\alpha e_i^2 - \overline{e}_i^2 - ke_i^2 + \frac{M}{2\delta}e_i^2 + \frac{M\delta}{2}\overline{e}_i^2 \\ &\quad +\beta n \max_{1\leq i,j\leq n}|\overline{c}_{ij}|\, e_i^2]dx \\ &\leq \sum_{i=1}^{n}\int_{\Omega}[(\alpha - k + \beta n \max_{1\leq i,j\leq n}|\overline{c}_{ij}| + \frac{M}{2\delta})e_i^2 \\ &\quad -(1-\frac{M\delta}{2})\overline{e}_i^2]dx \\ &\leq \sum_{i=1}^{n}\int_{\Omega}[-(k-\alpha-\beta n \max_{1\leq i,j\leq n}|\overline{c}_{ij}| - \frac{M}{2\delta})e_i^2 - (1-\frac{M\delta}{2})\overline{e}_i^2]dx.\end{aligned} \tag{20}$$

Chose $\delta > 0$ such that $1 - \frac{M\delta}{2} > 0$, and take $k > \alpha + \beta n \max_{1\leq i,j\leq n}|\overline{c}_{ij}| + \frac{M}{2\delta}$, then (20) can be estimated as:

$$\frac{dV(x,t)}{dt} \leq -\gamma\sum_{i=1}^{n}\int_{\Omega}\left[\frac{1}{2}(e_i^2+\overline{e}_i^2)\right]dx, \tag{21}$$

where

$$\gamma = \min\left\{2(k-\alpha-\beta n \max_{1\leq i,j\leq n}|\overline{c}_{ij}| - \frac{M}{2\delta}); 2(1-\frac{M\delta}{2})\right\}.$$

From (21), it implies that $0 \leq V(x,t) \leq V(x,0)$, and together with (14) it signifies that $V(x,t)$ is bounded. Based on the stability theory of Lyapunov and the invariance principle of LaSalle Aeyels (1995), we have:

$$\lim_{t\to+\infty}\sum_{i=1}^{n}\left(\|e_i\|_{L^2(\Omega)} + \|\overline{e}_i\|_{L^2(\Omega)}\right) = 0.$$

Then it implies that the drive-response networks of reaction-diffusion systems (3) and (6) achieve identical synchronization in the sense of Definition 1. In other words, we complete the proof. □

3 Illustrative Numerical Examples

In this section, the author concretely shows the numerical results through two examples of drive and response networks of reaction-diffusion systems of HR with different network topologies to check if the proposed method in the previous section is effective. We use C++ to integrate the reaction-diffusion systems and the pattern results are represented by Gnuplot.

Some parameters are fixed as (Belykh et al., 2005; Corson, 2009; Phan, 2022, 2023a):

$$g(X) = -X^3 + aX^2, a = 3, b = 5,$$

$$I_i = 0,\ d_i = 0.05,\ i = 1,2,...,n,$$

$$[0;T]\times\Omega = [0;1000]\times[0;100]\times[0;100],$$

$$\lambda = 10,\ V_{syn} = 2,\ \theta_{syn} = -0,25.$$

3.1 Example 1

In this example, we consider a drive network of 3 nodes that has the structure called a ring network with unidirectionally linear coupling as follows:

$$\begin{cases} X_{1t} = g(X_1) + Y_1 + I_1 + d_1\Delta X_1 - g_{syn}(X_1 - X_2), \\ Y_{1t} = 1 - bX_1^2 - Y_1, \\ X_{2t} = g(X_2) + Y_2 + I_2 + d_2\Delta X_2 - g_{syn}(X_2 - X_3), \\ Y_{2t} = 1 - bX_2^2 - Y_2, \\ X_{3t} = g(X_3) + Y_3 + I_3 + d_3\Delta X_3 - g_{syn}(X_3 - X_1), \\ Y_{3t} = 1 - bX_3^2 - Y_3, \end{cases} \tag{22}$$

and the response network of 3 nodes that have the structure called a complete (full) network with linear coupling is described by:

$$\begin{cases} \overline{X}_{1t} = g(\overline{X}_1) + \overline{Y}_1 + I_1 + d_1\Delta\overline{X}_1 \\ \qquad\qquad -g_{syn}(\overline{X}_1 - \overline{X}_2) - g_{syn}(\overline{X}_1 - \overline{X}_3) + w_1, \\ \overline{Y}_{1t} = 1 - b\overline{X}_1^2 - \overline{Y}_1, \\ \overline{X}_{2t} = g(\overline{X}_2) + \overline{Y}_2 + I_2 + d_2\Delta\overline{X}_2 \\ \qquad\qquad -g_{syn}(\overline{X}_2 - \overline{X}_3) - g_{syn}(\overline{X}_2 - \overline{X}_1) + w_2, \\ \overline{Y}_{2t} = 1 - b\overline{X}_2^2 - \overline{Y}_2, \\ \overline{X}_{3t} = g(\overline{X}_3) + \overline{Y}_3 + I_3 + d_3\Delta\overline{X}_3 \\ \qquad\qquad -g_{syn}(\overline{X}_3 - \overline{X}_1) - g_{syn}(\overline{X}_3 - \overline{X}_2) + w_3, \\ \overline{Y}_{3t} = 1 - b\overline{X}_3^2 - \overline{Y}_3, \end{cases} \tag{23}$$

with the adaptive controllers defined as:

$$\begin{cases} w_1 = X_{1t} - g(X_1) - Y_1 - I_1 - d_1\Delta X_1 \\ \qquad\qquad +g_{syn}(X_1 - X_2) + g_{syn}(X_1 - X_3) - k_1(\overline{X}_1 - X_1), \\ w_2 = X_{2t} - g(X_2) - Y_2 - I_2 - d_2\Delta X_2 \\ \qquad\qquad +g_{syn}(X_2 - X_1) + g_{syn}(X_2 - X_3) - k_2(\overline{X}_2 - X_2), \\ w_3 = X_{3t} - g(X_3) - Y_3 - I_3 - d_3\Delta X_3 \\ \qquad\qquad +g_{syn}(X_3 - X_1) + g_{syn}(X_3 - X_2) - k_3(\overline{X}_3 - X_3), \end{cases} \tag{24}$$

where the updated rules are defined as:

$$\begin{cases} k_{1t} = r_1(\overline{X}_1 - X_1)^2, \\ k_{2t} = r_2(\overline{X}_2 - X_2)^2, \\ k_{3t} = r_3(\overline{X}_3 - X_3)^2. \end{cases} \tag{25}$$

Notice that the adaptive controllers (24) and the updated rules (25) are defined as (10) and (11), respectively.

Figure 2 below illustrates the synchronization errors of drive-response networks of reaction-diffusion systems (22) and (23) without the adaptive controllers (24) and the updated rules (25). Specifically, Figure 2(a), 2(b), 2(c) display the synchronization errors of the coupled solutions, respectively:

$$\left(X_1(x_1,x_2,t), \overline{X}_1(x_1,x_2,t)\right), \ \left(X_2(x_1,x_2,t), \overline{X}_2(x_1,x_2,t)\right),$$

$$\text{and } \left(X_3(x_1,x_2,t), \overline{X}_3(x_1,x_2,t)\right),$$

where $t \in [0;T]$ and for all $(x_1,x_2) \in \Omega$, with $g_{syn} = 0.1$. This figure shows that the synchronization errors do not reach zero, which means the drive-response networks of reaction-diffusion systems of HR (22) and (23) can not achieve identical synchronization without the adaptive controllers (24) and the updated rules (25).

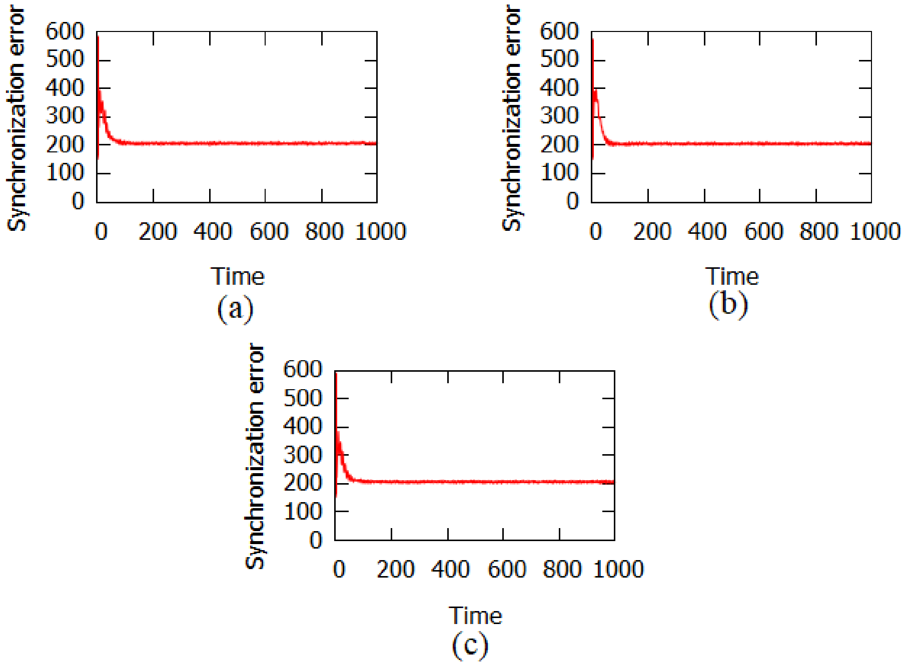

Figure 2: Synchronization errors of the drive-response networks of HR (22) and (23) without controllers (24). We can see that the synchronization errors do not reach zero which means there is not the synchronization.

Figure 3 below illustrates the synchronization errors of drive-response networks of reaction-diffusion systems of HR (22) and (23) with the adaptive controllers (24) and the updated rules (25). The simulations show that this adaptive controller is effective and we can obtain:

$$\lim_{t\to+\infty}\sum_{i=1}^{3}\left(\|e_i\|_{L^2(\Omega)}+\|\overline{e}_i\|_{L^2(\Omega)}\right)=0,$$

where $e_i = \overline{X}_i - X_i, \overline{e}_i = \overline{Y}_i - Y_i$, $i = 1, 2, 3$. Specifically, Figure 3(a), 3(b), 3(c) display the synchronization errors of the coupled solutions, respectively:

$$\left(X_1(x_1,x_2,t),\overline{X}_1(x_1,x_2,t)\right),\ \left(X_2(x_1,x_2,t),\overline{X}_2(x_1,x_2,t)\right),$$

$$\text{and}\ \left(X_3(x_1,x_2,t),\overline{X}_3(x_1,x_2,t)\right),$$

where $t \in [0;T]$ and for all $(x_1, x_2) \in \Omega$, with $g_{syn} = 0.1, r_1 = 0.1, r_2 = 0.2, r_3 = 0.3$. In this figure, we can see that the synchronization errors asymptotically reach zero, which means:

$$X_1(x_1,x_2,t)\approx\overline{X}_1(x_1,x_2,t),\ X_2(x_1,x_2,t)\approx\overline{X}_2(x_1,x_2,t)$$

$$\text{and}\ X_3(x_1,x_2,t)\approx\overline{X}_3(x_1,x_2,t),$$

for all $(x_1, x_2) \in \Omega$.

Figure 4(a), 4(b), 4(c) represent the solutions $X_i(x_1, x_2, 999), i = 1, 2, 3$, of the drive network (22) and Figure 4(d), 4(e), 4(f) perform the solutions $\overline{X}_i(x_1, x_2, 999)$, $i = 1, 2, 3$, of the response network (23). We can see that they have the same shape, i.e., the identical synchronization between drive and response networks is performed.

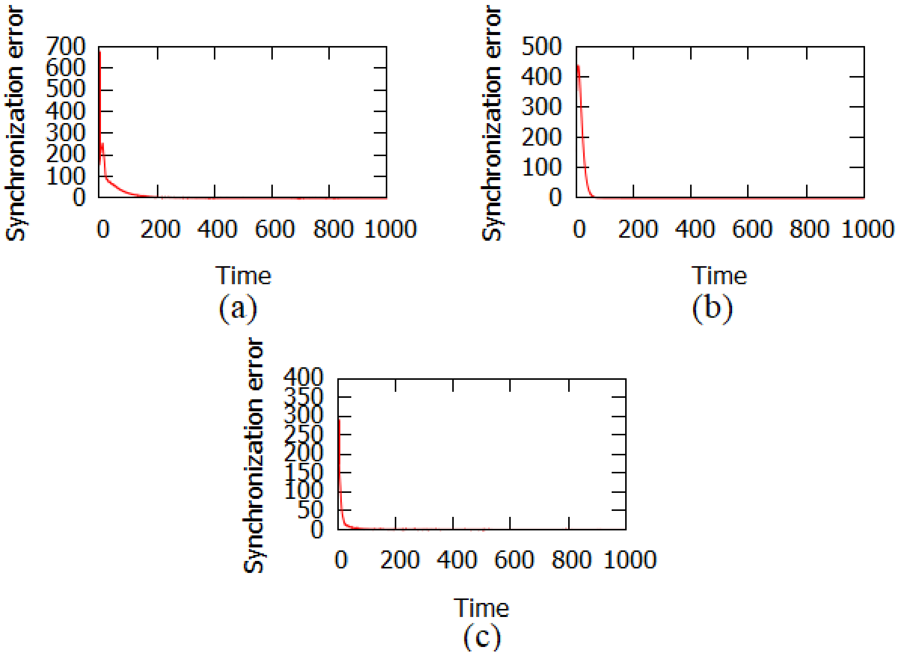

Figure 3: Synchronization errors of the drive-response networks of reaction-diffusion systems of HR (22) and (23) with controllers (24). We can see that the synchronization errors asymptotically reaches zero which means the synchronization occurs.

3.2 Example 2

Consider a drive network of 2 nodes that has the structure called a complete (full) network with linear coupling as follows:

$$\begin{cases} X_{1t} = g(X_1) + Y_1 + I_1 + d_1 \Delta X_1 - g_{syn}(X_1 - X_2), \\ Y_{1t} = 1 - bX_1^2 - Y_1, \\ X_{2t} = g(X_2) + Y_2 + I_2 + d_2 \Delta X_2 - g_{syn}(X_2 - X_1), \\ Y_{2t} = 1 - bX_2^2 - Y_2, \end{cases} \tag{26}$$

and the response network of 2 nodes that have the structure called a chain network with nonlinear coupling is described by:

$$\begin{cases} \overline{X}_{1t} = g(\overline{X}_1) + \overline{Y}_1 + I_1 + d_1 \Delta \overline{X}_1 \\ \qquad\qquad -g_{syn}(\overline{X}_1 - V_{syn}) \dfrac{1}{1 + \exp(-\lambda(\overline{X}_2 - \theta_{syn}))} + w_1, \\ \overline{Y}_{1t} = 1 - b\overline{X}_1^2 - \overline{Y}_1, \\ \overline{X}_{2t} = g(\overline{X}_2) + \overline{Y}_2 + I_2 + d_2 \Delta \overline{X}_2 + w_2, \\ \overline{Y}_{2t} = 1 - b\overline{X}_2^2 - \overline{Y}_2, \end{cases} \tag{27}$$

with the adaptive controllers defined as:

$$\begin{cases} w_1 = X_{1t} - g(X_1) - Y_1 - I_1 - d_1 \Delta X_1 \\ \qquad + g_{syn}(X_1 - V_{syn}) \dfrac{1}{1 + \exp(-\lambda(X_2 - \theta_{syn}))} - k_1(\overline{X}_1 - X_1), \\ w_2 = X_{2t} - g(X_2) - Y_2 - I_2 - d_2 \Delta X_2 - k_2(\overline{X}_2 - X_2), \end{cases} \tag{28}$$

where the updated rules are defined as:

$$\begin{cases} k_{1t} = r_1(\overline{X}_1 - X_1)^2, \\ k_{2t} = r_2(\overline{X}_2 - X_2)^2. \end{cases} \tag{29}$$

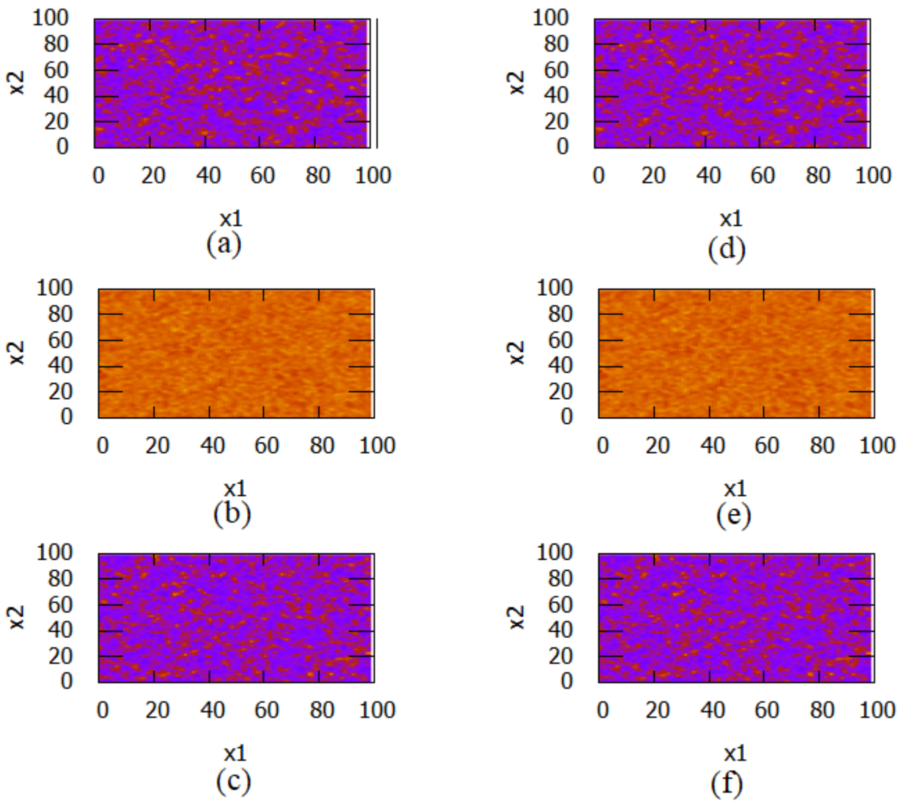

Figure 4: Synchronization patterns of the drive-response networks of reaction-diffusion systems of HR (22) and (23) with controllers (24). Figure 4(a), 4(b), 4(c) represent the solutions $u_i(x_1, x_2, 499), i = 1, 2, 3$, of the drive network (22), and Figure 4(d), 4(e), 4(f), perform the solutions $\overline{u}_i(x_1, x_2, 499)$, $i = 1, 2, 3$, of the response network (23). We can see that the patterns of the second column have the same shape with the patterns of the first column, respectively. In other words, the response network (22) synchronizes with the drive network (23).

Notice that the adaptive controllers (26) and the updated rules (27) are defined as (10) and (11), respectively.

Figure 5 below illustrates the synchronization errors of drive-response networks of reaction-diffusion systems of HR (26) and (27) without the adaptive controllers (28) and the updated rules (29). Specifically, Figure 5(a), 5(b) represent the synchronization errors of the coupled solutions, respectively:

$$\left(X_1(x_1, x_2, t), \overline{X}_1(x_1, x_2, t)\right), \text{ and } \left(X_2(x_1, x_2, t), \overline{X}_2(x_1, x_2, t)\right),$$

where $t \in [0; T]$ and for all $(x_1, x_2) \in \Omega$, with $g_{syn} = 0.1$. This figure shows that the synchronization errors do not reach zero, which means the drive-response networks of reaction-diffusion systems of HR (26) and (27) can not achieve identical synchronization without the adaptive controllers (28) and the updated rules (29).

Figure 6 below illustrates the synchronization errors of drive-response networks of reaction-diffusion systems of HR (26) and (27) with the adaptive controllers (28) and the updated rules (29). The simulations show that this adaptive controller is effective and we can obtain:

$$\lim_{t \to +\infty} \sum_{i=1}^{2} \left(\|e_i\|_{L^2(\Omega)} + \|\overline{e}_i\|_{L^2(\Omega)} \right) = 0,$$

where $e_i = \overline{X}_i - X_i, \overline{e}_i = \overline{Y}_i - Y_i$, $i = 1, 2$. Specifically, Figure 6(a), 6(b) represent the synchro-

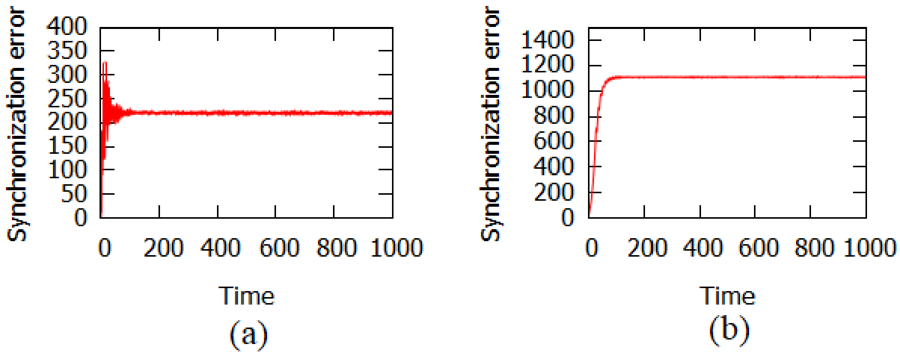

Figure 5: Synchronization errors of the drive-response networks (26) and (27) without controllers (28). We can see that the synchronization errors do not reach zero which means there is not the synchronization.

nization errors of the coupled solutions, respectively:

$$\left(X_1(x_1,x_2,t),\overline{X}_1(x_1,x_2,t)\right), \text{ and } \left(X_2(x_1,x_2,t),\overline{X}_2(x_1,x_2,t)\right),$$

where $t \in [0;T]$ and for all $(x_1,x_2) \in \Omega$, with $g_{syn} = 0.1, r_1 = 0.1, r_2 = 0.2$. In this figure, we can see that the synchronization errors asymptotically reach zero, which means:

$$X_1(x_1,x_2,t) \approx \overline{X}_1(x_1,x_2,t), \text{ and } X_2(x_1,x_2,t) \approx \overline{X}_2(x_1,x_2,t),$$

for all $(x_1,x_2) \in \Omega$.

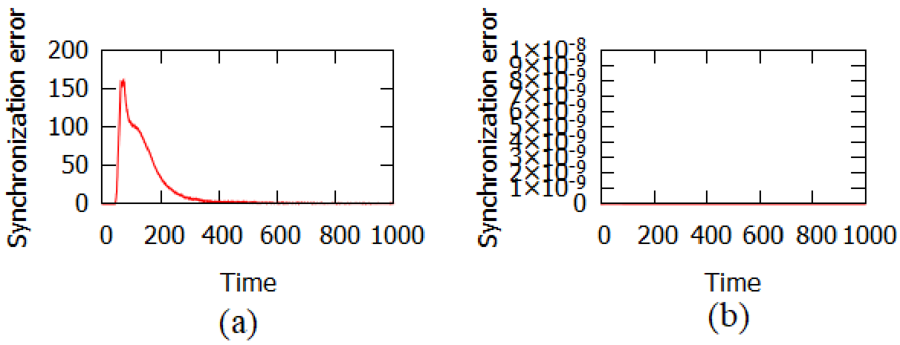

Figure 6: Synchronization errors of the drive-response networks of reaction-diffusion systems of HR (26) and (27) with controllers (28). We can see that the synchronization errors asymptotically reaches zero which means the synchronization occurs.

Figure 7(a), 7(b) display the solutions $X_i(x_1,x_2,999), i = 1,2$, of the drive network (26) and Figure 7(c), 7(d) perform the solutions $\overline{X}_i(x_1,x_2,999)$, $i = 1,2$, of the response network (27). We can see that they have the same shape, i.e., the identical synchronization between drive and response networks is performed.

Remark 5. *Notice that the synchronization above is the identical synchronization between two networks of cells. It means the ith cell of the first network will have the same pattern as the ith cell of the second network. In other words, one network will copy the behavior of the other. Meanwhile, the patterns of cells in the same network can be different (see Figure 4 and Figure 7). However, if we want all cells of both networks to have the same pattern, we need to increase the value of the coupling strength such that it is larger than the necessary threshold value (Belykh et al., 2005; Corson, 2009). Then all cells in both networks will have the same pattern (see Figure*

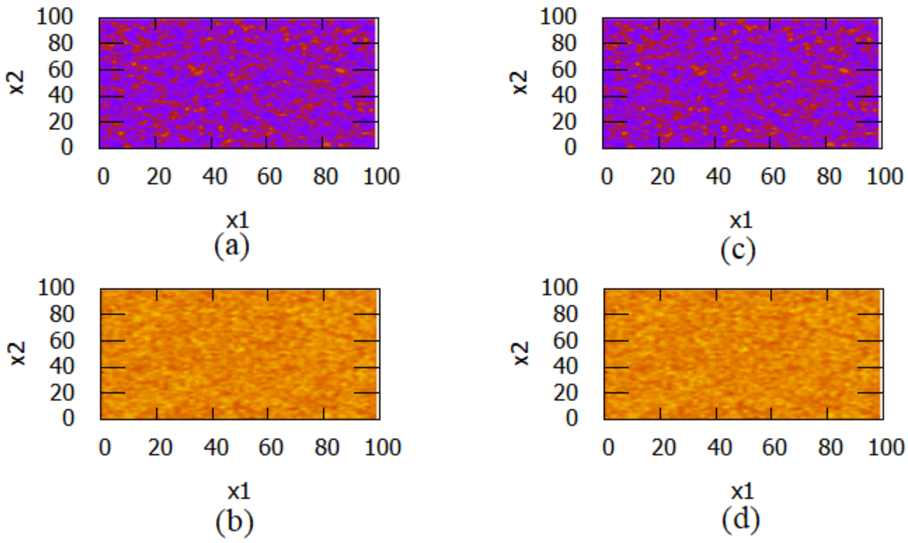

Figure 7: Synchronization patterns of the drive-response networks of reaction-diffusion systems of HR (26) and (27) with controllers (28). Figure 7(a), 7(b) represent the solutions $u_i(x_1, x_2, 499), i = 1, 2$, of the drive network (26), and Figure 7(c), 7(d) perform the solutions $\overline{u}_i(x_1, x_2, 499)$, $i = 1, 2$, of the response network (27). We can see that the patterns of the second column have the same shape with the patterns of the first column, respectively. In other words, the response network (26) synchronizes with the drive network (27).

9). In Figure 9, we take $g_{syn} = 2$, this value is big enough to get the same behavior for all cells of networks (26) *and* (27)*.*

Figure 8(a), 8(b), 8(c), and 8(d) represent the synchronization errors of the coupled solutions, respectively:

$$\left(X_1(x_1, x_2, t), \overline{X}_1(x_1, x_2, t)\right), \text{ and } \left(X_2(x_1, x_2, t), \overline{X}_2(x_1, x_2, t)\right),$$

$$\left(X_1(x_1, x_2, t), X_2(x_1, x_2, t)\right), \text{ and } \left(\overline{X}_1(x_1, x_2, t), \overline{X}_2(x_1, x_2, t)\right),$$

where $t \in [0; T]$ and for all $(x_1, x_2) \in \Omega$, with $g_{syn} = 2, r_1 = 0.1, r_2 = 0.2$. In this figure, we can see the synchronization errors asymptotically reach zero, which means:

$$X_1(x_1, x_2, t) \approx \overline{X}_1(x_1, x_2, t), \text{ and } X_2(x_1, x_2, t) \approx \overline{X}_2(x_1, x_2, t)$$

$$X_1(x_1, x_2, t) \approx X_2(x_1, x_2, t), \text{ and } \overline{X}_1(x_1, x_2, t) \approx \overline{X}_2(x_1, x_2, t),$$

for all $(x_1, x_2) \in \Omega$.

Figure 9(a), 9(b) display the solutions $X_i(x_1, x_2, 999), i = 1, 2$, of the drive network (26), and Figure 9(c), 9(d) perform the solutions $\overline{X}_i(x_1, x_2, 999)$, $i = 1, 2$, of the response network (27). We can see that they have the same shape, i.e., identical synchronization is achieved for all cells of both networks.

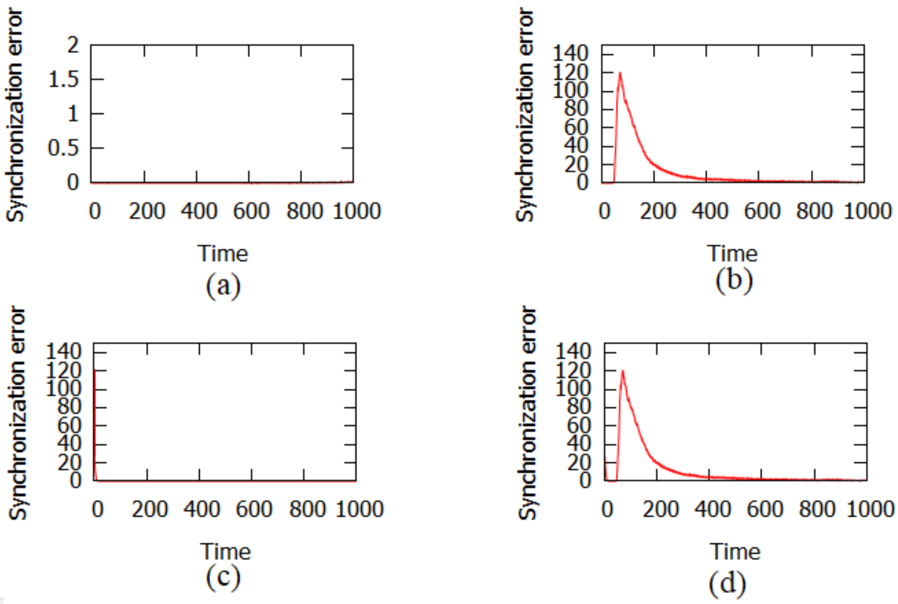

Figure 8: Synchronization errors of the drive-response networks of reaction-diffusion systems of HR (26) and (27) with controllers (28) and big enough coupling strength. We can see that the synchronization errors asymptotically reaches zero which means the synchronization occurs.

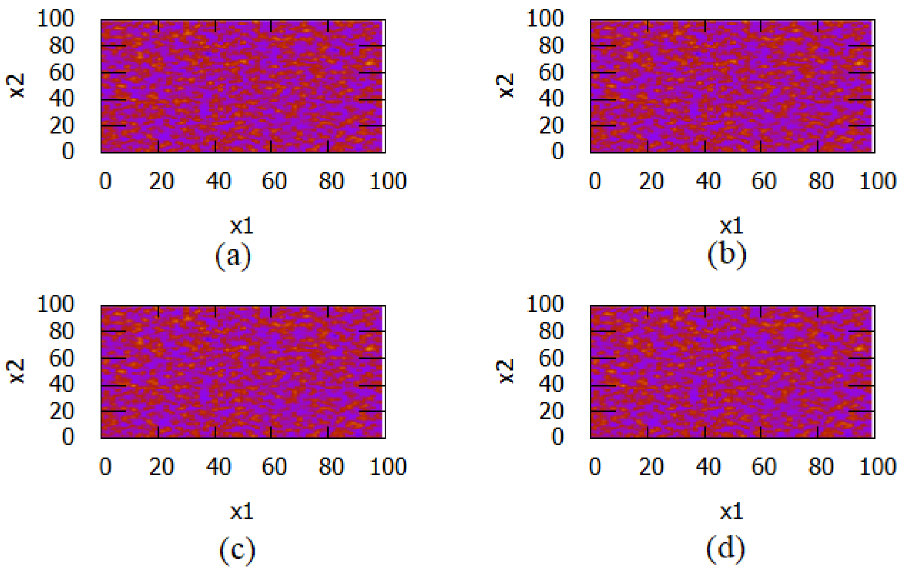

Figure 9: Synchronization patterns of the drive-response networks of reaction-diffusion systems of HR (26) and (27) with controllers and big enough coupling strength. We can see that all patterns of both networks have the same shape which means the identical synchronization is achieved for all cells of both networks.

4 Conclusion

This paper explores the identical synchronization of drive-response neural networks consisting of n reaction-diffusion systems of Hindmarsh-Rose type with arbitrary topological structures. Achieving the desired synchronization is challenging due to this distinction. In this study, we introduce nonlinear adaptive controllers and develop a suitable Lyapunov function to attain the desired results. Notably, simulation results demonstrate the effectiveness of the proposed method. The controller for identical synchronization in this paper is complex. However, it not only yields theoretical results but also numerical ones as desired. Simplifying the controller to have a more straightforward form will be a topic for future study.

Acknowledgments

This research is funded by Vietnam National University HoChiMinh City (VNU-HCM) under grant number C2024-16-03.

References

Ambrosio, B., Aziz-Alaoui, M.A. (2012). Synchronization and control of coupled reaction-diffusion systems of the FitzHugh-Nagumo-type. *Computers and Mathematics with Applications*, *64*, 934-943.

Ambrosio, B., Aziz-Alaoui, M.A. (2013). Synchronization and control of a network of coupled reaction-diffusion systems of generalized FitzHugh-Nagumo type. *ESAIM: Proceedings*, *39*, 15-24.

Aeyels, D. (1995). Asymptotic Stability of Nonautonomous Systems by Lyapunov's Direct Method. *Systems and Control Letters, 25*, 273-280.

Aziz-Alaoui, M.A. (2006). Synchronization of Chaos. *Encyclopedia of Mathematical Physics, Elsevier*, *5*, 213-226.

Belykh, I., De Lange, E. & Hasler, M. (2005). Synchronization of bursting neurons: What matters in the network topology. *Phys. Rev. Lett.*, 188101.

Corson, N. (2009). *Dynamics of a neural model, synchronization and complexity.* PhD Thesis, University of Le Havre, France.

Ermentrout, G.B., Terman, D.H. (2009). Mathematical Foundations of Neurosciences. Springer.

Hindmarsh, J.L., Rose, R.M. (1982). A model of the nerve impulse using two first order differential equations. *Nature*, *296*, 162-164.

Hodgkin, A.L., Huxley, A.F. (1952). A quantitative description of membrane current and its application to conduction and excitation in nerve. *J. Physiol., 117*, 500-544.

Izhikevich, E.M. (2007). *Dynamical Systems in Neuroscience.* The MIT Press.

Jordan, D.W., Smith, P. (2007). *Nonlinear Ordinary Differential Equations, An Introduction for Scientists and Engineers (4th Edition).* Oxford.

Keener, J.P., Sneyd, J. (2009). *Mathematical Physiology.* Springer.

Murray, J.D. (2010). *Mathematical Biology.* Springer.

Nithya Raj, R., Sundara Rajan, R., Ragunathan, T., Muthuvairavan Pillai, N. & S. Balamuralitharan (2024). Partition dimension for line graph of honeycomb networks and aztec diamond networks. *Advanced Mathematical Models & Applications, 9*(1), 121-129 https://doi.org/10.62476/amma9121

Phan, V.L.E. (2022). Sufficient condition for synchronization in complete networks of reaction-diffusion equations of Hindmarsh-Rose type with linear coupling. *IAENG International Journal of Applied Mathematics*, *52*(2), 315-319.

Phan, V.L.E. (2023a). Sufficient condition for synchronization in complete networks of n reaction-diffusion systems of Hindmarsh-Rose type with nonlinear coupling. *Engineering Letters, 31*(1), 413-418.

Phan, V.L.E. (2023b). Global attractor of networks of n coupled reaction-diffusion systems of Hindmarsh-Rose type. *Engineering Letters, 31*(3), 1215-1220.

Phan, V.L.E. (2017). Identical synchronization in complete network of reaction-diffusion equations of Fitzhugh-Nagumo. *An Giang University International Journal of Sciences, 5*, 51-58.

Phan, V.L.E. (2021a). Synchronization in complete network of reaction-diffusion equations of Fitzhugh-Nagumo type with nonlinear coupling. *Can Tho University Journal of Science, 13*(2), 43-51.

Phan, V.L.E. (2021b). Synchronization in complete networks of ordinary differential equations of Fitzhugh-Nagumo type with nonlinear coupling. *Dong Thap University Journal of Science, 10*(5), 3-9.

Pikovsky, A., Rosenblum, M. & Kurths, J. (2001). *Synchronization, A Universal Concept in Nonlinear Science.* Cambridge University Press.

Raj, R.N., Rajan, R.S. & Cangul, I.N.N. (2024). Investigation of the partition dimension in chemical networks and its application in chemistry. *Iranian Journal of Mathematical Chemistry, 15*(2), 51-63.