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THE m-DIMENSIONAL GENERALIZATIONS OF DARBO'S FIXED POINT THEOREM AND ITS APPLICATIONS

Mozhgan Amirsoleimani[1], **Jamal Reazei Roshan**[1*], **Mohammad Mursaleen**[2,3]

[1]Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran
[2]Department of Mathematical Sciences, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602105, Tamilnadu, India
[3]Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India

Abstract. This study aims to present a practical model for solving a system of implicit integral equations using theoretical results in generalizations of Darbo's fixed point theorem, in which the functions and operators involved in the system belong to certain classes. We give several extensions of Darbo's fixed point theorem in m-dimensional space using some classes of functions. We also investigate the existence of a solution for m operators with m variables that satisfy Darbo's condition. In addition, we investigate the feasibility of a solution to a system of implicit integral equations. An example is presented to illustrate the practical application of our findings.

Keywords: Measure of noncompactness, Darbo's fixed point theorem, systems of functional integral equations.
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*__Corresponding author:__ Jamal, Reazei Roshan, Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran, e-mail: *Jmlroshan@gmail.com, Jml.roshan@iau.ac.ir*
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1 Introduction

Researchers in the field of nonlinear analysis are interested in studying functional integral equations (FIEs). Many problems in applied mathematics, physics, and other sciences lead to the production of FIEs. On the other hand, there are different methods in applied and pure mathematics to check the existence of solutions for FIEs and systems of functional integral equations (SFIEs). The idea of the measure of non-compactness (MNC), which many academics have used to research and examine FIEs and SFIEs, is one of the most potent instruments in this subject. In Allahyari et al. (2015) the authors introduced a new measure of noncompactness on the space consisting of all real functions that are n times bounded and continuously differentiable on $\mathbb{R}_+$, and they investigated the problem of the existence of solutions for some classes of nonlinear integro-differential equations. The Darbo-Sadovskii fixed point theorem and measures of noncompactness are used by authors in Alvarez & Lizama (2016), to discover the necessary conditions for the presence of at least one solution of a nonlinear integral equation with a general kernel. In Banaei et al. (2021) the authors by applying the concept of the measure of noncompactness, proved some fixed point theorems in the Fréchet space $L^\infty(\mathcal{O})$ (where $\mathcal{O} \subseteq \mathbb{R}^\omega$). They handled their obtained consequences to inquire about the existence of solutions for infinite systems of Urysohn-type integral equations. In Das et al. (2023), the existence of a solution to generalized fractional integral equations with two variables was studied by authors. They developed a new fixed point theorem that generalized Darbo's fixed point theorem, utilizing a measure of noncompactness and a new contraction operator to accomplish their main goal. In Das &

Deuri (2021a) authors analyzed the solvability of an integral equation of the Hammerstein type in two variables by using the generalization of Darbos' Theoerem. In Das et al. (2022) authors proved some generalizations of the Darbo-type theorem to show that implicit fractional integral equations can be solved in the space $C\left(I, \ell_p^\alpha\right)$, which is composed of all continuous functions from $I=[0, a]$ $(a>0)$ to ℓ_p^α, where ℓ_p^α is a tempered sequence space. In Das et al. (2021b), the solvability of a generalized proportional fractional integral equation in Banach space was investigated by authors. In Deuri & Das (2022a); Deuri et al. (2022b) authors investigated the solvability of classes of fractional integral equations by presenting versions of the generalization of Darbo's fixed point theorem. In Mursaleen & Mohiuddine (2012) Measures of noncompactness were used by authors. To study the theory of infinite systems of differential equations in the Banach sequence spaces $\ell_p(1 \leq p<\infty)$. In Sen et al. (2019), authors looked at the Hadamard fractional operator nonlinear functional-integral equation. Additionally, utilizing the concept of the measure of noncompactness, they investigated the existence of solutions in Banach algebra under specific pertinent assumptions and in connection with fixed point theory. These are only a few instances of how the idea of measurements of noncompactness might be used to solve problems with nonlinear analysis. Contrarily, a lot of scholars tried to broaden and generalize Darbo's fixed point theorem for this reason, and recently, a lot of publications in this area have been written. Aghajani et al. (2014, 2013), were among the first researchers who did wonderful work in this direction. In Falset et al. (2015), without utilizing the maximum property, authors studied the axioms of a measure of (weak) non-compactness and came up with some expansions of the celebrated Darbo and Sadovskii fixed-point theorems. In Hajji (2013), Darbo's and Sadovski's fixed-point theorems were expanded by author into common fixed-point theorems for commuting operators. He investigated the existence of typical solutions to equations in Banach spaces, using the measure of noncompactness as an illustration and application. In Nashine & Das (2022) authors studied the solution of an infinite system of mixed-type fractional integral equations. To do this, they first established a novel version of Darbo's fixed point theorem using the shifting distance function, and then they applied it to the fractional integral equations to prove the existence of a solution on tempered sequence space. In Zarinfar et al. (2018), authors introduced a generalization of Darbo's fixed point theorem based on $SR-$ functions. Then, the integral equations, which are related to $L-$ functions, are solved by their results. For additional literature in this area, we recommend readers to look into references Deep et al. (2021); Kumar et al. (2022); Matani et al. (2020); Nasiri et al. (2021); Roshan (2017).

This paper aims to extend and generalize Darbo's fixed-point theorem differently. By employing a class of functions, we aim to derive τ-dimensional extensions of Darbo's theorem. These extensions enable us to determine if a specific class of SFIEs has a solution in a particular circumstance. Additionally, we investigate the possibility of a solution for m operators and m variables that satisfies Darbo's criterion. Finally, for the particular 2-dimensional case, we examine the possibility of a solution for the following SFIEs:

$$
\left\{\begin{array}{ll}
\varrho(\iota, \jmath)= & a_1(\iota, \jmath)+h_1(\iota, \jmath, \varrho(\iota, \jmath), \wp(\iota, \jmath)) \int_0^\iota g_1(\iota, \jmath, \upsilon) f_1(\upsilon, \varrho(\upsilon, \jmath), \wp(\upsilon, \jmath)) d \upsilon \\
& +p_1(\iota, \jmath, \varrho(\iota, \jmath), \wp(\iota, \jmath)) \int_0^\jmath \int_0^\iota q_1(\iota, \jmath, \upsilon, \nu) r_1(\upsilon, \nu, \varrho(\upsilon, \nu), \wp(\upsilon, \jmath)) d \upsilon d \nu \\
\\
\wp(\iota, \jmath)= & a_2(\iota, \jmath)+h_2(\iota, \jmath, \varrho(\iota, \jmath), \wp(\iota, \jmath)) \int_0^\iota g_2(\iota, \jmath, \upsilon) f_2(\upsilon,, \varrho(\upsilon, \jmath), \wp(\upsilon, \jmath)) d \upsilon \\
& +p_2(\iota, \jmath, \varrho(\iota, \jmath), \wp(\iota, \jmath)) \int_0^\jmath \int_0^\iota q_2(\iota, \jmath, \upsilon, \nu) r_2(\upsilon, \nu, \varrho(\upsilon, \nu), \wp(\upsilon, \jmath)) d \upsilon d \nu,
\end{array}\right. \tag{1}
$$

where $\varrho, \wp: \mathbb{R}_{+}^2 \rightarrow \mathbb{R}$ are continuous bounded functions, and the functions $a_i,\ h_i,\ g_i,\ f_i,\ p_i,$ $q_i,\ r_i\ (i=1,2)$ for $\iota, \jmath \in \mathbb{R}_{+}$ fulfill certain requirements. For more explanation and details, please see Theorem 9.

2 Preliminaries

This section recalls some notation, definitions, and theorems to achieve all work outcomes. Let $\mathbb{N}$ be the collection of natural numbers, $\mathbb{R}$ be the collection of real numbers, and $\mathbb{R}_+ = [0, +\infty)$. The symbol $(\mathbb{B}, \|.\|)$ denotes a Banach space with norm $\|.\|$ and element $\mathbf{0}$ as the zero value, and $\overline{B_\hbar}\ (\varrho, \hbar)$ signifies the closed ball in $\mathbb{B}$. We also use $\overline{B}_\hbar$ to express $\overline{B}(\mathbf{0}, \hbar)$. We use $\overline{\mathcal{K}}$ and $Conv\mathcal{K}$ to denote the closure of $\mathcal{K}$ and the closed convex hull of $\mathcal{K}$, respectively, for $\varnothing \neq \mathcal{K} \subset \mathbb{B}$. Additionally, $\mathcal{M}_\mathbb{B}$ is the family of all bounded and non-empty sets, and $\mathcal{N}_\mathbb{B} \subset \mathcal{M}_\mathbb{B}$ is the family of moderately compact sets. We take it for granted that $\Im$ is an arbitrary MNC in Banach space $\mathbb{B}$ and that $\mathcal{D}$ is a convex, bounded, closed, and non-empty subset of Banach space $\mathbb{B}$ throughout this article. Also, $\mathcal{L}$ is an arbitrary MNC in Banach space $\mathbb{B}_1 \times \mathbb{B}_2 \times \cdots \times \mathbb{B}_\tau$ with a norm of $\|.\| = \|.\|_1 + \|.\|_2 + \cdots + \|.\|_\tau$, where $\|.\|_1, \|.\|_2, \cdots, \|.\|_\tau$ are the respective norms on $\mathbb{B}_1, \mathbb{B}_2, \cdots, \mathbb{B}_\tau$. The following definitions will be needed in the sequel:

Definition 1. *(Banaš et al., 1980) The function $\Im : \mathcal{M}_\mathbb{B} \to \mathbb{R}_+$ is referred to as an MNC in $\mathbb{B}$ if it meets the criteria listed below:*

$(A1)$ The family $\ker \Im = \{\mathcal{K} \in \mathcal{M}_\mathbb{B} : \Im(\mathcal{K}) = 0\} \neq \varnothing$ and $\ker \Im \subset \mathcal{N}_\mathbb{B}$;

$(A2)$ $\mathcal{K}_1 \subset \mathcal{K}_2 \Rightarrow \Im(\mathcal{K}_1) \leq \Im(\mathcal{K}_2)$;

$(A3)$ $\Im\left(\overline{\mathcal{K}}\right) = \Im(conv\mathcal{K}) = \Im(\mathcal{K})$;

$(A4)$ $\Im(\jmath\mathcal{K}_1 + (1-\jmath)\mathcal{K}_2) \leq \jmath\Im(\mathcal{K}_1) + (1-\jmath)\Im(\mathcal{K}_2)$ for $\jmath \in [0,1]$;

$(A5)$ IF $(\mathcal{K}_k) \subset \mathcal{M}_\mathbb{B}$ be a sequence such that $\overline{\mathcal{K}}_k = \mathcal{K}_k$ and $\mathcal{K}_{k+1} \subset \mathcal{K}_k$, for $k = 1, 2, ..., \infty$; where $\lim_{k\to\infty} \Im(\mathcal{K}_k) = 0$, then $\mathcal{K}_\infty = \cap_{k=1}^{\infty} \mathcal{K}_k \neq \varnothing$. The subset $\ker \Im$ stated in $(A1)$ stands for the kernel of $\Im$, and since $\Im(\mathcal{K}_\infty) = \Im(\cap_{k=1}^{\infty} \mathcal{K}_k) \leq \Im(\mathcal{K}_k)$, we see that $\Im(\mathcal{K}_\infty) = 0$. Therefore, $\mathcal{K}_\infty \in \ker \Im$.

Theorem 1. *(Akmerov et al., 1992) Assume that MNC for the Banach spaces $\mathbb{B}_1, \mathbb{B}_2, ..., \mathbb{B}_\tau$ are respectively, $\Im_1, \Im_2, ..., \Im_\tau$. Assume as well that $\lambda(\varrho_{1,}\varrho_{2,}\varrho_{3,...,}\varrho_\tau) = 0 \Leftrightarrow \varrho_i = 0, (i = 1, 2, 3, ..., \tau)$, for the convex function $\lambda : \mathbb{R}_+^\tau \to \mathbb{R}_+$. Then,*

$$\Im(\mathcal{K}) = \lambda(\Im_1(\mathcal{K}_1(\mathcal{K})), \Im_2(\mathcal{K}_2(\mathcal{K})), ..., \Im_\tau(\mathcal{K}_\tau(\mathcal{K}))),$$

defines an MNC in $\mathbb{B}_1 \times \mathbb{B}_2 \times ... \times \mathbb{B}_\tau$, where $\mathcal{K}_i : \mathcal{K} \to \mathbb{B}_i$ is defined by $\mathcal{K}_i(b_1, b_2, \ldots, b_\tau) = b_i$ for $i = 1, 2, 3, ..., \tau$.

In the following, we refer to particular cases of the Theorem 1.

Example 1. *Assume that MNC for the Banach spaces $\mathbb{B}_1, \mathbb{B}_2, ..., \mathbb{B}_\tau$ are respectively, $\Im_1, \Im_2, ..., \Im_\tau$.*

We define $\lambda(\varrho_1, \varrho_2, \ldots, \varrho_\tau) = \varrho_1 + \varrho_2 + \ldots + \varrho_\tau$ for each $(\varrho_1, \varrho_2, \ldots, \varrho_\tau) \in \mathbb{R}_+^\tau$.

Then, for each $\mathcal{K} \subseteq \mathbb{B}_1 \times \mathbb{B}_2 \times ... \times \mathbb{B}_\tau$, the function $\mathcal{L}(\mathcal{K}) = \Im_1(\mathcal{K}_1(\mathcal{K})) + \Im_2(\mathcal{K}_2(\mathcal{K})) + \cdots + \Im_\tau(\mathcal{K}_\tau(\mathcal{K}))$

is an MNC in $\mathbb{B}_1 \times \mathbb{B}_2 \times ... \times \mathbb{B}_\tau$, where $\mathcal{K}_i : \mathcal{K} \to \mathbb{B}_i$ is defined by $\mathcal{K}_i(b_1, b_2, \ldots, b_\tau) = b_i$ for $i = 1, 2, 3, ..., \tau$.

Example 2. *Assume that $\Im$ is a MNC in Banach space $\mathbb{B}$, and $\lambda : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a convex function, and*

$$\lambda(\varrho_{1,}\varrho_2) = 0 \Leftrightarrow \varrho_i = 0, (i = 1, 2).$$

Then $\Im(\mathcal{K}) = \lambda(\Im(\mathcal{K}_1(\mathcal{K})), \Im(\mathcal{K}_2(\mathcal{K})))$, defines an MNC in $\mathbb{B} \times \mathbb{B}$, where $\mathcal{K}_i : \mathcal{K} \to \mathbb{B}$ is defined by $\mathcal{K}_i(b_1, b_2) = b_i$ for $i = 1, 2$.

Example 3. *(Akmerov et al., 1992) Assume that $\Im$ is a MNC on a Banach space $\mathbb{B}$; consider $\lambda(\varrho,\wp) = \max\{\varrho,\wp\}$ for every $(\varrho,\wp) \in \mathbb{R}_{+}^{2}$. Then $\mathcal{L}(\mathcal{K}) = max\{\Im(\mathcal{K}_1(\mathcal{K})), \Im(\mathcal{K}_2(\mathcal{K}))\}$ for each $\mathcal{K} \subseteq \mathbb{B}\times\mathbb{B}$, is a MNC in the space $\mathbb{B}\times\mathbb{B}$, where $\mathcal{K}_i : \mathcal{K} \to \mathbb{B}$ is defined by $\mathcal{K}_i(b_1,b_2) = b_i$ for $i = 1,2$.*

Example 4. *(Akmerov et al., 1992) Let $\Im$ be a MNC on a Banach space $\mathbb{B}$; consider $\lambda(\varrho,\wp) = \varrho + \wp$ for every $(\varrho,\wp) \in \mathbb{R}_{+}^{2}$.*

Then $\mathcal{L}(\mathcal{K}) = \Im(\mathcal{K}_1(\mathcal{K})) + \Im(\mathcal{K}_2(\mathcal{K}))$, for each $\mathcal{K} \subseteq \mathbb{B}\times\mathbb{B}$, is the MNC in $\mathbb{B}\times\mathbb{B}$, where $\mathcal{K}_i : \mathcal{K} \to \mathbb{B}$ is defined by $\mathcal{K}_i(b_1,b_2) = b_i$ for $i = 1,2$.

Theorem 2. *(Schauder) (Aghajani et al., 2013) Let $\mathcal{O} : \mathcal{D} \to \mathcal{D}$ be a compact and continuous operator. Then the set of fixed points $\mathcal{O}$ is non-empty.*

Theorem 3. *(Darbo) (Darbo, 1955) Let $\mathcal{O} : \mathcal{D} \to \mathcal{D}$ be a continuous operator. Suppose that $\Im(\mathcal{O}\mathcal{Q}) \leq \jmath\Im(\mathcal{Q})$ for each non-empty $\mathcal{Q} \subseteq \mathcal{D}$ and for some $\jmath \in [0,1)$. Then the set of fixed points $\mathcal{O}$ is non-empty.*

Theorem 4. *(Aghajani et al., 2013) Let $\mathcal{O} : \mathcal{D} \to \mathcal{D}$ be a continuous operator satisfying $\Im(\mathcal{O}\mathcal{Q}) \leq \psi(\Im(\mathcal{Q}))$, for each non-empty $\mathcal{Q} \subseteq \mathcal{D}$, and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing function, and $\lim_{n\to\infty}\psi^n(\iota) = 0$ for each $\iota \geq 0$. Then the set of fixed points $\mathcal{O}$ is non-empty.*

3 Main results

Before obtaining the main results, we introduce three classes of functions as follows:

$$\begin{aligned}
\Upsilon_\tau &= \left\{\beta : \mathbb{R}_+^\tau \to [0,1) : \beta(\iota_1^n, \iota_2^n, \ldots, \iota_\tau^n) \to 1 \text{ as } n\to\infty \ \Rightarrow\ \iota_i^n \to 0 \text{ as } n \to \infty \ \text{ for } i = 1,2,\ldots\tau\right\},\\
\Gamma_\tau &= \left\{\varphi : \mathbb{R}_+^\tau \to \mathbb{R}_+ : \varphi \text{ is continuous, non-decreasing, sublinear, and } \ \varphi\ (0,0,\ldots,0) = 0\right\}.\\
\Psi &= \left\{\psi : \mathbb{R}_+ \to \mathbb{R}_+ : \psi \text{ is non-decreasing, providing that } \lim_{n\to\infty}\psi^n(\iota) = 0 \text{ for each } \iota \geq 0\right\}.
\end{aligned}$$

For instance, it is simple to observe that the function $\beta(\iota_1,\iota_2,\ldots,\iota_\tau) = \frac{e^{-\left(\frac{\iota_1+\iota_2+\ldots+\iota_\tau}{\tau}\right)}}{1+\frac{\iota_1+\iota_2+\ldots+\iota_\tau}{\tau}}$ belongs to Υ_τ. We also remind that if $\psi \in \Psi$ then $\psi(\iota) < \iota$ for each $\iota > 0$. We are now in a position to present our first main result:

Theorem 5. *Assume that the continuous operator $\mathcal{O} : \mathcal{D}_1\times\mathcal{D}_2\times\cdots\times\mathcal{D}_\tau \longrightarrow \mathcal{D}_1\times\mathcal{D}_2\times\cdots\times\mathcal{D}_\tau$ meets the following contraction:*

$$\mathcal{L}(\mathcal{O}(\mathcal{K})) + \varphi(\mathcal{L}(\mathcal{O}(\mathcal{K})), \mathcal{L}(\mathcal{O}(\mathcal{K})), \cdots, \mathcal{L}(\mathcal{O}(\mathcal{K}))) \leq \beta(\mathcal{L}(\mathcal{K}), \mathcal{L}(\mathcal{K}), \cdots, \mathcal{L}(\mathcal{K}))$$

$$[\mathcal{L}(\mathcal{K}) + \varphi(\mathcal{L}(\mathcal{K}), \mathcal{L}(\mathcal{K}), \cdots, \mathcal{L}(\mathcal{K}))]$$

for any $\mathcal{K} \subseteq \mathcal{D}_1\times\mathcal{D}_2\times\cdots\times\mathcal{D}_\tau$, where $\beta \in \Upsilon_\tau$, φ is continuous, and $\varphi(0,0,\cdots,0) = 0$. Then the set of fixed points $\mathcal{O}$ is non-empty.

Proof. Let $\mathcal{D}_1^0\times\mathcal{D}_2^0\times\cdots\times\mathcal{D}_\tau^0 = \mathcal{D}_1\times\mathcal{D}_2\times\cdots\times\mathcal{D}_\tau$, we define a sequence $\{\mathcal{D}_1^n\times\mathcal{D}_2^n\times\cdots\times\mathcal{D}_\tau^n\}_{n=1}^\infty$, such that $\mathcal{D}_1^n\times\mathcal{D}_2^n\times\cdots\times\mathcal{D}_\tau^n = ConvT\left(\mathcal{D}_1^{n-1}\times\mathcal{D}_2^{n-1}\times\cdots\times\mathcal{D}_\tau^{n-1}\right)$. Considering that $\mathcal{D}_1\times\mathcal{D}_2\times\cdots\times\mathcal{D}_\tau$ is closed, convex, and invariant under $\mathcal{O}$, it can be easily shown that

$$\cdots \subseteq \mathcal{D}_1^{n+1}\times\mathcal{D}_2^{n+1}\times\cdots\times\mathcal{D}_\tau^{n+1} \subseteq \mathcal{D}_1^n\times\mathcal{D}_2^n\times\cdots\times\mathcal{D}_\tau^n \subseteq \cdots \subseteq \mathcal{D}_1^1\times\mathcal{D}_2^1\times\cdots\times\mathcal{D}_\tau^1 \subseteq \mathcal{D}_1^0\times\mathcal{D}_2^0\times\cdots\times\mathcal{D}_\tau^0.$$

If there is $N \in \mathbb{N}$ such that

$$\mathcal{L}\left(\mathcal{D}_1^N\times\mathcal{D}_2^N\times\cdots\times\mathcal{D}_\tau^N\right) + \varphi\left(\mathcal{L}\left(\mathcal{D}_1^N\times\mathcal{D}_2^N\times\cdots\times\mathcal{D}_\tau^N\right), \ldots, \mathcal{L}\left(\mathcal{D}_1^N\times\mathcal{D}_2^N\times\cdots\times\mathcal{D}_\tau^N\right)\right) = 0,$$

i.e., $\mathcal{L}\left(\mathcal{D}_1^N \times \mathcal{D}_2^N \times \cdots \times \mathcal{D}_\tau^N\right) = 0$, then $\mathcal{D}_1^N \times \mathcal{D}_2^N \times \cdots \times \mathcal{D}_\tau^N$ is relatively compact, convex, and closed. Also

$$\mathcal{O}\left(\mathcal{D}_1^N \times \mathcal{D}_2^N \times \cdots \times \mathcal{D}_\tau^N\right) \subseteq \mathcal{D}_1^{N+1} \times \mathcal{D}_2^{N+1} \times \cdots \times \mathcal{D}_\tau^{N+1} \subseteq \mathcal{D}_1^N \times \mathcal{D}_2^N \times \cdots \times \mathcal{D}_\tau^N.$$

Thus, Theorem 2 implies that $\mathcal{O}$ has at least one fixed point. Now assume that

$$\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right) + \varphi\left(\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right), \ldots, \ \mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)\right) > 0,$$

for all $n \in \mathbb{N}$, by our assumption, we have

$$\begin{aligned}
&\mathcal{L}\left(\mathcal{D}_1^{n+1} \times \mathcal{D}_2^{n+1} \times \cdots \times \mathcal{D}_\tau^{n+1}\right) + \varphi\left(\mathcal{L}\left(\mathcal{D}_1^{n+1} \times \mathcal{D}_2^{n+1} \times \cdots \times \mathcal{D}_\tau^{n+1}\right), \ldots, \mathcal{L}\left(\mathcal{D}_1^{n+1} \times \mathcal{D}_2^{n+1} \times \cdots \times \mathcal{D}_\tau^{n+1}\right)\right) \\
&\leq \mathcal{L}\left(\mathcal{O}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)\right) + \varphi\left(\mathcal{L}\left(\mathcal{O}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)\right), \ldots, \mathcal{L}\left(\mathcal{O}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)\right)\right) \\
&\leq \beta\left(\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right), \ldots, \mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)\right) \times \\
&\left[\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right) + \varphi\left(\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right), \ldots, \mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)\right)\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\mathcal{L}\left(\mathcal{D}_1^{n+1} \times \mathcal{D}_2^{n+1} \times \cdots \times \mathcal{D}_\tau^{n+1}\right) + \varphi\left(\mathcal{L}\left(\mathcal{D}_1^{n+1} \times \mathcal{D}_2^{n+1} \times \cdots \times \mathcal{D}_\tau^{n+1}\right), \ldots, \mathcal{L}\left(\mathcal{D}_1^{n+1} \times \mathcal{D}_2^{n+1} \times \cdots \times \mathcal{D}_\tau^{n+1}\right)\right)}{\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right) + \varphi\left(\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right), \ldots, \mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)\right)} \\
&\leq \beta\left(\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right), \ldots, \mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)\right) \qquad (2) \\
&\leq 1.
\end{aligned}$$

This suggests that for every $n \in \mathbb{N}$,

$$\begin{aligned}
&\mathcal{L}\left(\mathcal{D}_1^{n+1} \times \mathcal{D}_2^{n+1} \times \cdots \times \mathcal{D}_\tau^{n+1}\right) + \varphi\left(\mathcal{L}\left(\mathcal{D}_1^{n+1} \times \mathcal{D}_2^{n+1} \times \cdots \times \mathcal{D}_\tau^{n+1}\right), \ldots, \mathcal{L}\left(\mathcal{D}_1^{n+1} \times \mathcal{D}_2^{n+1} \times \cdots \times \mathcal{D}_\tau^{n+1}\right)\right) \\
&\leq \mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right) + \varphi\left(\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right), \ldots, \mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)\right).
\end{aligned}$$

Thus, the sequence $\{\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right) + \varphi\left(\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right), \ldots,\right.$ $\left.\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)\right)\}_{n=1}^{\infty}$ is nonincreasing and non-negative, so there exists $\gamma \in \mathbb{R}_+$ such that

$$\lim_{n\to\infty} \left[\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right) + \varphi\left(\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right), \ldots, \mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)\right)\right] = \gamma.$$

We establish that $\gamma = 0$. Let's assume instead that $\gamma > 0$. Then, by (2) letting $n \to \infty$, we have

$$\lim_{n\to\infty} \beta\left(\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right), \ldots, \mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)\right) = 1.$$

Since $\beta \in \Upsilon_\tau$, so $\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right) \to 0$ as $n \to \infty$. Since φ is continuous and $\varphi(0, 0, \ldots, 0) = 0$, we conclude that $\gamma = 0$, which is a contradiction.

Hence,

$$\lim_{n\to\infty} \left[\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right) + \varphi\left(\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right), \ldots, \mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)\right)\right] = 0,$$

therefore, $\lim_{n\to\infty} \mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right) = 0$.

Since $\mathcal{D}_1^{n+1} \times \mathcal{D}_2^{n+1} \times \cdots \times \mathcal{D}_\tau^{n+1} \subseteq \mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n$ and $\mathcal{O}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right) \subseteq \mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n$ for each $n \in \mathbb{N}$, therefore, $(A6)$ indicates that $\mathcal{D}_1^\infty \times \mathcal{D}_2^\infty \times \cdots \times \mathcal{D}_\tau^\infty = \cap_{n=1}^\infty \mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n$ is a closed, convex, non-empty set that is invariant under $\mathcal{O}$ and a member of the $\ker \mathcal{L}$. Consequently, Theorem 2 completes the proof. $\square$

Taking $\varphi(\iota_1, \iota_2, \ldots, \iota_\tau) = \iota_1 + \iota_2 + \cdots + \iota_\tau$ and $\beta(\iota_1, \iota_2, \ldots, \iota_\tau) = k$, $0 \leq k < 1$ in Theorem 5, we obtain the τ-dimensional version of Darbo's fixed point theorem.

Corollary 1. *Assume that the continuous operator $\mathcal{O} : \mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_\tau \longrightarrow \mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_\tau$ meets the following contraction:*

$$\mathcal{L}(\mathcal{O}(\mathcal{K})) \leq k\mathcal{L}(\mathcal{K})$$

for any $\mathcal{K} \subseteq \mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_\tau$. Then the set of fixed points $\mathcal{O}$ is non-empty.

Theorem 6. *Assume that the continuous operators $\mathcal{O}_i : \mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_\tau \longrightarrow \mathcal{D}_i$, $i = 1, 2, \ldots, \tau$, meet the following contractions:*

$$
\begin{aligned}
&\Im_i\left(\mathcal{O}_i\left(\mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_\tau\right)\right) + \varphi\left(\Im_i\left(\mathcal{O}_i\left(\mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_\tau\right)\right), \ldots, \Im_i\left(\mathcal{O}_i\left(\mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_\tau\right)\right)\right) \\
&\leq \beta\left(\Im_1\left(\mathcal{K}_1\right) + \cdots + \Im_\tau\left(\mathcal{K}_\tau\right), \ldots, \Im_1\left(\mathcal{K}_1\right) + \cdots + \Im_\tau\left(\mathcal{K}_\tau\right)\right) \times \\
&\left[\frac{\Im_1\left(\mathcal{K}1\right) + \cdots + \Im_\tau\left(\mathcal{K}_\tau\right)}{\tau} + \frac{1}{\tau}\varphi\left(\Im_1\left(\mathcal{K}_1\right), \Im_2\left(\mathcal{K}_2\right) \ldots, \Im_\tau\left(\mathcal{K}_\tau\right)\right)\right]
\end{aligned}
$$

for each $\mathcal{K}_i \subseteq \mathcal{D}_i$ with $i = 1, 2, \ldots, \tau$, where $\beta \in \Upsilon_\tau$ and $\varphi \in \Gamma_\tau$. Then, in the space $\mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_\tau$, there is at least one point $(z_1^*, z_2^*, \cdots, z_\tau^*)$ such that

$$
\left\{
\begin{array}{c}
\mathcal{O}_1\left(z_1^*, z_2^*, \cdots, z_\tau^*\right) = z_1^* \\
\mathcal{O}_2\left(z_1^*, z_2^*, \cdots, z_\tau^*\right) = z_2^* \\
\vdots \\
\mathcal{O}_\tau\left(z_1^*, z_2^*, \cdots, z_\tau^*\right) = z_\tau^*.
\end{array}
\right. \tag{3}
$$

Proof. Suppose $\mathcal{L}$ is the MNC defined in Example 1. We define the mapping $\widetilde{\mathcal{O}} : \mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_\tau \to \mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_\tau$ by

$$
\widetilde{\mathcal{O}}\left(z_1, z_2, \ldots, z_\tau\right) = \left(\mathcal{O}_1\left(z_1, z_2, \ldots, z_\tau\right), \mathcal{O}_2\left(z_1, z_2, \ldots, z_\tau\right), \ldots, \mathcal{O}_\tau\left(z_1, z_2, \ldots, z_\tau\right)\right).
$$

$\widetilde{\mathcal{O}}$ is continuous, and by our contraction, for each $\mathcal{K} \subseteq \mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_\tau$, we have

$$
\begin{aligned}
&\mathcal{L}\left(\widetilde{\mathcal{O}}\left(\mathcal{K}\right)\right) + \varphi\left(\mathcal{L}\left(\widetilde{\mathcal{O}}\left(\mathcal{K}\right)\right), \ldots, \mathcal{L}\left(\widetilde{\mathcal{O}}\left(\mathcal{K}\right)\right)\right) \\
&\leq \mathcal{L}\left(\mathcal{O}_1\left(\mathcal{K}_1\left(\mathcal{K}\right) \times \ldots \times \mathcal{K}_\tau\left(\mathcal{K}\right)\right) \times \cdots \times \mathcal{O}_\tau\left(\mathcal{K}_1\left(\mathcal{K}\right) \times \ldots \times \mathcal{K}_\tau\left(\mathcal{K}\right)\right)\right) \\
&+ \varphi\begin{pmatrix} \mathcal{L}\left(\mathcal{O}_1\left(\mathcal{K}_1\left(\mathcal{K}\right) \times \ldots \times \mathcal{K}_\tau\left(\mathcal{K}\right)\right) \times \cdots \times \mathcal{O}_\tau\left(\mathcal{K}_1\left(\mathcal{K}\right) \times \ldots \times \mathcal{K}_\tau\left(\mathcal{K}\right)\right)\right), \ldots, \\ \mathcal{L}\left(\mathcal{O}_1\left(\mathcal{K}_1\left(\mathcal{K}\right) \times \ldots \times \mathcal{K}_\tau\left(\mathcal{K}\right)\right) \times \cdots \times \mathcal{O}_\tau\left(\mathcal{K}_1\left(\mathcal{K}\right) \times \ldots \times \mathcal{K}_\tau\left(\mathcal{K}\right)\right)\right) \end{pmatrix} \\
&\leq \Im_1\left(\mathcal{O}_1\left(\mathcal{K}_1\left(\mathcal{K}\right) \times \ldots \times \mathcal{K}_\tau\left(\mathcal{K}\right)\right)\right) + \cdots + \Im_\tau\left(\mathcal{O}_\tau\left(\mathcal{K}_1\left(\mathcal{K}\right) \times \ldots \times \mathcal{K}_\tau\left(\mathcal{K}\right)\right)\right) \\
&+ \varphi\begin{pmatrix} \Im_1\left(\mathcal{O}_1\left(\mathcal{K}_1\left(\mathcal{K}\right) \times \ldots \times \mathcal{K}_\tau\left(\mathcal{K}\right)\right)\right) + \cdots + \Im_\tau\left(\mathcal{O}_\tau\left(\mathcal{K}_1\left(\mathcal{K}\right) \times \ldots \times \mathcal{K}_\tau\left(\mathcal{K}\right)\right)\right), \ldots, \\ \Im_1\left(\mathcal{O}_1\left(\mathcal{K}_1\left(\mathcal{K}\right) \times \ldots \times \mathcal{K}_\tau\left(\mathcal{K}\right)\right)\right) + \cdots + \Im_\tau\left(\mathcal{O}_\tau\left(\mathcal{K}_1\left(\mathcal{K}\right) \times \ldots \times \mathcal{K}_\tau\left(\mathcal{K}\right)\right)\right) \end{pmatrix} \\
&\leq \left[\Im_1\left(\mathcal{O}_1\left(\mathcal{K}_1\left(\mathcal{K}\right) \times \ldots \times \mathcal{K}_\tau\left(\mathcal{K}\right)\right)\right) + \varphi\begin{pmatrix} \Im_1\left(\mathcal{O}_1\left(\mathcal{K}_1\left(\mathcal{K}\right) \times \ldots \times \mathcal{K}_\tau\left(\mathcal{K}\right)\right)\right), \ldots, \\ \Im_1\left(\mathcal{O}_1\left(\mathcal{K}_1\left(\mathcal{K}\right) \times \ldots \times \mathcal{K}_\tau\left(\mathcal{K}\right)\right)\right) \end{pmatrix}\right] \\
&+ \left[\Im_2\left(\mathcal{O}_2\left(\mathcal{K}_1\left(\mathcal{K}\right) \times \ldots \times \mathcal{K}_\tau\left(\mathcal{K}\right)\right)\right) + \varphi\begin{pmatrix} \Im_2\left(\mathcal{O}_2\left(\mathcal{K}_1\left(\mathcal{K}\right) \times \ldots \times \mathcal{K}_\tau\left(\mathcal{K}\right)\right)\right), \ldots, \\ \Im_2\left(\mathcal{O}_2\left(\mathcal{K}_1\left(\mathcal{K}\right) \times \ldots \times \mathcal{K}_\tau\left(\mathcal{K}\right)\right)\right) \end{pmatrix}\right] \\
&\vdots \\
&+ \left[\Im_\tau\left(\mathcal{O}_\tau\left(\mathcal{K}_1\left(\mathcal{K}\right) \times \ldots \times \mathcal{K}_\tau\left(\mathcal{K}\right)\right)\right) + \varphi\begin{pmatrix} \Im_\tau\left(\mathcal{O}_\tau\left(\mathcal{K}_1\left(\mathcal{K}\right) \times \ldots \times \mathcal{K}_\tau\left(\mathcal{K}\right)\right)\right), \ldots, \\ \Im_\tau\left(\mathcal{O}_\tau\left(\mathcal{K}_1\left(\mathcal{K}\right) \times \ldots \times \mathcal{K}_\tau\left(\mathcal{K}\right)\right)\right) \end{pmatrix}\right] \\
&\leq \beta\begin{pmatrix} \Im_1\left(\mathcal{K}_1\left(\mathcal{K}\right)\right) + \Im_2\left(\mathcal{K}_2\left(\mathcal{K}\right)\right) + \cdots + \Im_\tau\left(\mathcal{K}_\tau\left(\mathcal{K}\right)\right), \ldots, \\ \Im_1\left(\mathcal{K}_1\left(\mathcal{K}\right)\right) + \Im_2\left(\mathcal{K}_2\left(\mathcal{K}\right)\right) + \cdots + \Im_\tau\left(\mathcal{K}_\tau\left(\mathcal{K}\right)\right) \end{pmatrix} \\
&\times \left[\frac{\Im_1\left(\mathcal{K}1\right) + \cdots + \Im_\tau\left(\mathcal{K}_\tau\right)}{\tau} + \frac{1}{\tau}\varphi\left(\Im_1\left(\mathcal{K}_1\left(\mathcal{K}\right)\right), \Im_2\left(\mathcal{K}_2\left(\mathcal{K}\right)\right), \ldots, \Im_\tau\left(\mathcal{K}_\tau\left(\mathcal{K}\right)\right)\right)\right] \\
&+ \beta\begin{pmatrix} \Im_1\left(\mathcal{K}_1\left(\mathcal{K}\right)\right) + \Im_2\left(\mathcal{K}_2\left(\mathcal{K}\right)\right) + \cdots + \Im_\tau\left(\mathcal{K}_\tau\left(\mathcal{K}\right)\right), \ldots, \\ \Im_1\left(\mathcal{K}_1\left(\mathcal{K}\right)\right) + \Im_2\left(\mathcal{K}_2\left(\mathcal{K}\right)\right) + \cdots + \Im_\tau\left(\mathcal{K}_\tau\left(\mathcal{K}\right)\right) \end{pmatrix} \\
&\times \left[\frac{\Im_1\left(\mathcal{K}1\right) + \cdots + \Im_\tau\left(\mathcal{K}_\tau\right)}{\tau} + \frac{1}{\tau}\varphi\left(\Im_1\left(\mathcal{K}_1\left(\mathcal{K}\right)\right), \Im_2\left(\mathcal{K}_2\left(\mathcal{K}\right)\right), \ldots, \Im_\tau\left(\mathcal{K}_\tau\left(\mathcal{K}\right)\right)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\vdots\\
&+\beta\begin{pmatrix}\Im_1(\mathcal{K}_1(\mathcal{K}))+\Im_2(\mathcal{K}_2(\mathcal{K}))+\cdots+\Im_\tau(\mathcal{K}_\tau(\mathcal{K})),\ldots,\\ \Im_1(\mathcal{K}_1(\mathcal{K}))+\Im_2(\mathcal{K}_2(\mathcal{K}))+\cdots+\Im_\tau(\mathcal{K}_\tau(\mathcal{K}))\end{pmatrix}\\
&\times\left[\frac{\Im_1(\mathcal{K}1)+\cdots+\Im_\tau(\mathcal{K}_\tau)}{\tau}+\frac{1}{\tau}\varphi(\Im_1(\mathcal{K}_1(\mathcal{K})),\Im_2(\mathcal{K}_2(\mathcal{K})),\ldots,\Im_\tau(\mathcal{K}_\tau(\mathcal{K})))\right]\\
&\le\beta\begin{pmatrix}\Im_1(\mathcal{K}_1(\mathcal{K}))+\Im_2(\mathcal{K}_2(\mathcal{K}))+\cdots+\Im_\tau(\mathcal{K}_\tau(\mathcal{K})),\ldots,\\ \Im_1(\mathcal{K}_1(\mathcal{K}))+\Im_2(\mathcal{K}_2(\mathcal{K}))+\cdots+\Im_\tau(\mathcal{K}_\tau(\mathcal{K}))\end{pmatrix}\\
&\times[\Im_1(\mathcal{K}_1(\mathcal{K}))+\cdots+\Im_\tau(\mathcal{K}_2(\mathcal{K}))+\varphi(\Im_1(\mathcal{K}_1(\mathcal{K})),\Im_2(\mathcal{K}_2(\mathcal{K})),\ldots,\Im_\tau(\mathcal{K}_\tau(\mathcal{K})))]\\
&\le\beta\begin{pmatrix}\Im_1(\mathcal{K}_1(\mathcal{K}))+\Im_2(\mathcal{K}_2(\mathcal{K}))+\cdots+\Im_\tau(\mathcal{K}_\tau(\mathcal{K})),\ldots,\\ \Im_1(\mathcal{K}_1(\mathcal{K}))+\Im_2(\mathcal{K}_2(\mathcal{K}))+\cdots+\Im_\tau(\mathcal{K}_\tau(\mathcal{K}))\end{pmatrix}\times\\
&[\Im_1(\mathcal{K}_1(\mathcal{K}))+\cdots+\Im_\tau(\mathcal{K}_\tau(\mathcal{K}))+\varphi(\Im_1(\mathcal{K}_1(\mathcal{K}))+\cdots+\Im_\tau(\mathcal{K}_\tau(\mathcal{K})),\ldots,\Im_1(\mathcal{K}_1(\mathcal{K}))+\cdots+\Im_\tau(\mathcal{K}_\tau(\mathcal{K})))]\\
&\le\beta(\mathcal{L}(\mathcal{K}),\mathcal{L}(\mathcal{K}),\ldots,\mathcal{L}(\mathcal{K}))\times[\mathcal{L}(\mathcal{K})+\varphi(\mathcal{L}(\mathcal{K}),\mathcal{L}(\mathcal{K}),\ldots,\mathcal{L}(\mathcal{K}))].
\end{aligned}
$$

Then $\widetilde{\mathcal{O}}$ satisfies all the conditions of Theorem 5; therefore, $\widetilde{\mathcal{O}}$ has at least one fixed point $(z_1^*, z_2^*, \cdots, z_\tau^*)$ such that $\widetilde{\mathcal{O}}(z_1^*, z_2^*, \cdots, z_\tau^*) = (z_1^*, z_2^*, \cdots, z_\tau^*)$. Hence, the proof is complete. □

Corollary 2. *Assume that the continuous operators $\mathcal{O}_i : \mathcal{D}_1 \times \mathcal{D}_2 \longrightarrow \mathcal{D}_i$, $i = 1, 2$ meet the following contractions:*

$$
\begin{aligned}
\Im_1(\mathcal{O}_1(\mathcal{K}_1\times\mathcal{K}_2))+\varphi(\Im_1(\mathcal{O}_1(\mathcal{K}_1\times\mathcal{K}_2)),\Im_1(\mathcal{O}_1(\mathcal{K}_1\times\mathcal{K}_2)))&\le\beta(\Im_1(\mathcal{K}_1)+\Im_2(\mathcal{K}_2),\Im_1(\mathcal{K}_1)+\Im_2(\mathcal{K}_2))\\
&\times\left[\frac{\Im_1(\mathcal{K}_1)+\Im_2(\mathcal{K}_2)}{2}+\frac{1}{2}\varphi(\Im_1(\mathcal{K}_1),\Im_2(\mathcal{K}_2))\right],
\end{aligned}
$$

$$
\begin{aligned}
\Im_2(\mathcal{O}_2(\mathcal{K}_1\times\mathcal{K}_2))+\varphi(\Im_2(\mathcal{O}_2(\mathcal{K}_1\times\mathcal{K}_2)),\Im_2(\mathcal{O}_2(\mathcal{K}_1\times\mathcal{K}_2)))&\le\beta(\Im_1(\mathcal{K}_1)+\Im_2(\mathcal{K}_2),\Im_1(\mathcal{K}_1)+\Im_2(\mathcal{K}_2))\\
&\times\left[\frac{\Im_1(\mathcal{K}_1)+\Im_2(\mathcal{K}_2)}{2}+\frac{1}{2}\varphi(\Im_1(\mathcal{K}_1),\Im_2(\mathcal{K}_2))\right]
\end{aligned}
$$

for each $\mathcal{K}_i \subseteq \mathcal{D}_i$ with $i = 1, 2$, where $\beta \in \Upsilon_2$, and $\varphi \in \Gamma_2$. Then, in the space $\mathcal{D}_1 \times \mathcal{D}_2$, there is at least one point (z_1^*, z_2^*), such that

$$
\begin{cases}\mathcal{O}_1(z_1^*, z_2^*) = z_1^*\\ \mathcal{O}_2(z_1^*, z_2^*) = z_2^*.\end{cases}
$$

Theorem 7. *Assume that the continuous operator $\mathcal{O} : \mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_\tau \longrightarrow \mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_\tau$ meets the following contraction:*

$$
\mathcal{L}(\mathcal{O}(\mathcal{K}))+\varphi(\mathcal{L}(\mathcal{O}(\mathcal{K})),\ldots,\mathcal{L}(\mathcal{O}(\mathcal{K})))\le\psi(\mathcal{L}(\mathcal{K})+\varphi(\mathcal{L}(\mathcal{K}),\ldots,\mathcal{L}(\mathcal{K}))),
$$

for any $\mathcal{K} \subseteq \mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_\tau$, where $\psi \in \Psi$, $\varphi : \mathbb{R}_+^\tau \to \mathbb{R}_+$ is continuous, and $\varphi(0, 0, \cdots, 0) = 0$. Then, in the space $\mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_\tau$, the set of fixed points $\mathcal{O}$ is non-empty.

Proof. In a manner similar to Theorem 5, we define $\{\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\}_{n=1}^\infty$ such that $\mathcal{D}_1^0 \times \mathcal{D}_2^0 \times \cdots \times \mathcal{D}_\tau^0 = \mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_\tau$, and $\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n = ConvT\left(\mathcal{D}_1^{n-1} \times \mathcal{D}_2^{n-1} \times \cdots \times \mathcal{D}_\tau^{n-1}\right)$, also

$$
\cdots\subseteq\mathcal{D}_1^{n+1}\times\mathcal{D}_2^{n+1}\times\cdots\times\mathcal{D}_\tau^{n+1}\subseteq\mathcal{D}_1^{n}\times\mathcal{D}_2^{n}\times\cdots\times\mathcal{D}_\tau^{n}\subseteq\cdots\subseteq\mathcal{D}_1^{1}\times\mathcal{D}_2^{1}\times\cdots\times\mathcal{D}_\tau^{1}\subseteq\mathcal{D}_1^{0}\times\mathcal{D}_2^{0}\times\cdots\times\mathcal{D}_\tau^{0}.
$$

If there is $N \in \mathbb{N}$, such that

$$
\mathcal{L}\left(\mathcal{D}_1^N\times\mathcal{D}_2^N\times\cdots\times\mathcal{D}_\tau^N\right)+\varphi\left(\mathcal{L}\left(\mathcal{D}_1^N\times\mathcal{D}_2^N\times\cdots\times\mathcal{D}_\tau^N\right),\ldots,\mathcal{L}\left(\mathcal{D}_1^N\times\mathcal{D}_2^N\times\cdots\times\mathcal{D}_\tau^N\right)\right)=0,
$$

i.e. $\mathcal{L}\left(\mathcal{D}_1^N \times \mathcal{D}_2^N \times \cdots \times \mathcal{D}_\tau^N\right)=0$, then $\mathcal{D}_1^N \times \mathcal{D}_2^N \times \cdots \times \mathcal{D}_\tau^N$ is relatively compact, convex, and closed. Since

$$\mathcal{O}\left(\mathcal{D}_1^N \times \mathcal{D}_2^N \times \cdots \times \mathcal{D}_\tau^N\right) \subseteq \mathcal{D}_1^{N+1} \times \mathcal{D}_2^{N+1} \times \cdots \times \mathcal{D}_\tau^{N+1} \subseteq \mathcal{D}_1^N \times \mathcal{D}_2^N \times \cdots \times \mathcal{D}_\tau^N .$$

Thus, Theorem 2 implies that $\mathcal{O}$ has at least one fixed point. So we assume that for all $n \in \mathbb{N}$

$$\mathcal{L}\left(\mathcal{D}_1^N \times \mathcal{D}_2^N \times \cdots \times \mathcal{D}_\tau^N\right)+\varphi\left(\mathcal{L}\left(\mathcal{D}_1^N \times \mathcal{D}_2^N \times \cdots \times \mathcal{D}_\tau^N\right), \ldots, \mathcal{L}\left(\mathcal{D}_1^N \times \mathcal{D}_2^N \times \cdots \times \mathcal{D}_\tau^N\right)\right)>0.$$

By our assumption, we have

$$\begin{aligned}
&\mathcal{L}\left(\mathcal{D}_1^{n+1} \times \mathcal{D}_2^{n+1} \times \cdots \times \mathcal{D}_\tau^{n+1}\right)+\varphi\left(\mathcal{L}\left(\mathcal{D}_1^{n+1} \times \mathcal{D}_2^{n+1} \times \cdots \times \mathcal{D}_\tau^{n+1}\right), \ldots, \mathcal{L}\left(\mathcal{D}_1^{n+1} \times \mathcal{D}_2^{n+1} \times \cdots \times \mathcal{D}_\tau^{n+1}\right)\right) \\
&\leq \mathcal{L}\left(\mathcal{O}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)\right)+\varphi\left(\mathcal{L}\left(\mathcal{O}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)\right), \ldots, \mathcal{L}\left(\mathcal{O}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)\right)\right) \\
&\leq \psi\left(\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)+\varphi\left(\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right), \ldots, \mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)\right)\right) \\
&<\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)+\varphi\left(\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right), \ldots, \mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)\right).
\end{aligned}$$

Therefore, the sequence

$$\left\{\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)+\varphi\left(\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right), \ldots, \mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)\right)\right\}_{n=1}^{\infty}$$

is decreasing sequence of real numbers that is bounded from below, so there exists $\gamma \in \mathbb{R}_+$ such that

$$\lim_{n \to \infty}\left[\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)+\varphi\left(\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right), \ldots, \mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)\right)\right]=\gamma.$$

We now demonstrate that $\gamma=0$. Let's assume instead that $\gamma>0$

$$\begin{aligned}
&\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)+\varphi\left(\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right), \ldots, \mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)\right) \\
&\leq \mathcal{L}\left(\mathcal{O}\left(\mathcal{D}_1^{n-1} \times \mathcal{D}_2^{n-1} \times \cdots \times \mathcal{D}_\tau^{n-1}\right)\right) \\
&+\varphi\left(\mathcal{L}\left(\mathcal{O}\left(\mathcal{D}_1^{n-1} \times \mathcal{D}_2^{n-1} \times \cdots \times \mathcal{D}_\tau^{n-1}\right)\right), \ldots, \mathcal{L}\left(\mathcal{O}\left(\mathcal{D}_1^{n-1} \times \mathcal{D}_2^{n-1} \times \cdots \times \mathcal{D}_\tau^{n-1}\right)\right)\right) \\
&\leq \psi\left(\mathcal{L}\left(\mathcal{D}_1^{n-1} \times \mathcal{D}_2^{n-1} \times \cdots \times \mathcal{D}_\tau^{n-1}\right)+\varphi\left(\mathcal{L}\left(\mathcal{D}_1^{n-1} \times \mathcal{D}_2^{n-1} \times \cdots \times \mathcal{D}_\tau^{n-1}\right), \ldots, \mathcal{L}\left(\mathcal{D}_1^{n-1} \times \mathcal{D}_2^{n-1} \times \cdots \times \mathcal{D}_\tau^{n-1}\right)\right)\right) \\
&\leq \psi^2\left(\mathcal{L}\left(\mathcal{D}_1^{n-2} \times \mathcal{D}_2^{n-2} \times \cdots \times \mathcal{D}_\tau^{n-2}\right)+\varphi\left(\mathcal{L}\left(\mathcal{D}_1^{n-2} \times \mathcal{D}_2^{n-2} \times \cdots \times \mathcal{D}_\tau^{n-2}\right), \ldots, \mathcal{L}\left(\mathcal{D}_1^{n-2} \times \mathcal{D}_2^{n-2} \times \cdots \times \mathcal{D}_\tau^{n-2}\right)\right)\right) \\
&\vdots \\
&\leq \psi^n\left(\mathcal{L}\left(\mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_\tau\right)+\varphi\left(\mathcal{L}\left(\mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_\tau\right), \ldots, \mathcal{L}\left(\mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_\tau\right)\right)\right).
\end{aligned}$$

Since $\psi \in \Psi$, this implies that $\gamma=0$, and we obtain

$$\lim_{n \to \infty}\left[\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)+\varphi\left(\mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right), \ldots, \mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)\right)\right]=0.$$

Hence, $\lim_{n \to \infty} \mathcal{L}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right)=0$.

On the other hand, since for each $n=1,2,3...$; we have

$$\begin{aligned}
\mathcal{D}_1^{n+1} \times \mathcal{D}_2^{n+1} \times \cdots \times \mathcal{D}_\tau^{n+1} &\subseteq \mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n, \\
\mathcal{O}\left(\mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n\right) &\subseteq \mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n.
\end{aligned}$$

Then, starting with $(A5)$, $\mathcal{D}_1^{\infty} \times \mathcal{D}_2^{\infty} \times \cdots \times \mathcal{D}_\tau^{\infty}=\cap_{n=1}^{\infty} \mathcal{D}_1^n \times \mathcal{D}_2^n \times \cdots \times \mathcal{D}_\tau^n$ is a convex closed and non-empty set that is invariant under $\mathcal{O}$ and belongs to $\ker \mathcal{L}$. The proof is concluded by Theorem 2. □

Theorem 8. *Assume that the continuous operators $\mathcal{O}_i: \mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_\tau \longrightarrow \mathcal{D}_i$, $i=1,2,\ldots,\tau$, meet the following contractions:*

$$\begin{aligned}
&\Im_i\left(\mathcal{O}_i\left(\mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_\tau\right)\right)+\varphi\left(\Im_i\left(\mathcal{O}_i\left(\mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_\tau\right)\right), \ldots, \Im_i\left(\mathcal{O}_i\left(\mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_\tau\right)\right)\right) \\
&\leq \frac{1}{\tau} \psi\left(\Im_1(\mathcal{K}1)+\cdots+\Im_\tau\left(\mathcal{K}_\tau\right)+\varphi\left(\Im_1\left(\mathcal{K}_1\right), \Im_2\left(\mathcal{K}_2\right) \ldots, \Im_\tau\left(\mathcal{K}_\tau\right)\right)\right)
\end{aligned}$$

for each $\mathcal{K}_i \subseteq \mathcal{D}_i$ with $i = 1, 2, \ldots, \tau$, where $\varphi \in \Gamma_\tau$ and $\psi \in \Psi$. Then, in the space $\mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_\tau$, there is at least one point $(z_1^*, z_2^*, \cdots, z_\tau^*)$ such that

$$\begin{cases} \mathcal{O}_1(z_1^*, z_2^*, \cdots, z_\tau^*) = z_1^* \\ \mathcal{O}_2(z_1^*, z_2^*, \cdots, z_\tau^*) = z_2^* \\ \vdots \\ \mathcal{O}_\tau(z_1^*, z_2^*, \cdots, z_\tau^*) = z_\tau^*. \end{cases}$$

Proof. Suppose $\mathcal{L}$ is the MNC defined in Example 1. We define the mapping $\widetilde{\mathcal{O}} : \mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_\tau \to \mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_\tau$ by

$$\widetilde{\mathcal{O}}(z_1, z_2, \ldots, z_\tau) = (\mathcal{O}_1(z_1, z_2, \ldots, z_\tau), \mathcal{O}_2(z_1, z_2, \ldots, z_\tau), \ldots, \mathcal{O}_\tau(z_1, z_2, \ldots, z_\tau)).$$

$\widetilde{\mathcal{O}}$ is continuous, and by our contraction, for each $\mathcal{K} \subseteq \mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_\tau$, we have

$$\begin{aligned}
&\mathcal{L}\left(\widetilde{\mathcal{O}}(\mathcal{K})\right) + \varphi\left(\mathcal{L}\left(\widetilde{\mathcal{O}}(\mathcal{K})\right), \ldots, \mathcal{L}\left(\widetilde{\mathcal{O}}(\mathcal{K})\right)\right) \\
&\leq \mathcal{L}(\mathcal{O}_1(\mathcal{K}_1(\mathcal{K}) \times \ldots \times \mathcal{K}_\tau(\mathcal{K})) \times \cdots \times \mathcal{O}_\tau(\mathcal{K}_1(\mathcal{K}) \times \ldots \times \mathcal{K}_\tau(\mathcal{K}))) \\
&+ \varphi\begin{pmatrix} \mathcal{L}(\mathcal{O}_1(\mathcal{K}_1(\mathcal{K}) \times \ldots \times \mathcal{K}_\tau(\mathcal{K})) \times \cdots \times \mathcal{O}_\tau(\mathcal{K}_1(\mathcal{K}) \times \ldots \times \mathcal{K}_\tau(\mathcal{K}))), \ldots, \\ \mathcal{L}(\mathcal{O}_1(\mathcal{K}_1(\mathcal{K}) \times \ldots \times \mathcal{K}_\tau(\mathcal{K})) \times \cdots \times \mathcal{O}_\tau(\mathcal{K}_1(\mathcal{K}) \times \ldots \times \mathcal{K}_\tau(\mathcal{K}))) \end{pmatrix} \\
&\leq \Im_1(\mathcal{O}_1(\mathcal{K}_1(\mathcal{K}) \times \ldots \times \mathcal{K}_\tau(\mathcal{K}))) + \cdots + \Im_\tau(\mathcal{O}_\tau(\mathcal{K}_1(\mathcal{K}) \times \ldots \times \mathcal{K}_\tau(\mathcal{K}))) \\
&+ \varphi\begin{pmatrix} \Im_1(\mathcal{O}_1(\mathcal{K}_1(\mathcal{K}) \times \ldots \times \mathcal{K}_\tau(\mathcal{K}))) + \cdots + \Im_\tau(\mathcal{O}_\tau(\mathcal{K}_1(\mathcal{K}) \times \ldots \times \mathcal{K}_\tau(\mathcal{K}))), \ldots, \\ \Im_1(\mathcal{O}_1(\mathcal{K}_1(\mathcal{K}) \times \ldots \times \mathcal{K}_\tau(\mathcal{K}))) + \cdots + \Im_\tau(\mathcal{O}_\tau(\mathcal{K}_1(\mathcal{K}) \times \ldots \times \mathcal{K}_\tau(\mathcal{K}))) \end{pmatrix} \\
&\leq \left[\Im_1(\mathcal{O}_1(\mathcal{K}_1(\mathcal{K}) \times \ldots \times \mathcal{K}_\tau(\mathcal{K}))) + \varphi\begin{pmatrix} \Im_1(\mathcal{O}_1(\mathcal{K}_1(\mathcal{K}) \times \ldots \times \mathcal{K}_\tau(\mathcal{K}))), \ldots, \\ \Im_1(\mathcal{O}_1(\mathcal{K}_1(\mathcal{K}) \times \ldots \times \mathcal{K}_\tau(\mathcal{K}))) \end{pmatrix}\right] \\
&+ \left[\Im_2(\mathcal{O}_2(\mathcal{K}_1(\mathcal{K}) \times \ldots \times \mathcal{K}_\tau(\mathcal{K}))) + \varphi\begin{pmatrix} \Im_2(\mathcal{O}_2(\mathcal{K}_1(\mathcal{K}) \times \ldots \times \mathcal{K}_\tau(\mathcal{K}))), \ldots, \\ \Im_2(\mathcal{O}_2(\mathcal{K}_1(\mathcal{K}) \times \ldots \times \mathcal{K}_\tau(\mathcal{K}))) \end{pmatrix}\right] \\
&\vdots \\
&+ \left[\Im_\tau(\mathcal{O}_\tau(\mathcal{K}_1(\mathcal{K}) \times \ldots \times \mathcal{K}_\tau(\mathcal{K}))) + \varphi\begin{pmatrix} \Im_\tau(\mathcal{O}_\tau(\mathcal{K}_1(\mathcal{K}) \times \ldots \times \mathcal{K}_\tau(\mathcal{K}))), \ldots, \\ \Im_\tau(\mathcal{O}_\tau(\mathcal{K}_1(\mathcal{K}) \times \ldots \times \mathcal{K}_\tau(\mathcal{K}))) \end{pmatrix}\right] \\
&\leq \frac{1}{\tau}\psi(\Im_1(\mathcal{K}_1(\mathcal{K})) + \cdots + \Im_\tau(\mathcal{K}_\tau(\mathcal{K})) + \varphi(\Im_1(\mathcal{K}_1), \Im_2(\mathcal{K}_2) \ldots, \Im_\tau(\mathcal{K}_\tau))) \\
&+ \frac{1}{\tau}\psi(\Im_1(\mathcal{K}_1(\mathcal{K})) + \cdots + \Im_\tau(\mathcal{K}_\tau(\mathcal{K})) + \varphi(\Im_1(\mathcal{K}_1), \Im_2(\mathcal{K}_2) \ldots, \Im_\tau(\mathcal{K}_\tau))) \\
&\vdots \\
&+ \frac{1}{\tau}\psi(\Im_1(\mathcal{K}_1(\mathcal{K})) + \cdots + \Im_\tau(\mathcal{K}_\tau(\mathcal{K})) + \varphi(\Im_1(\mathcal{K}_1), \Im_2(\mathcal{K}_2) \ldots, \Im_\tau(\mathcal{K}_\tau))) \\
&\leq \psi(\Im_1(\mathcal{K}_1(\mathcal{K})) + \cdots + \Im_\tau(\mathcal{K}_\tau(\mathcal{K})) + \varphi(\Im_1(\mathcal{K}_1), \Im_2(\mathcal{K}_2) \ldots, \Im_\tau(\mathcal{K}_\tau))) \\
&\leq \psi\left(\Im_1(\mathcal{K}_1(\mathcal{K})) + \cdots + \Im_\tau(\mathcal{K}_\tau(\mathcal{K})) + \varphi\begin{pmatrix} \Im_1(\mathcal{K}_1(\mathcal{K})) + \cdots + \Im_\tau(\mathcal{K}_\tau(\mathcal{K})), \ldots, \\ \Im_1(\mathcal{K}_1(\mathcal{K})) + \cdots + \Im_\tau(\mathcal{K}_\tau(\mathcal{K})) \end{pmatrix}\right) \\
&\leq \psi(\mathcal{L}(\mathcal{K}) + \varphi(\mathcal{L}(\mathcal{K}), \mathcal{L}(\mathcal{K}), \ldots, \mathcal{L}(\mathcal{K}))).
\end{aligned}$$

Then $\widetilde{\mathcal{O}}$ satisfies all the conditions of Theorem 7; therefore, $\widetilde{\mathcal{O}}$ has at least one fixed point $(z_1^*, z_2^*, \cdots, z_\tau^*)$ such that $\widetilde{\mathcal{O}}(z_1^*, z_2^*, \cdots, z_\tau^*) = (z_1^*, z_2^*, \cdots, z_\tau^*)$. Hence, the proof is complete. □

In order to apply the above theorem in the application section, we need the two-dimensional version of it in the following:

Corollary 3. *Assume that the continuous operators $\mathcal{O}_i : \mathcal{D}_1 \times \mathcal{D}_2 \longrightarrow \mathcal{D}_i$, $i = 1, 2$ meet the following contractions:*

$$\begin{aligned}&\Im_1(\mathcal{O}_1(\mathcal{K}_1\times\mathcal{K}_2))+\varphi(\Im_1(\mathcal{O}_1(\mathcal{K}_1\times\mathcal{K}_2)),\Im_1(\mathcal{O}_1(\mathcal{K}_1\times\mathcal{K}_2)))\\&\leq\frac{1}{2}\psi(\Im_1(\mathcal{K}_1)+\Im_2(\mathcal{K}_2)+\varphi(\Im_1(\mathcal{K}_1),\Im_2(\mathcal{K}_2))),\end{aligned}$$

$$\begin{aligned}&\Im_2(\mathcal{O}_2(\mathcal{K}_1\times\mathcal{K}_2))+\varphi(\Im_2(\mathcal{O}_2(\mathcal{K}_1\times\mathcal{K}_2)),\Im_2(\mathcal{O}_2(\mathcal{K}_1\times\mathcal{K}_2)))\\&\leq\frac{1}{2}\psi(\Im_1(\mathcal{K}1)+\Im_2(\mathcal{K}_2)+\varphi(\Im_1(\mathcal{K}_1),\Im_2(\mathcal{K}_2)))\end{aligned}$$

for each $\mathcal{K}_i\subseteq\mathcal{D}_i$ with $i=1,2$, where $\varphi\in\Gamma_2$ and $\psi\in\Psi$. Then, in the space $\mathcal{D}_1\times\mathcal{D}_2$, there is at least one point (z_1^*,z_2^*), such that

$$\begin{cases}\mathcal{O}_1(z_1^*,z_2^*)=z_1^*\\\mathcal{O}_2(z_1^*,z_2^*)=z_2^*.\end{cases}.$$

Corollary 4. *Assume that the continuous operators $\mathcal{O}_i:\mathcal{D}\times\mathcal{D}\longrightarrow\mathcal{D}$ meet the following contractions:*

$$\begin{aligned}&\Im(\mathcal{O}_1(\mathcal{K}_1\times\mathcal{K}_2))+\varphi(\Im(\mathcal{O}_1(\mathcal{K}_1\times\mathcal{K}_2)),\Im(\mathcal{O}_1(\mathcal{K}_1\times\mathcal{K}_2)))\\&\leq\frac{1}{2}\psi(\Im(\mathcal{K}1)+\Im(\mathcal{K}_2)+\varphi(\Im(\mathcal{K}_1),\Im(\mathcal{K}_2))),\end{aligned}$$

$$\begin{aligned}&\Im(\mathcal{O}_2(\mathcal{K}_1\times\mathcal{K}_2))+\varphi(\Im(\mathcal{O}_2(\mathcal{K}_1\times\mathcal{K}_2)),\Im(\mathcal{O}_2(\mathcal{K}_1\times\mathcal{K}_2)))\\&\leq\frac{1}{2}\psi(\Im(\mathcal{K}1)+\Im(\mathcal{K}_2)+\varphi(\Im(\mathcal{K}_1),\Im(\mathcal{K}_2)))\end{aligned}$$

for each $\mathcal{K}_i\subseteq\mathcal{D}_i$ with $i=1,2$ where $\varphi\in\Gamma_2$ and $\psi\in\Psi$. Then, in the space $\mathcal{D}_1\times\mathcal{D}_2$, there is at least one point (z_1^*,z_2^*), such that

$$\begin{cases}\mathcal{O}_1(z_1^*,z_2^*)=z_1^*\\\mathcal{O}_2(z_1^*,z_2^*)=z_2^*.\end{cases}.$$

Taking $\mathcal{D}_1=\mathcal{D}_2=\mathcal{D}$, $\Im_1=\Im_2=\Im$, in Corollary 3, the anticipated outcome is attained.

Corollary 5. *Assume that the continuous operators $\mathcal{O}_i:\mathcal{D}\times\mathcal{D}\longrightarrow\mathcal{D}$ meet the following contractions:*

$$\Im(\mathcal{O}_1(\mathcal{K}_1\times\mathcal{K}_2))\leq\alpha[\Im(\mathcal{K}1)+\Im(\mathcal{K}_2)],$$

$$\Im(\mathcal{O}_2(\mathcal{K}_1\times\mathcal{K}_2))\leq\alpha[\Im(\mathcal{K}1)+\Im(\mathcal{K}_2)],$$

for each $\mathcal{K}_i\subseteq\mathcal{D}$ $(i=1,2)$, where $0\leq\alpha<\frac{1}{2}$. Then, in the space $\mathcal{D}_1\times\mathcal{D}_2$, there is at least one point (z_1^*,z_2^*), such that

$$\begin{cases}\mathcal{O}_1(z_1^*,z_2^*)=z_1^*\\\mathcal{O}_2(z_1^*,z_2^*)=z_2^*.\end{cases}.$$

Proof. Taking $\varphi(\iota,\jmath)=0$ and $\psi(\iota)=2\alpha t$ with $0\leq\alpha<\frac{1}{2}$ in Corollary 4, the anticipated outcome is attained. $\square$

4 Application and Example

We demonstrate the existence of solutions for SFIEs in this part as an application of our findings. We consider the Banach space $BC\left(\mathbb{R}_+^2\right)$, which consists of all continuous real functions that are bounded on $\mathbb{R}_+^2$.

$BC\left(\mathbb{R}_+^2\right)$ is equipped with the following norm:

$$\|\varrho\| = \sup\left\{|\varrho(\iota,\jmath)| : \iota,\jmath \geq 0\right\}.$$

We select a fixed $\mathcal{K} \in \mathcal{M}_{BC\left(\mathbb{R}_+^2\right)}$ and a number $\xi > 0$. For $\varepsilon > 0$ and $\varrho \in \mathcal{K}$, the modulus of continuity ϱ:$[0,\xi] \to \mathbb{R}$ is indicated by $\kappa^{\xi}(\varrho,\varepsilon)$, where:

$$\kappa^{\xi}(\varrho,\epsilon) = \sup\left\{|\varrho(\iota,\jmath) - \varrho(\upsilon,\nu)| : \iota,\jmath,\upsilon,\nu \in [0,\xi], |\iota-\upsilon| \leq \epsilon, |\jmath-\nu| \leq \epsilon\right\}.$$

Now, we define

$$\begin{aligned}
\kappa^{\xi}(\mathcal{K},\epsilon) &= \sup\left\{\kappa^{\xi}(\varrho,\epsilon) : \varrho \in \mathcal{K}\right\},\\
\kappa_0^{\xi}(\mathcal{K}) &= \lim_{\epsilon\to 0}\kappa^{\xi}(\mathcal{K},\epsilon),\\
\kappa_0(\mathcal{K}) &= \lim_{\xi\to\infty}\kappa_0^{\xi}(\mathcal{K}).
\end{aligned}$$

Moreover, for two fixed numbers $\iota,\jmath \in \mathbb{R}_+$, we denote $\mathcal{K}(\iota,\jmath) = \{\varrho(\iota,\jmath) : \varrho \in \mathcal{K}\}$ and define

$$diam\mathcal{K}(\iota,\jmath) = \sup\left\{|\varrho(\iota,\jmath) - \wp(\iota,\jmath)| : \varrho,\wp \in \mathcal{K}\right\}.$$

Finally, we define the MNC $\Im$: $M_{BC\left(\mathbb{R}_+^2\right)} \to [0,\infty)$ as follows:

$$\Im(\mathcal{K}) = \kappa_0(\mathcal{K}) + \lim_{\iota,\jmath\to\infty}\sup diam\mathcal{K}(\iota,\jmath). \tag{4}$$

We analyze the existence of solutions to the SFIEs *Eq.* (1) in order to provide an application of the results.

Theorem 9. *Let the following conditions be satisfied:*

(I) There are the continuous bounded functions $a_i : \mathbb{R}_+^2 \to \mathbb{R}$ $(i = 1,2)$ with

$$A_0 = \max\left\{\sup_{\iota,\jmath\in\mathbb{R}_+}|a_1(\iota,\jmath)|, \sup_{\iota,\jmath\in\mathbb{R}_+}|a_2(\iota,\jmath)|\right\}.$$

(II) There are the continuous functions $h_i : \mathbb{R}_+^2 \times \mathbb{R}^2 \to \mathbb{R}$, and the bounded functions $k_i : \mathbb{R}_+^2 \to \mathbb{R}$
$(i = 1,2)$ such that,

$$\begin{aligned}
|h_1(\iota,\jmath,\varrho_2,\wp_2) - h_1(\iota,\jmath,\varrho_1,\wp_1)| &\leq k_1(\iota,\jmath)\left(|\varrho_2-\varrho_1| + |\wp_2-\wp_1|\right),\\
|h_2(\iota,\jmath,\varrho_2,\wp_2) - h_2(\iota,\jmath,\varrho_1,\wp_1)| &\leq k_2(\iota,\jmath)\left(|\varrho_2-\varrho_1| + |\wp_2-\wp_1|\right),
\end{aligned}$$

with

$$K_1 = \sup_{\iota,\jmath\in\mathbb{R}_+} k_1(\iota,\jmath),\ K_2 = \sup_{\iota,\jmath\in\mathbb{R}_+} k_2(\iota,\jmath).$$

(III) There are the bounded functions $H_i : \mathbb{R}_+^2 \to \mathbb{R}_+$ $(i = 1,2)$ in such a way that,

$$H_1(\iota,\jmath) = |h_1(\iota,\jmath,0,0)\ |,\ H_2(\iota,\jmath) = |h_2(\iota,\jmath,0,0)\ |,$$

with

$$\hat{H}_1 = \sup_{\iota,\jmath\in\mathbb{R}_+} H_1(\iota,\jmath), \ \hat{H}_2 = \sup_{\iota,\jmath\in\mathbb{R}_+} H_2(\iota,\jmath).$$

(IV) There are the continuous functions $p_i : \mathbb{R}_+^2 \times \mathbb{R}^2 \to \mathbb{R}$ and the bounded functions $\nu_i : \mathbb{R}_+^2 \to \mathbb{R}$

$(i = 1, 2)$ in such a way that,

$$\begin{aligned}|p_1(\iota,\jmath,\varrho_2,\wp_2) - p_1(\iota,\jmath,\varrho_1,\wp_1)| &\le \nu_1(\iota,\jmath)(|\varrho_2-\varrho_1| + |\wp_2-\wp_1|),\\ |p_2(\iota,\jmath,\varrho_2,\wp_2) - p_2(\iota,\jmath,\varrho_1,\wp_1)| &\le \nu_2(\iota,\jmath)(|\varrho_2-\varrho_1| + |\wp_2-\wp_1|),\end{aligned}$$

with

$$V_1 = \sup_{\iota,\jmath\in\mathbb{R}_+} \nu_1(\iota,\jmath), \ V_2 = \sup_{\iota,\jmath\in\mathbb{R}_+} \nu_2(\iota,\jmath).$$

(V) There are the bounded functions $U_i : \mathbb{R}_+^2 \to \mathbb{R}_+$ $(i = 1, 2)$ in such a way that,

$$U_1(\iota,\jmath) = |p_1(\iota,\jmath,0,0)|, \ U_2(\iota,\jmath) = |p_2(\iota,\jmath,0,0)|,$$

with

$$\hat{U}_1 = \sup_{\iota,\jmath\in\mathbb{R}_+} U_1(\iota,\jmath), \ \hat{U}_2 = \sup_{\iota,\jmath\in\mathbb{R}_+} U_2(\iota,\jmath),$$

for each $\iota,\jmath \in \mathbb{R}_+$. We also define

$$\begin{aligned}\widetilde{H}_1 &= \max\left\{\hat{H}_1, \hat{U}_1\right\}, \ \widetilde{H}_2 = \max\left\{\hat{H}_2, \hat{U}_2\right\},\\ \widetilde{K}_1 &= \max\{K_1, V_1\}, \ \widetilde{K}_2 = \max\{K_2, V_2\}.\end{aligned}$$

(VI) There are continuous functions $f_i : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ and the bounded continuous functions $m_i : \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 1, 2)$ such that, for each $\iota \in \mathbb{R}_+$ and $\varrho, \wp \in \mathbb{R}$

$$|f_1(\iota,\varrho,\wp)| \le m_1(\iota),$$

$$|f_2(\iota,\varrho,\wp)| \le m_2(\iota).$$

Moreover, for each $\iota \in \mathbb{R}_+$ and $\varrho_1, \wp_1, \varrho_2, \wp_2 \in \mathbb{R}$

$$\begin{aligned}|f_1(\iota,\varrho_2,\wp_2) - f_1(\iota,\varrho_1,\wp_1)| &\le m_1(\iota)(|\varrho_2-\varrho_1| + |\wp_2-\wp_1|),\\ |f_2(\iota,\varrho_2,\wp_2) - f_2(\iota,\varrho_1,\wp_1)| &\le m_2(\iota)(|\varrho_2-\varrho_1| + |\wp_2-\wp_1|).\end{aligned}$$

(VII) There are continuous functions $r_i : \mathbb{R}_+^2 \times \mathbb{R}^2 \to \mathbb{R}$ $(i = 1, 2)$ and the bounded continuous functions $n_i : \mathbb{R}_+^2 \to \mathbb{R}$ $(i = 1, 2)$ such that, for all $\iota,\jmath \in \mathbb{R}_+$ and $\varrho, \wp \in \mathbb{R}$

$$\begin{aligned}|r_1(\iota,\jmath,\varrho,\wp)| &\le n_1(\iota,\jmath),\\ |r_1(\iota,\jmath,\varrho,\wp)| &\le n_1(\iota,\jmath).\end{aligned}$$

Moreover, for each $\iota,\jmath \in \mathbb{R}_+$ and $\varrho_1, \wp_1, \varrho_2, \wp_2 \in \mathbb{R}$

$$\begin{aligned}|r_1(\iota,\jmath,\varrho_2,\wp_2) - r_1(\iota,\jmath,\varrho_1,\wp_1)| &\le n_1(\iota,\jmath)(|\varrho_2-\varrho_1| + |\wp_2-\wp_1|),\\ |r_2(\iota,\jmath,\varrho_2,\wp_2) - r_2(\iota,\jmath,\varrho_1,\wp_1)| &\le n_2(\iota,\jmath)(|\varrho_2-\varrho_1| + |\wp_2-\wp_1|).\end{aligned}$$

$(VIII)$ There are continuous functions $g_i : \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ and positive constants G_i $(i = 1, 2)$

in such a way that,

$$\sup_{\iota,\jmath\in\mathbb{R}_+} \left\{ \int_0^\iota |g_1(\iota,\jmath,\upsilon)| \, m_1(\upsilon) \, d\upsilon \right\} \leq G_1,$$
$$\sup_{\iota,\jmath\in\mathbb{R}_+} \left\{ \int_0^\iota |g_2(\iota,\jmath,\upsilon)| \, m_2(\upsilon) \, d\upsilon \right\} \leq G_2,$$

moreover,

$$\lim_{\iota,\jmath\to\infty} \int_0^\iota |g_1(\iota,\jmath,\upsilon)| \, m_1(\upsilon) \, d\upsilon = 0,$$
$$\lim_{\iota,\jmath\to\infty} \int_0^\iota |g_2(\iota,\jmath,\upsilon)| \, m_2(\upsilon) \, d\upsilon = 0.$$

(IX) There are continuous functions $q_i : \mathbb{R}_+^2 \times \mathbb{R}^2 \to \mathbb{R}$ $(i = 1, 2)$ and positive constants Q_i $(i = 1, 2)$ in such a way that,

$$\sup_{\iota,\jmath\in\mathbb{R}_+} \left\{ \int_0^\jmath \int_0^\iota |q_1(\iota,\jmath,\upsilon,\nu)| \, n_1(\upsilon,\nu) \, d\upsilon d\nu \right\} \leq Q_1,$$
$$\sup_{\iota,\jmath\in\mathbb{R}_+} \left\{ \int_0^\jmath \int_0^\iota |q_2(\iota,\jmath,\upsilon,\nu)| \, n_2(\upsilon,\nu) \, d\upsilon d\nu \right\} \leq Q_2,$$

and

$$\lim_{\iota,\jmath\to\infty} \int_0^\jmath \int_0^\iota |q_1(\iota,\jmath,\upsilon,\nu)| \, n_1(\upsilon,\nu) \, d\upsilon d\nu = 0,$$
$$\lim_{\iota,\jmath\to\infty} \int_0^\jmath \int_0^\iota |q_2(\iota,\jmath,\upsilon,\nu)| \, n_2(\upsilon,\nu) \, d\upsilon d\nu = 0,$$

and

$$\begin{array}{ll} K_1 G_1 + V_1 Q_1 & < \frac{1}{2}, \\ K_2 G_2 + V_2 Q_2 & < \frac{1}{2}, \end{array}$$

and

$$\begin{array}{ll} \widetilde{K}_1 (G_1 + Q_1) & < \frac{1}{2}, \\ \widetilde{K}_2 (G_2 + Q_2) & < \frac{1}{2}. \end{array}$$

Then, the SFIEs *Eq.* (1) has at least one solution in $BC\left(\mathbb{R}_+^2\right) \times BC\left(\mathbb{R}_+^2\right)$.

Proof. Let, we define the operators $\mathcal{O}_i$, $\mathcal{P}_i$, and $\mathcal{U}_i$: $BC\left(\mathbb{R}_+^2\right) \times BC\left(\mathbb{R}_+^2\right) \to BC\left(\mathbb{R}_+^2\right)$, $(i = 1, 2)$ by

$$\begin{aligned} \mathcal{O}_1(\varrho(\iota,\jmath), \wp(\iota,\jmath)) &= a_1(\iota,\jmath) + h_1(\iota,\jmath,\varrho(\iota,\jmath), \wp(\iota,\jmath)) \, \mathcal{U}_1(\varrho(\iota,\jmath), \wp(\iota,\jmath)) \\ &\quad + p_1(\iota,\jmath,\varrho(\iota,\jmath), \wp(\iota,\jmath)) \, \mathcal{P}_1(\varrho(\iota,\jmath), \wp(\iota,\jmath)), \end{aligned}$$

$$\begin{aligned} \mathcal{O}_2(\varrho(\iota,\jmath), \wp(\iota,\jmath)) &= a_2(\iota,\jmath) + h_2(\iota,\jmath,\varrho(\iota,\jmath), \wp(\iota,\jmath)) \, \mathcal{U}_2(\varrho(\iota,\jmath), \wp(\iota,\jmath)) \\ &\quad + p_2(\iota,\jmath,\varrho(\iota,\jmath), \wp(\iota,\jmath)) \, \mathcal{P}_2(\varrho(\iota,\jmath), \wp(\iota,\jmath)), \end{aligned}$$

$$\mathcal{U}_1(\varrho(\iota,\jmath), \wp(\iota,\jmath)) = \int_0^\iota g_1(\iota,\jmath,\upsilon) \, f_1(\upsilon, \varrho(\upsilon,\jmath), \wp(\upsilon,\jmath)) \, d\upsilon,$$

$$\mathcal{U}_2(\varrho(\iota,\jmath), \wp(\iota,\jmath)) = \int_0^\iota g_2(\iota,\jmath,\upsilon) \, f_2(\upsilon, \varrho(\upsilon,\jmath), \wp(\upsilon,\jmath)) \, d\upsilon,$$

$$\mathcal{P}_1(\varrho(\iota,\jmath),\wp(\iota,\jmath)) = \int_0^{\jmath}\int_0^{\iota} q_1(\iota,\jmath,\upsilon,\nu)\, r_1(\upsilon,\nu,\varrho(\upsilon,\nu),\wp(\upsilon,\jmath))\, d\upsilon d\nu,$$
$$\mathcal{P}_2(\varrho(\iota,\jmath),\wp(\iota,\jmath)) = \int_0^{\jmath}\int_0^{\iota} q_2(\iota,\jmath,\upsilon,\nu)\, r_2(\upsilon,\nu,\varrho(\upsilon,\nu),\wp(\upsilon,\jmath))\, d\upsilon d\nu.$$

Step1. First, we check the well-definedness of these operators. According to the assumptions, the functions $\mathcal{O}_i(\varrho(\iota,\jmath),\wp(\iota,\jmath)) : \mathbb{R}_+^2 \to \mathbb{R}$ for $i = 1,2$ are continuous.

Now, we show that these functions are bounded. By assumptions $(I)-(IX)$, we have

$$\begin{aligned}
|\mathcal{O}_1(\varrho(\iota,\jmath),\wp(\iota,\jmath))| &\le |a_1(\iota,\jmath)| + |h_1(\iota,\jmath,\varrho(\iota,\jmath),\wp(\iota,\jmath)) - h_1(\iota,\jmath,0,0)|\,.\,|\mathcal{U}_1(\varrho(\iota,\jmath),\wp(\iota,\jmath))| \\
&+ |h_1(\iota,\jmath,0,0)|\,.\,|\mathcal{U}_1(\varrho(\iota,\jmath),\wp(\iota,\jmath))| + |p_1(\iota,\jmath,0,0)|\,.\,|\mathcal{P}_1(\varrho(\iota,\jmath),\wp(\iota,\jmath))| \\
&+ |p_1(\iota,\jmath,\varrho(\iota,\jmath),\wp(\iota,\jmath)) - p_1(\iota,\jmath,0,0)|\,.\,|\mathcal{P}_1(\varrho(\iota,\jmath),\wp(\iota,\jmath))| \\
&\le A_0 + k_1(\iota,\jmath)\left(|\varrho(\iota,\jmath)| + |\wp(\iota,\jmath)|\right).\int_0^{\iota} |g_1(\iota,\jmath,\upsilon)|\,.\,|f_1(\upsilon,\varrho(\upsilon,\jmath),\wp(\upsilon,\jmath))|\, d\upsilon \\
&+ \hat{H}_1.\int_0^{\iota} |g_1(\iota,\jmath,\upsilon)|\,.\,|f_1(\upsilon,\varrho(\upsilon,\jmath),\wp(\upsilon,\jmath))|\, d\upsilon \\
&+ \hat{U}_1.\int_0^{\jmath}\int_0^{\iota} |q_1(\iota,\jmath,\upsilon,\nu)|\,.\,|r_1(\upsilon,\nu,\varrho(\upsilon,\nu),\wp(\upsilon,\jmath))|\, d\upsilon d\nu \\
&+ \nu_1(\iota,\jmath)\left(|\varrho(\iota,\jmath)| + |\wp(\iota,\jmath)|\right).\int_0^{\jmath}\int_0^{\iota} |q_1(\iota,\jmath,\upsilon,\nu)|\,.\,|r_1(\upsilon,\nu,\varrho(\upsilon,\nu),\wp(\upsilon,\jmath))|\, d\upsilon d\nu \\
&\le A_0 + K_1(\|\varrho\| + \|\wp\|)\int_0^{\iota} |g_1(\iota,\jmath,\upsilon)|\,.m_1(\upsilon)\, d\upsilon \\
&+ \hat{H}_1.\int_0^{\iota} |g_1(\iota,\jmath,\upsilon)|\,.m_1(\upsilon)\, d\upsilon \\
&+ \hat{U}_1.\int_0^{\jmath}\int_0^{\iota} |q_1(\iota,\jmath,\upsilon,\nu)|\,.\,|n_1(\upsilon,\nu)|\, d\upsilon d\nu \\
&+ V_1(\|\varrho\| + \|\wp\|).\int_0^{\jmath}\int_0^{\iota} |q_1(\iota,\jmath,\upsilon,\nu)|\,.\,|n_1(\upsilon,\nu)|\, d\upsilon d\nu \\
&\le A_0 + K_1 G_1(\|\varrho\| + \|\wp\|) + \hat{H}_1 G_1 + V_1 Q_1(\|\varrho\| + \|\wp\|) + \hat{U}_1 Q_1 \\
&\le A_0 + \hat{H}_1 G_1 + \hat{U}_1 Q_1 + (K_1 G_1 + V_1 Q_1)(\|\varrho\| + \|\wp\|) \\
&\le M_1 + \widetilde{K}_1(G_1 + Q_1)(\|\varrho\| + \|\wp\|).
\end{aligned}$$

Hence

$$|\mathcal{O}_1(\varrho(\iota,\jmath),\wp(\iota,\jmath))| \le M_1 + \widetilde{K}_1(G_1 + Q_1)(\|\varrho\| + \|\wp\|), \tag{5}$$

where

$$M_1 = A_0 + (Q_1 + G_1)\widetilde{H}_1.$$

In the same way, we can get

$$|\mathcal{O}_2(\varrho(\iota,\jmath),\wp(\iota,\jmath))| \le M_2 + \widetilde{K}_2(G_2 + Q_2)(\|\varrho\| + \|\wp\|), \tag{6}$$

where

$$M_2 = A_0 + (Q_2 + G_2).\widetilde{H}_2.$$

Now from (5) and (6), we have

$$\begin{aligned}
\|\mathcal{O}_1\| &= \sup\{|\mathcal{O}_1(\varrho(\iota,\jmath),\wp(\iota,\jmath))| : \iota,\jmath \ge 0\} &\le M_1 + \widetilde{K}_1(G_1 + Q_1)(\|\varrho\| + \|\wp\|) < \infty, \\
\|\mathcal{O}_2\| &= \sup\{|\mathcal{O}_2(\varrho(\iota,\jmath),\wp(\iota,\jmath))| : \iota,\jmath \ge 0\} &\le M_2 + \widetilde{K}_2(G_2 + Q_2)(\|\varrho\| + \|\wp\|) < \infty,
\end{aligned}$$

These results demonstrate that $\mathcal{O}_1$and $\mathcal{O}_2$ are well defined.

Step2. We show that $\mathcal{O}_i : \overline{B_\hbar}\times\overline{B_\hbar} \to \overline{B_\hbar}, (i=1,2)$. Let $\hbar \geq \max\left\{\frac{M_1}{1-2\widetilde{K}_1(G_1+Q_1)}, \frac{M_2}{1-2\widetilde{K}_2(G_2+Q_2)}\right\}$, then for each $(\varrho,\wp) \in \overline{B_\hbar}\times\overline{B_\hbar}$, from (5) and (6) we have

$$|\mathcal{O}_1(\varrho(\iota,\jmath),\wp(\iota,\jmath))| \leq M_1 + \widetilde{K}_1(G_1+Q_1)(\|\varrho\|+\|\wp\|) \leq M_1 + 2\hbar\widetilde{K}_1(G_1+Q_1) \leq \hbar,$$
$$|\mathcal{O}_2(\varrho(\iota,\jmath),\wp(\iota,\jmath))| \leq M_2 + \widetilde{K}_2(G_2+Q_2)(\|\varrho\|+\|\wp\|) \leq M_2 + 2\hbar\widetilde{K}_2(G_2+Q_2) \leq \hbar.$$

That shows $\mathcal{O}_i\left(\overline{B_\hbar}\times\overline{B_\hbar}\right) \subseteq \overline{B_\hbar}, (i=1,2)$.

Step3. We show that the operators $\mathcal{O}_i : \overline{B_\hbar}\times\overline{B_\hbar} \to \overline{B_\hbar}, (i=1,2)$ are continuous.

Let $\epsilon > 0$ arbitrarily and $(\varrho_1,\wp_1) \in \overline{B_\hbar}\times\overline{B_\hbar}$, $(\varrho_2,\wp_2) \in \overline{B_\hbar}\times\overline{B_\hbar}$ such that $\|\varrho_2-\varrho_1\| \leq \epsilon$, $\|\wp_2-\wp_1\| \leq \epsilon$, indeed $\|(\varrho_2,\wp_2)-(\varrho_1,\wp_1)\| \leq \epsilon$.

Then, we have:

$$\begin{aligned}
&|\mathcal{O}_1(\varrho_2,\wp_2)(\iota,\jmath) - \mathcal{O}_1(\varrho_1,\wp_1)(\iota,\jmath)| \\
&\leq \left|\begin{array}{c} h_1(\iota,\jmath,\varrho_2,\wp_2)\mathcal{U}_1(\varrho_2,\wp_2) + P_1(\iota,\jmath,\varrho_2,\wp_2)\mathcal{P}_1(\varrho_2,\wp_2) \\ -h_1(\iota,\jmath,\varrho_1,\wp_1)\mathcal{U}_1(\varrho_1,\wp_1) - P_1(\iota,\jmath,\varrho_1,\wp_1)\mathcal{P}_1(\varrho_1,\wp_1) \end{array}\right| \\
&\leq |h_1(\iota,\jmath,\varrho_2,\wp_2)\mathcal{U}_1(\varrho_2,\wp_2) - h_1(\iota,\jmath,\varrho_1,\wp_1)\mathcal{U}_1(\varrho_1,\wp_1)| \\
&+ |P_1(\iota,\jmath,\varrho_2,\wp_2)\mathcal{P}_1(\varrho_2,\wp_2) - P_1(\iota,\jmath,\varrho_1,\wp_1)\mathcal{P}_1(\varrho_1,\wp_1)| \\
&\leq \left|\begin{array}{c} h_1(\iota,\jmath,\varrho_2,\wp_2)\mathcal{U}_1(\varrho_2,\wp_2) - h_1(\iota,\jmath,\varrho_1,\wp_1)\mathcal{U}_1(\varrho_2,\wp_2) \\ +h_1(\iota,\jmath,\varrho_1,\wp_1)\mathcal{U}_1(\varrho_2,\wp_2) - h_1(\iota,\jmath,\varrho_1,\wp_1)\mathcal{U}_1(\varrho_1,\wp_1) \end{array}\right| \\
&+ \left|\begin{array}{c} P_1(\iota,\jmath,\varrho_2,\wp_2)\mathcal{P}_1(\varrho_2,\wp_2) - P_1(\iota,\jmath,\varrho_1,\wp_1)\mathcal{P}_1(\varrho_2,\wp_2) \\ +P_1(\iota,\jmath,\varrho_1,\wp_1)\mathcal{P}_1(\varrho_2,\wp_2) - P_1(\iota,\jmath,\varrho_1,\wp_1)\mathcal{P}_1(\varrho_1,\wp_1) \end{array}\right| \\
&\leq |h_1(\iota,\jmath,\varrho_2,\wp_2) - h_1(\iota,\jmath,\varrho_1,\wp_1)| . |\mathcal{U}_1(\varrho_2,\wp_2)| \\
&+ |h_1(\iota,\jmath,\varrho_1,\wp_1)| . |\mathcal{U}_1(\varrho_2,\wp_2) - \mathcal{U}_1(\varrho_1,\wp_1)| \\
&+ |\mathcal{P}_1(\varrho_2,\wp_2)| . |P_1(\iota,\jmath,\varrho_2,\wp_2) - P_1(\iota,\jmath,\varrho_1,\wp_1)| \\
&+ |P_1(\iota,\jmath,\varrho_1,\wp_1)| . |\mathcal{P}_1(\varrho_2,\wp_2) - \mathcal{P}_1(\varrho_1,\wp_1)| \\
&\leq G_1K_1(\|\varrho_2-\varrho_1\|+\|\wp_2-\wp_1\|) \\
&+ \left(K_1(\|\varrho_1\|+\|\wp_1\|)+\hat{H}_1\right) . \int_0^\iota |g_1(\iota,\jmath,\upsilon)| \left(f_1(\upsilon,\varrho_2(\upsilon,\jmath),\wp_2(\upsilon,\jmath) - f_1(\upsilon,\varrho_1(\upsilon,\jmath),\wp_1(\upsilon,\jmath)))\right) d\upsilon \\
&+ V_1Q_1(\|\varrho_2-\varrho_1\|+\|\wp_2-\wp_1\|) \\
&+ \left(V_1(\|\varrho_1\|+\|\wp_1\|)+\hat{U}_1\right) . \int_0^\jmath\int_0^\iota |q_1(\iota,\jmath,\upsilon,\nu)| \left(r_1(\upsilon,\nu,\varrho_2(\upsilon,\nu),\wp_2(\upsilon,\jmath)) - r_1(\upsilon,\nu,\varrho_1(\upsilon,\nu),\wp_1(\upsilon,\jmath))\right) d\upsilon d\nu \\
&\leq (G_1K_1+V_1Q_1)(\|\varrho_2-\varrho_1\|+\|\wp_2-\wp_1\|) \\
&+ \left(K_1(\|\varrho_1\|+\|\wp_1\|)+\hat{H}_1\right) . \int_0^\iota |g_1(\iota,\jmath,\upsilon)| . |f_1(\upsilon,\varrho_2(\upsilon,\jmath),\wp_2(\upsilon,\jmath) - f_1(\upsilon,\varrho_1(\upsilon,\jmath),\wp_1(\upsilon,\jmath)))| d\upsilon \\
&+ \left(V_1(\|\varrho_1\|+\|\wp_1\|)+\hat{U}_1\right) . \int_0^\jmath\int_0^\iota |q_1(\iota,\jmath,\upsilon,\nu)| . |r_1(\upsilon,\nu,\varrho_2(\upsilon,\nu),\wp_2(\upsilon,\jmath)) - r_1(\upsilon,\nu,\varrho(\upsilon,\nu),\wp(\upsilon,\jmath))| d\upsilon d\nu
\end{aligned} \tag{7}$$

So, we get

$$\begin{aligned}
&|\mathcal{O}_1(\varrho_2,\wp_2)(\iota,\jmath) - \mathcal{O}_1(\varrho_1,\wp_1)(\iota,\jmath)| \\
&\leq (G_1K_1+V_1Q_1)\epsilon \\
&+ \left(2\hbar K_1+\hat{H}_1\right) \times \int_0^\iota |g_1(\iota,\jmath,\upsilon)| . |f_1(\upsilon,\varrho_2(\upsilon,\jmath),\wp_2(\upsilon,\jmath) - f_1(\upsilon,\varrho_1(\upsilon,\jmath),\wp_1(\upsilon,\jmath)))| d\upsilon \\
&+ \left(2\hbar V_1+\hat{U}_1\right) \times \int_0^\jmath\int_0^\iota |q_1(\iota,\jmath,\upsilon,\nu)| . |r_1(\upsilon,\nu,\varrho_2(\upsilon,\nu),\wp_2(\upsilon,\jmath)) - r_1(\upsilon,\nu,\varrho(\upsilon,\nu),\wp(\upsilon,\jmath))| d\upsilon d\nu.
\end{aligned} \tag{8}$$

Also, by establishing the conditions of $(VIII)$ and (IX), there exists $\xi > 0$, such that for

$\iota, \jmath > \xi$, we have

$$\int_0^\iota |g_1(\iota,\jmath,\upsilon)| . |f_1(\upsilon, \varrho_2(\upsilon,\jmath), \wp_2(\upsilon,\jmath)) - f_1(\upsilon, \varrho_1(\upsilon,\jmath), \wp_1(\upsilon,\jmath))| \, d\upsilon \le \epsilon,$$

$$\int_0^\jmath \int_0^\iota |q_1(\iota,\jmath,\upsilon,\nu)| . |r_1(\upsilon,\nu, \varrho_2(\upsilon,\nu), \wp_2(\upsilon,\jmath)) - r_1(\upsilon,\nu, \varrho_1(\upsilon,\nu), \wp_1(\upsilon,\jmath))| \, d\upsilon d\nu \le \epsilon. \tag{9}$$

Suppose that $\iota, \jmath > \xi$, it follows (8) and (9) that

$$|\mathcal{O}_1(\varrho_2,\wp_2)(\iota,\jmath) - \mathcal{O}_1(\varrho_1,\wp_1)(\iota,\jmath)| \le \left(G_1K_1 + V_1Q_1 + \hat{H}_1 + \hat{U}_1 + 2\hbar K_1 + 2\hbar V_1\right)\epsilon.$$

So, in this case, $\mathcal{O}_1$ is a continuous operator.

In the other case, that is, if $\iota, \jmath \in [0,\xi]$, then we get

$$\int_0^\iota |g_1(\iota,\jmath,\upsilon)| . |f_1(\upsilon, \varrho_2(\upsilon,\jmath), \wp_2(\upsilon,\jmath)) - f_1(\upsilon, \varrho_1(\upsilon,\jmath), \wp_1(\upsilon,\jmath))| \, d\upsilon \le \xi G_3 \kappa^\xi(f_1,\epsilon), \tag{10}$$

$$\int_0^\jmath \int_0^\iota |q_1(\iota,\jmath,\upsilon,\nu)| . |r_1(\upsilon,\nu, \varrho_2(\upsilon,\nu), \wp_2(\upsilon,\jmath)) - r_1(\upsilon,\nu, \varrho_1(\upsilon,\nu), \wp_1(\upsilon,\jmath))| \, d\upsilon d\nu \le \xi^2 Q_3 \kappa^\xi(r_1,\epsilon), \tag{11}$$

where

$$\begin{aligned}
\kappa^\xi(f_1,\epsilon) &= \sup_{\iota,\jmath\in\mathbb{R}_+} \left\{ \begin{array}{c} |f_1(\iota, \varrho_2(\iota,\jmath), \wp_2(\iota,\jmath)) - f_1(\iota, \varrho_1(\iota,\jmath), \wp_1(\iota,\jmath))| : \iota \in [0,\xi] \\ (\varrho_2,\wp_2), (\varrho_1,\wp_1) \in \overline{B_\hbar} \times \overline{B_\hbar}, \ |\varrho_2 - \varrho_1| \le \epsilon, \ |\wp_1 - \wp_2| \le \epsilon \end{array} \right\}, \\
\kappa^\xi(r_1,\epsilon) &= \sup_{\iota,\jmath\in\mathbb{R}_+} \left\{ \begin{array}{c} |r_1(\iota,\jmath, \varrho_2(\iota,\jmath), \wp_2(\iota,\jmath)) - r_1(\iota,\jmath, \varrho_1(\iota,\jmath), \wp_1(\iota,\jmath))| : \iota,\jmath \in [0,\xi] \\ (\varrho_2,\wp_2), (\varrho_1,\wp_1) \in \overline{B_\hbar} \times \overline{B_\hbar}, \ |\varrho_2 - \varrho_1| \le \epsilon, \ |\wp_1 - \wp_2| \le \epsilon \end{array} \right\}, \\
G_3 &= \max\left\{ |g_1(\iota,\jmath,\upsilon)| : (\iota,\jmath,\upsilon) \in [0,\xi]^3 \right\}, \\
Q_3 &= \max\left\{ |q_1(\iota,\jmath,\upsilon,\nu)| : (\iota,\jmath,\upsilon,\nu) \in [0,\xi]^4 \right\}.
\end{aligned}$$

Now, due to inequalities (8), (10), and (11), we have

$$\begin{aligned}
|\mathcal{O}_1(\varrho_2,\wp_2)(\iota,\jmath) - \mathcal{O}_1(\varrho_1,\wp_1)(\iota,\jmath)| \le{}& (G_1K_1 + V_1Q_1)\epsilon + \left(2\hbar K_1 + \hat{H}_1\right) \xi G_3 \kappa^\xi(f_1,\epsilon) \\
&+ \left(2\hbar V_1 + \hat{U}_1\right) \xi^2 Q_3 \kappa^\xi(\hbar_1,\epsilon)
\end{aligned}$$

But we note that the functions f_1 on the set $[0,\xi] \times [-\hbar,\hbar] \times [-\hbar,\hbar]$ and r_1 on the set $[0,\xi] \times [0,\xi] \times [-\hbar,\hbar] \times [-\hbar,\hbar]$ are uniformly continuous. Therefore, we conclude that $\kappa^\xi(f_1,\epsilon) \to 0$ and $\kappa^\xi(r_1,\epsilon) \to 0$ as $\epsilon \to 0$. This shows that $\mathcal{O}_1$ is a continuous operator. Similarly, it can be shown that $\mathcal{O}_2$ is also a continuous operator.

Step4. In the sequel, we show that for each $\mathcal{K}_1, \mathcal{K}_2 \subseteq \overline{B_\hbar}$

$$\begin{aligned}
\Im(\mathcal{O}_1(\mathcal{K}_1 \times \mathcal{K}_2)) &\le (K_1G_1 + V_1Q_1)(\Im(\mathcal{K}_1) + \Im(\mathcal{K}_2)), \\
\Im(\mathcal{O}_2(\mathcal{K}_1 \times \mathcal{K}_2)) &\le (K_2G_2 + V_2Q_2)(\Im(\mathcal{K}_1) + \Im(\mathcal{K}_2)).
\end{aligned}$$

Let $\xi > 0$, $\epsilon > 0$. Take $\iota_1, \iota_2, \jmath_1$, and $\jmath_2 \in [0,\xi]$ with $\iota_1 \le \iota_2$ and $\jmath_1 \le \jmath_2$ such that $\iota_2 - \iota_1 < \epsilon$, $\jmath_2 - \jmath_1 < \epsilon$. Then for $(\varrho,\wp) \in \mathcal{K}_1 \times \mathcal{K}_2$, we get

$$
\begin{aligned}
&|\mathcal{O}_1(\varrho,\wp)(\iota_2,\jmath_2)-\mathcal{O}_1(\varrho,\wp)(\iota_1,\jmath_1)| \\
&\le\ |a_1(\iota_2,\jmath_2)-a_1(\iota_1,\jmath_1)| \\
&+|h_1(\iota_2,\jmath_2,\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2)).\mathcal{U}_1(\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2))-h_1(\iota_1,\jmath_1,\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1)).\mathcal{U}_1(\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1))| \\
&+|p_1(\iota_2,\jmath_2,\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2)).\mathcal{P}_1(\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2))-p_1(\iota_1,\jmath_1,\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1)).\mathcal{P}_1(\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1))| \\
&\le|h_1(\iota_2,\jmath_2,\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2)).\mathcal{U}_1(\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2))-h_1(\iota_2,\jmath_2,\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1)).\mathcal{U}_1(\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2))| \\
&+|h_1(\iota_2,\jmath_2,\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1)).\mathcal{U}_1\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2)-h_1(\iota_1,\jmath_1,\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1)).\mathcal{U}_1(\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2))| \\
&+|h_1(\iota_1,\jmath_1,\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1)).\mathcal{U}_1(\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2))-h_1(\iota_1,\jmath_1,\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1)).\mathcal{U}_1(\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1))| \\
&+|p_1(\iota_2,\jmath_2,\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2)).\mathcal{P}_1(\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2))-p_1(\iota_2,\jmath_2,\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1)).\mathcal{P}_1(\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2))| \\
&+|p_1(\iota_2,\jmath_2,\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1)).\mathcal{P}_1(\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2))-p_1(\iota_1,\jmath_1,\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1)).\mathcal{P}_1(\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2))| \\
&+|p_1(\iota_1,\jmath_1,\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1)).\mathcal{P}_1(\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2))-p_1(\iota_1,\jmath_1,\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1)).\mathcal{P}_1(\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1))| \\
&\le\kappa^\xi(a,\epsilon)+k_1(\iota_2,\jmath_2)(|\varrho(\iota_2,\jmath_2)-\varrho(\iota_1,\jmath_1)|+|\wp(\iota_2,\jmath_2)-\wp(\iota_1,\jmath_1)|).G_1+\kappa_\hbar^\xi(h_1,\epsilon).G_1 \\
&+|h_1(\iota_1,\jmath_1,\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1))-h_1(\iota_1,\jmath_1,0,0)|.|\mathcal{U}_1(\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2))-\mathcal{U}_1(\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1))| \\
&+|h_1(\iota_1,\jmath_1,0,0)|.|\mathcal{U}_1(\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2))-\mathcal{U}_1(\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1))| \\
&+\nu_1(\iota_2,\jmath_2)(|\varrho(\iota_2,\jmath_2)-\varrho(\iota_1,\jmath_1)|+|\wp(\iota_2,\jmath_2)-\wp(\iota_1,\jmath_1)|).|\mathcal{P}_1(\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2))|+\kappa_\hbar^\xi(p_1,\epsilon).Q_1 \\
&+|p_1(\iota_1,\jmath_1,\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1))-p_1(\iota_1,\jmath_1,0,0)|.|\mathcal{P}_1(\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2))-\mathcal{P}_1(\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1))| \\
&+|p_1(\iota_1,\jmath_1,0,0)|.|\mathcal{P}_1(\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2))-\mathcal{P}_1(\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1))| \\
&\le\kappa^\xi(a,\epsilon)+K_1\left(\kappa^\xi(\varrho,\epsilon)+\kappa^\xi(\wp,\epsilon)\right)G_1+\kappa_\hbar^\xi(h_1,\epsilon).G_1 \\
&+\left(2\hbar K_1+\hat{H}_1\right).|\mathcal{U}_1(\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2))-\mathcal{U}_1(\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1))| \\
&+V_1\left(\kappa^\xi(\varrho,\epsilon)+\kappa^\xi(\wp,\epsilon)\right)Q_1+\kappa_\hbar^\xi(p_1,\epsilon).Q_1 \\
&+\left(2\hbar V_1+\hat{U}_1\right).|\mathcal{P}_1(\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2))-\mathcal{P}_1(\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1))| \qquad (12)
\end{aligned}
$$

where

$$
\begin{aligned}
&\kappa^\xi(a,\epsilon)=\sup\{|a_1(\iota_2,\jmath_2)-a_1(\iota_1,\jmath_1)|:\iota_1,\iota_2,\jmath_1,\jmath_2\in[0,\xi],\ |\iota_2-\iota_1|\le\epsilon,\ |\jmath_2-\jmath_1|\le\epsilon\}, \\
&\kappa_\hbar^\xi(h_1,\epsilon)=\sup\left\{\begin{array}{c}|h_1(\iota_2,\jmath_2,\varrho,\wp)-h_1(\iota_1,\jmath_1,\varrho,\wp)|:\iota_1,\iota_2,\jmath_1,\jmath_2\in[0,\xi],\ |\iota_2-\iota_1|\le\epsilon,\ |\jmath_2-\jmath_1|\le\epsilon, \\ \varrho,\wp\in[-\hbar,\hbar]\end{array}\right\}, \\
&\kappa_\hbar^\xi(p_1,\epsilon)=\sup\left\{\begin{array}{c}|p_1(\iota_2,\jmath_2,\varrho,\wp)-p_1(\iota_1,\jmath_1,\varrho,\wp)|:\iota_1,\iota_2,\jmath_1,\jmath_2\in[0,\xi],\ |\iota_2-\iota_1|\le\epsilon,\ |\jmath_2-\jmath_1|\le\epsilon, \\ \varrho,\wp\in[-\hbar,\hbar],\end{array}\right\}.
\end{aligned}
$$

On the other hand, the following estimate is available:

$$
\begin{aligned}
&|\mathcal{U}_1(\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2))-\mathcal{U}_1(\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1))| \\
&=\left|\int_0^{\iota_2}g_1(\iota_2,\jmath_2,\upsilon)f_1(\upsilon,\varrho(\upsilon,\jmath_2),\wp(\upsilon,\jmath_2))d\upsilon-\int_0^{\iota_1}g_1(\iota_1,\jmath_1,\upsilon)f_1(\upsilon,\varrho(\upsilon,\jmath_1),\wp(\upsilon,\jmath_1))d\upsilon\right| \\
&\le\int_0^{\iota_1}|g_1(\iota_2,\jmath_2,\upsilon)f_1(\upsilon,\varrho(\upsilon,\jmath_2),\wp(\upsilon,\jmath_2))-g_1(\iota_1,\jmath_1,\upsilon)f_1(\upsilon,\varrho(\upsilon,\jmath_1),\wp(\upsilon,\jmath_1))|d\upsilon \\
&+\int_{\iota_1}^{\iota_2}|g_1(\iota_2,\jmath_2,\upsilon)f_1(\upsilon,\varrho(\upsilon,\jmath_2),\wp(\upsilon,\jmath_2))|d\upsilon \\
&\le\int_0^{\iota_1}|g_1(\iota_2,\jmath_2,\upsilon)f_1(\upsilon,\varrho(\upsilon,\jmath_2),\wp(\upsilon,\jmath_2))-g_1(\iota_1,\jmath_1,\upsilon)f_1(\upsilon,\varrho(\upsilon,\jmath_2),\wp(\upsilon,\jmath_2))|d\upsilon \\
&+\int_0^{\iota_1}|g_1(\iota_1,\jmath_1,\upsilon)f_1(\upsilon,\varrho(\upsilon,\jmath_2),\wp(\upsilon,\jmath_2))-g_1(\iota_1,\jmath_1,\upsilon)f_1(\upsilon,\varrho(\upsilon,\jmath_1),\wp(\upsilon,\jmath_1))|d\upsilon \\
&+\int_{\iota_1}^{\iota_2}|g_1(\iota_2,\jmath_2,\upsilon)f_1(\upsilon,\varrho(\upsilon,\jmath_2),\wp(\upsilon,\jmath_2))|d\upsilon \\
&\le\kappa^\xi(g_1,\epsilon).\int_0^{\iota_1}|f_1(\upsilon,\varrho(\upsilon,\jmath_2),\wp(\upsilon,\jmath_2))|d\upsilon+\kappa_\hbar^\xi(f_1,\epsilon).\int_0^{\iota_1}|g_1(\iota_1,\jmath_1,\upsilon)|d\upsilon+\epsilon BM_1 \\
&\le\xi A\omega^\xi(g_1,\epsilon)+\xi B\omega_\hbar^\xi(f_1,\epsilon)+\epsilon BM_1, \qquad (13)
\end{aligned}
$$

where

$$\kappa^{\xi}(g_1,\epsilon)=\sup\left\{\begin{array}{c}|g_1(\iota_2,\jmath_2,\upsilon)-g_1(\iota_1,\jmath_1,\upsilon)|:\iota_1,\iota_2,\jmath_1,\jmath_2,\upsilon\in[0,\xi],\\ |\iota_2-\iota_1|\le\epsilon,|\jmath_2-\jmath_1|\le\epsilon\end{array}\right\},$$
$$\kappa^{\xi}_{\hbar}(f_1,\epsilon)=\sup\left\{\begin{array}{c}|f_1(\upsilon,\varrho(\upsilon,\jmath_2),\wp(\upsilon,\jmath_2))-f_1(\upsilon,\varrho(\upsilon,\jmath_1),\wp(\upsilon,\jmath_1))|:\jmath_1,\jmath_2,\upsilon\in[0,\xi],\\ |\jmath_2-\jmath_1|\le\epsilon,\ \varrho,\wp\in[-\hbar,\hbar]\end{array}\right\},$$
$$A=\sup\{|f_1(\upsilon,\varrho(\upsilon,\jmath),\wp(\upsilon,\jmath))|:\jmath,\upsilon\in[0,\xi],\ \varrho,\wp\in[-\hbar,\hbar]\ \},$$
$$B=\sup\{|g_1(\iota,\jmath,\upsilon)|:\iota,\jmath,\upsilon\in[0,\xi]\},$$
$$M_1=\sup\{m_1(\iota):\iota\in[0,\xi]\}.$$

We also have the following estimate:

$$\begin{aligned}
&|\mathcal{P}_1(\varrho(\iota_2,\jmath_2),\wp(\iota_2,\jmath_2))-\mathcal{P}_1(\varrho(\iota_1,\jmath_1),\wp(\iota_1,\jmath_1))|\\
&=\left|\int_0^{\jmath_2}\int_0^{\iota_2}q_1(\iota_2,\jmath_2,\upsilon,\nu)\,r_1(\upsilon,\nu,\varrho(\upsilon,\nu),\wp(\upsilon,\jmath_2))\,d\upsilon d\nu-\int_0^{\jmath_1}\int_0^{\iota_1}q_1(\iota_1,\jmath_1,\upsilon,\nu)\,r_1(\upsilon,\nu,\varrho(\upsilon,\nu),\wp(\upsilon,\jmath_1))\,d\upsilon d\nu\right|\\
&\le\int_0^{\jmath_1}\int_0^{\iota_1}|q_1(\iota_2,\jmath_2,\upsilon,\nu)\,r_1(\upsilon,\nu,\varrho(\upsilon,\nu),\wp(\upsilon,\jmath_2))-q_1(\iota_1,\jmath_1,\upsilon,\nu)\,r_1(\upsilon,\nu,\varrho(\upsilon,\nu),\wp(\upsilon,\jmath_1))|\,d\upsilon d\nu\\
&+\int_0^{\jmath_1}\int_{\iota_1}^{\iota_2}|q_1(\iota_2,\jmath_2,\upsilon,\nu)|\,.\,|r_1(\upsilon,\nu,\varrho(\upsilon,\nu),\wp(\upsilon,\jmath_2))|\,d\upsilon d\nu\\
&+\int_{\jmath_1}^{\jmath_2}\int_0^{\iota_2}|q_1(\iota_2,\jmath_2,\upsilon,\nu)|\,.\,|r_1(\upsilon,\nu,\varrho(\upsilon,\nu),\wp(\upsilon,\jmath_2))|\,d\upsilon d\nu\\
&\le\int_0^{\jmath_1}\int_0^{\iota_1}|q_1(\iota_2,\jmath_2,\upsilon,\nu)-q_1(\iota_1,\jmath_1,\upsilon,\nu)|\,.\,|r_1(\upsilon,\nu,\varrho(\upsilon,\nu),\wp(\upsilon,\jmath_2))|\,d\upsilon d\nu\\
&+\int_0^{\jmath_1}\int_0^{\iota_1}|q_1(\iota_1,\jmath_1,\upsilon,\nu)|\,.\,|r_1(\upsilon,\nu,\varrho(\upsilon,\nu),\wp(\upsilon,\jmath_2))-r_1(\upsilon,\nu,\varrho(\upsilon,\nu),\wp(\upsilon,\jmath_1))|\,d\upsilon d\nu\\
&+\int_0^{\jmath_1}\int_{\iota_1}^{\iota_2}|q_1(\iota_2,\jmath_2,\upsilon,\nu)|\,.\,|r_1(\upsilon,\nu,\varrho(\upsilon,\nu),\wp(\upsilon,\jmath_2))|\,d\upsilon d\nu\\
&+\int_{\jmath_1}^{\jmath_2}\int_0^{\iota_2}|q_1(\iota_2,\jmath_2,\upsilon,\nu)|\,.\,|r_1(\upsilon,\nu,\varrho(\upsilon,\nu),\wp(\upsilon,\jmath_2))|\,d\upsilon d\nu\\
&\le\xi^2N_1\kappa^{\xi}(q_1,\epsilon)+\xi^2CN_1(|\wp(\upsilon,\jmath_2)-\wp(\upsilon,\jmath_1)|)+2\xi CN_1\epsilon,
\end{aligned}\tag{14}$$

where

$$\kappa^{\xi}(q_1,\epsilon)=\sup\{|q_1(\iota_2,\jmath_2,\upsilon,\nu)-q_1(\iota_1,\jmath_1,\upsilon,\nu)|:\iota_1,\iota_2,\jmath_1,\jmath_2,\upsilon,\nu\in[0,\xi],\ \ |\iota_2-\iota_1|\le\epsilon\ ,\ \ |\jmath_2-\jmath_1|\le\epsilon\},$$
$$C=\sup\{|q_1(\iota,\jmath,\upsilon,\nu)|:\iota,\jmath,\upsilon,\nu\in[0,\xi]\},$$
$$N_1=\sup\{n_1(\iota,\jmath):\iota,\jmath\in[0,\xi]\}.$$

Now, considering (12), (13) , and (14), we get

$$\begin{aligned}
\kappa^{\xi}_{\hbar}(\mathcal{O}_1(\mathcal{K}_1\times\mathcal{K}_2),\epsilon)\le{}&\kappa^{\xi}(a,\epsilon)+K_1\left(\kappa^{\xi}(\varrho,\epsilon)+\kappa^{\xi}(\wp,\epsilon)\right)G_1+\kappa^{\xi}_{\hbar}(h_1,\epsilon).G_1\\
&+\left(2\hbar K_1+\hat{H}_1\right).\left(\xi A\omega^{\xi}(g_1,\epsilon)+\xi B\omega^{\xi}_{\hbar}(f_1,\epsilon)+\epsilon BM_1\right)\\
&+V_1\left(\kappa^{\xi}(\varrho,\epsilon)+\kappa^{\xi}(\wp,\epsilon)\right)Q_1+\kappa^{\xi}_{\hbar}(p_1,\epsilon).Q_1\\
&+\left(2\hbar V_1+\hat{U}_1\right).\left(\xi^2N_1\kappa^{\xi}(q_1,\epsilon)+\xi^2CN_1(|\wp(\upsilon,\jmath_2)-\wp(\upsilon,\jmath_1)|)+2\xi CN_1\epsilon\right).
\end{aligned}\tag{15}$$

Note that assumpitions (I)–(IX) guarantee that the functions $a_1(\iota,\jmath)$, $f_1(\iota,\varrho,\wp)$, $g_1(\iota,\jmath,\varrho,\wp)$, $h_1(\iota,\jmath,\varrho,\wp)$, $p_1(\iota,\jmath,\varrho,\wp)$, $q_1(\iota,\jmath,\varrho,\wp)$, $r_1(\iota,\jmath,\varrho,\wp)$ and $\wp(\upsilon,\cdot)$ are uniformly continuous on the sets $[0,\xi]\times[0,\xi]$, $[0,\xi]\times[0,\xi]\times[-\hbar,\hbar]$, $[0,\xi]\times[0,\xi]\times[-\hbar,\hbar]\times[-\hbar,\hbar]$, $[0,\xi]\times[0,\xi]\times[-\hbar,\hbar]\times[-\hbar,\hbar]$, $[0,\xi]\times[0,\xi]\times[-\hbar,\hbar]\times[-\hbar,\hbar]$, $[0,\xi]\times[0,\xi]\times[-\hbar,\hbar]\times[-\hbar,\hbar]$, $[0,\xi]\times[0,\xi]\times[-\hbar,\hbar]\times[-\hbar,\hbar]$, and $[0,\xi]$ respectively. It follows that $\kappa^{\xi}(a,\epsilon)\to 0$, $\kappa^{\xi}_{\hbar}(h_1,\epsilon)\to 0$, $\kappa^{\xi}(g_1,\epsilon)\to 0$, $\kappa^{\xi}_{\hbar}(f_1,\epsilon)\to 0$, $\kappa^{\xi}_{\hbar}(p_1,\epsilon)\to 0$, $\kappa^{\xi}(q_1,\epsilon)\to 0$, and $|\wp(\upsilon,\jmath_2)-\wp(\upsilon,\jmath_1)|\to 0$, as $\epsilon\to 0$.

Consequently, combining the data generated above with the estimate (15), we get

$$\kappa^{\xi}_{0}\left(\mathcal{O}_1\left(\mathcal{K}_1\times\mathcal{K}_2\right)\right)\leq\left(K_1G_1+V_1Q_1\right)\left(\kappa^{\xi}_{0}\left(\mathcal{K}_1\right)+\kappa^{\xi}_{0}\left(\mathcal{K}_2\right)\right),$$

so by taking $\xi\to\infty$, we obtain

$$\kappa_0\left(\mathcal{O}_1\left(\mathcal{K}_1\times\mathcal{K}_2\right)\right)\leq\left(K_1G_1+V_1Q_1\right)\left(\kappa_0\left(\mathcal{K}_1\right)+\kappa_0\left(\mathcal{K}_2\right)\right). \tag{16}$$

Applying a similar method, we can demonstrate that

$$\kappa_0\left(\mathcal{O}_2\left(\mathcal{K}_1\times\mathcal{K}_2\right)\right)\leq\left(K_2G_2+V_2Q_2\right)\left(\kappa_0\left(\mathcal{K}_1\right)+\kappa_0\left(\mathcal{K}_2\right)\right). \tag{17}$$

Furthermore, the estimate made in (7) indicates that for arbitrary $(\varrho_2,\wp_2),(\varrho_1,\wp_1)\in\mathbb{R}^2$ and $\iota,\jmath\in\mathbb{R}_+$ we have

$$\begin{aligned}
&\left|\mathcal{O}_1\left(\varrho_2,\wp_2\right)(\iota,\jmath)-\mathcal{O}_1\left(\varrho_1,\wp_1\right)(\iota,\jmath)\right|\\
&\leq\left(K_1G_1+V_1Q_1\right)\left(\|\varrho_2-\varrho_1\|+\|\wp_2-\wp_1\|\right)\\
&+\left(K_1\left(\|\varrho_1\|+\|\wp_1\|\right)+\hat{H}_1\right).\int_0^{\iota}\left|g_1(\iota,\jmath,\upsilon)\right|.\left|f_1\left(\upsilon,\varrho_2(\upsilon,\jmath),\wp_2(\upsilon,\jmath)-f_1\left(\upsilon,\varrho_1(\upsilon,\jmath),\wp_1(\upsilon,\jmath)\right)\right)\right|d\upsilon\\
&+\left(V_1\left(\|\varrho_1\|+\|\wp_1\|\right)+\hat{U}_1\right).\int_0^{\jmath}\int_0^{\iota}\left|q_1(\iota,\jmath,\upsilon,\nu)\right|.\left|r_1\left(\upsilon,\nu,\varrho_2(\upsilon,\nu),\wp_2(\upsilon,\jmath)\right)-r_1\left(\upsilon,\nu,\varrho(\upsilon,\nu),\wp(\upsilon,\jmath)\right)\right|d\upsilon d\nu\\
&\leq\left(K_1G_1+V_1Q_1\right)\left(\|\varrho_2-\varrho_1\|+\|\wp_2-\wp_1\|\right)\\
&+\left(K_1\left(\|\varrho_1\|+\|\wp_1\|\right)+\hat{H}_1\right)\left(\|\varrho_2-\varrho_1\|+\|\wp_2-\wp_1\|\right).\int_0^{\iota}\left|g_1(\iota,\jmath,\upsilon)\right|.m_1(\upsilon)\,d\upsilon\\
&+\left(V_1\left(\|\varrho_1\|+\|\wp_1\|\right)+\hat{U}_1\right)\left(\|\varrho_2-\varrho_1\|+\|\wp_2-\wp_1\|\right).\int_0^{\jmath}\int_0^{\iota}\left|q_1(\iota,\jmath,\upsilon,\nu)\right|n_1(\upsilon,\nu)\,d\upsilon d\nu.
\end{aligned}$$

Now according to conditions $(VIII)$ and (IX) as $\iota,\jmath\to\infty$, we deduce that

$$\lim_{\iota,\jmath\to\infty}\sup diam\left(\mathcal{O}_1\left(\mathcal{K}_1\times\mathcal{K}_2\right)\right)(\iota,\jmath)\leq\left(K_1G_1+V_1Q_1\right)\left(\lim_{\iota,\jmath\to\infty}\sup diam\mathcal{K}_1(\iota,\jmath)+\lim_{\iota,\jmath\to\infty}\sup diam\mathcal{K}_2(\iota,\jmath)\right), \tag{18}$$

with a similar method, it can be shown that

$$\lim_{\iota,\jmath\to\infty}\sup diam\left(\mathcal{O}_2\left(\mathcal{K}_1\times\mathcal{K}_2\right)\right)(\iota,\jmath)\leq\left(K_2G_2+V_2Q_2\right)\left(\lim_{\iota,\jmath\to\infty}\sup diam\mathcal{K}_1(\iota,\jmath)+\lim_{\iota,\jmath\to\infty}\sup diam\mathcal{K}_2(\iota,\jmath)\right). \tag{19}$$

Now combining (16) with (18) and (17) with (19), respectively, we obtain

$$\begin{aligned}
\Im\left(\mathcal{O}_1\left(\mathcal{K}_1\times\mathcal{K}_2\right)\right)&\leq\left(K_1G_1+V_1Q_1\right)\left(\Im\left(\mathcal{K}_1\right)+\Im\left(\mathcal{K}_2\right)\right),\\
\Im\left(\mathcal{O}_2\left(\mathcal{K}_1\times\mathcal{K}_2\right)\right)&\leq\left(K_2G_2+V_2Q_2\right)\left(\Im\left(\mathcal{K}_1\right)+\Im\left(\mathcal{K}_2\right)\right).
\end{aligned}$$

So, the SFIEs *Eq.*(1) has at least one solution, according to Corollary 5, in the space $BC\left(\mathbb{R}^2_+\right)\times BC\left(\mathbb{R}^2_+\right)$. □

Now, we provide an example to show how our results function.

Example 5. *We analyze the system of functional-integral equations as follows:*

$$\left\{\begin{array}{ll} \varrho(\iota,\jmath)= & \cos\left(\frac{\iota\jmath}{1+\iota^3}\right)+\ln\left(1+\frac{2(1+|\varrho(\iota,\jmath)|+|\wp(\iota,\jmath)|)}{7+\iota^2\ +\jmath^2}\right)\int_0^\iota \ln\left(1+\frac{3\iota^2 e^{\upsilon}}{4(\iota^3\ +2)(3+\jmath)}\right)e^{-\upsilon}\frac{\arctan|\varrho(\upsilon,\jmath)|+\arctan|\wp(\upsilon,\jmath)|}{\pi+\upsilon^4}d\upsilon \\ & +\ln\left(3+\frac{2|\varrho(\iota,\jmath)|\cos\sqrt[3]{\jmath+\iota}+2|\wp(\iota,\jmath)|\sin\sqrt[3]{\jmath+\iota}}{5+\iota^4+\jmath^4}\right)\int_0^\jmath\int_0^\iota \frac{(1+\upsilon)(1+\nu)}{(2+\iota^2)(3+\jmath^2)}\cdot\frac{\arctan|\varrho(\upsilon,\nu)|+\arctan|\wp(\upsilon,\jmath)|}{(1+\jmath)(1+\iota)}d\upsilon d\nu \\ \wp(\iota,\jmath)= & \sin\left(\frac{\iota\jmath}{1+\jmath^3}\right)+\ln\left(1+\frac{3(1+arctg|\varrho(\iota,\jmath)|+arctg|\wp(\iota,\jmath)|)}{4+\iota^2\ +\jmath^2}\right)\int_0^\iota \ln\left(1+\frac{|\upsilon|}{\left(3+\iota^2\right)(\jmath+2)\sqrt[3]{1+\iota^2}}\right)\cdot \\ & \frac{(\sin\varrho(\upsilon,\jmath)+\cos\wp(\upsilon,\jmath))\sqrt[3]{1+\iota^2}}{2+\iota^6}d\upsilon \\ & +\ln\left(2+\frac{|\varrho(\iota,\jmath)|e^{\frac{-\iota\jmath}{2}}}{2\sqrt[5]{1+\iota\jmath}}+\frac{|\wp(\iota,\jmath)|e^{\frac{-\iota\jmath}{2}}}{2\sqrt[5]{1+\iota\jmath}}\right)\int_0^\jmath\int_0^\iota \frac{\upsilon^2\nu^2}{(4+\jmath^2)(5+\iota^2)}\cdot\frac{|\cos\varrho(\upsilon,\nu)|+|\sin\wp(\upsilon,\jmath)|}{(1+\jmath^2)(1+\iota^2)}d\upsilon d\nu. \end{array}\right. \tag{20}$$

We have $a_1(\iota,\jmath)=\cos\left(\frac{\iota\jmath}{1+\iota^3}\right)$, $a_2(\iota,\jmath)=\sin\left(\frac{\iota\jmath}{1+\jmath^3}\right)$ and these functions satisfy assumption (I) and $A_0=1$. On the other hand, the functions

$$h_1(\iota,\jmath,\varrho,\wp)=\ln\left(1+\frac{2(1+|\varrho|+|\wp|)}{7+\iota^2\ +\jmath^2}\right),$$

$$h_2(\iota,\jmath,\varrho,\wp)=\ln\left(1+\frac{3(1+arctg|\varrho|+arctg|\wp|)}{4+\iota^2\ +\jmath^2}\right),$$

satisfy the assumption (II).

Indeed,

$$\begin{aligned} &|h_1(\iota,\jmath,\varrho_2,\wp_2)-h_1(\iota,\jmath,\varrho_1,\wp_1)| \\ &\leq\left|\ln\left(1+\frac{2(1+|\varrho_2|+|\wp_2|)}{7+\iota^2\ +\jmath^2}\right)-\ln\left(1+\frac{2(1+|\varrho_1|+|\wp_1|)}{7+\iota^2\ +\jmath^2}\right)\right| \\ &\leq\left|\frac{2(|\varrho_2|+|\wp_2|)}{7+\iota^2\ +\jmath^2}-\frac{2(|\varrho_1|+|\wp_1|)}{7+\iota^2\ +\jmath^2}\right| \\ &\leq\left|\frac{2(|\varrho_2|-|\varrho_1|+|\wp_2|-|\wp_1|)}{7+\iota^2\ +\jmath^2}\right| \\ &\leq\frac{2}{7+\iota^2\ +\jmath^2}\,|||\varrho_2|-|\varrho_1||+||\wp_2|-|\wp_1||| \\ &\leq\frac{2}{7+\iota^2\ +\jmath^2}\left(|\varrho_2-\varrho_1|+|\wp_2-\wp_1|\right). \end{aligned}$$

And $H_1(\iota,\jmath)=|h_1(\iota,\jmath,0,0)\ |=\left|\ln\left(1+\frac{2}{7+\iota^2\ +\jmath^2}\right)\right|\leq\frac{2}{7+\iota^2\ +\jmath^2}$, and $\hat{H}_1=\sup_{\iota,\jmath\in\mathbb{R}_+}H_1(\iota,\jmath)=\frac{2}{7}$,

and $K_1=\sup_{\iota,\jmath\in\mathbb{R}_+}k_1(\iota,\jmath)=\sup_{\iota,\jmath\in\mathbb{R}_+}\frac{2}{7+\iota^2\ +\jmath^2}=\frac{2}{7}$.

In the same way,

$$\begin{aligned} &|h_2(\iota,\jmath,\varrho_2,\wp_2,)-h_2(\iota,\jmath,\varrho_1,\wp_1)| \\ &\leq\left|\ln\left(1+\frac{3(1+arctg|\varrho_2|+arctg|\wp_2|)}{4+\iota^2\ +\jmath^2}\right)-\ln\left(1+\frac{3(1+arctg|\varrho_1|+arctg|\wp_1|)}{4+\iota^2\ +\jmath^2}\right)\right| \\ &\leq\left|\frac{3(1+arctg|\varrho_2|+arctg|\wp_2|)}{4+\iota^2\ +\jmath^2}-\frac{3(1+arctg|\varrho_1|+arctg|\wp_1|)}{4+\iota^2\ +\jmath^2}\right| \\ &\leq\frac{3}{4+\iota^2\ +\jmath^2}\left(|arctg|\varrho_2|-arctg|\varrho_1||+|arctg|\wp_2|-arctg|\wp_1||\right) \\ &\leq\frac{3}{4+\iota^2\ +\jmath^2}\left(||\varrho_2|-|\varrho_1||+||\wp_2|-|\wp_1||\right) \\ &\leq\frac{3}{4+\iota^2\ +\jmath^2}\left(|\varrho_2-\varrho_1|+|\wp_2-\wp_1|\right). \end{aligned}$$

And $H_2(\iota,\jmath)=|h_2(\iota,\jmath,0,0)\ |=\left|\ln\left(1+\frac{3}{4+\iota^2\ +\jmath^2}\right)\right|\leq\frac{3}{4+\iota^2\ +\jmath^2}$, and $\hat{H}_2=\sup_{\iota,\jmath\in\mathbb{R}_+}H_2(\iota,\jmath)=\frac{3}{4}$,

and $K_2 = \sup_{\iota,\jmath\in\mathbb{R}_+} k_2(\iota,\jmath) = \sup_{\iota,\jmath\in\mathbb{R}_+} \frac{3}{4+\iota^2+\jmath^2} = \frac{3}{4}$.
Moreover, the following functions satisfy the assumption (IV).:

$$p_1(\iota,\jmath,\varrho,\wp) = \ln\left(3 + \frac{2|\varrho|\cos\sqrt[3]{\jmath+\iota} + 2|\wp|\sin\sqrt[3]{\jmath+\iota}}{5+\iota^4+\jmath^4}\right),$$

$$p_2(\iota,\jmath,\varrho,\wp) = \ln\left(2 + \frac{|\varrho|\, e^{\frac{-\iota\jmath}{2}}}{2\sqrt[5]{1+\iota\jmath}} + \frac{|\wp|\, e^{\frac{-\iota\jmath}{2}}}{2\sqrt[5]{1+\iota\jmath}}\right).$$

Indeed,

$$\begin{aligned}
&|p_1(\iota,\jmath,\varrho_2,\wp_2) - p_1(\iota,\jmath,\varrho_1,\wp_1)| \\
&\le \left|\ln\left(3 + \frac{2|\varrho_2|\cos\sqrt[3]{\jmath+\iota} + 2|\wp_2|\sin\sqrt[3]{\jmath+\iota}}{5+\iota^4+\jmath^4}\right) - \ln\left(3 + \frac{2|\varrho_1|\cos\sqrt[3]{\jmath+\iota} + 2|\wp_1|\sin\sqrt[3]{\jmath+\iota}}{5+\iota^4+\jmath^4}\right)\right| \\
&\le \left|\frac{2|\varrho_2|\cos\sqrt[3]{\jmath+\iota} + 2|\wp_2|\sin\sqrt[3]{\jmath+\iota}}{5+\iota^4+\jmath^4} - \frac{2|\varrho_1|\cos\sqrt[3]{\jmath+\iota} + 2|\wp_1|\sin\sqrt[3]{\jmath+\iota}}{5+\iota^4+\jmath^4}\right| \\
&\le \left|\left(\frac{2|\varrho_2| - 2|\varrho_1|}{5+\iota^4+\jmath^4}\right)\cos\sqrt[3]{\jmath+\iota} + \left(\frac{2|\wp_2| - 2|\wp_1|}{5+\iota^4+\jmath^4}\right)\sin\sqrt[3]{\jmath+\iota}\right| \\
&\le \frac{2}{5+\iota^4+\jmath^4}\left(||\varrho_2| - |\varrho_1|| + ||\wp_2| - |\wp_1||\right) \\
&\le \frac{2}{5+\iota^4+\jmath^4}\left(|\varrho_2 - \varrho_1| + |\wp_2 - \wp_1|\right).
\end{aligned}$$

And $U_1(\iota,\jmath) = |p_1(\iota,\jmath,0,0)| = \ln 3$, and $\hat{U}_1 = \ln 3$, and $V_1 = \sup\limits_{\iota,\jmath\in\mathbb{R}_+} \nu_1(\iota,\jmath) = \sup\limits_{\iota,\jmath\in\mathbb{R}_+} \frac{2}{5+\iota^4+\jmath^4} = \frac{2}{5}$.
Moreover,

$$\begin{aligned}
|p_2(\iota,\jmath,\varrho_2,\wp_2) - p_2(\iota,\jmath,\varrho_1,\wp_1)| &= \left|\ln\left(2 + \frac{|\varrho_2|\, e^{\frac{-\iota\jmath}{2}}}{2\sqrt[5]{1+\iota\jmath}} + \frac{|\wp_2|\, e^{\frac{-\iota\jmath}{2}}}{2\sqrt[5]{1+\iota\jmath}}\right) - \ln\left(2 + \frac{|\varrho_1|\, e^{\frac{-\iota\jmath}{2}}}{2\sqrt[5]{1+\iota\jmath}} + \frac{|\wp_1|\, e^{\frac{-\iota\jmath}{2}}}{2\sqrt[5]{1+\iota\jmath}}\right)\right| \\
&\le \left|\left(\frac{|\varrho_2|\, e^{\frac{-\iota\jmath}{2}}}{2\sqrt[5]{1+\iota\jmath}} + \frac{|\wp_2|\, e^{\frac{-\iota\jmath}{2}}}{\sqrt[5]{1+\iota\jmath}}\right) - \left(\frac{|\varrho_1|\, e^{\frac{-\iota\jmath}{2}}}{2\sqrt[5]{1+\iota\jmath}} + \frac{|\wp_1|\, e^{\frac{-\iota\jmath}{2}}}{2\sqrt[5]{1+\iota\jmath}}\right)\right| \\
&\le \frac{e^{\frac{-\iota\jmath}{2}}}{2\sqrt[5]{1+\iota\jmath}}\left(||\varrho_2| - |\varrho_1|| + ||\wp_2| - |\wp_1||\right) \\
&\le \frac{e^{\frac{-\iota\jmath}{2}}}{2\sqrt[5]{1+\iota\jmath}}\left(|\varrho_2 - \varrho_1| + |\wp_2 - \wp_1|\right).
\end{aligned}$$

And $U_2(\iota,\jmath) = |p_2(\iota,\jmath,0,0)| = \ln 2$, and $\hat{U}_2 = \ln 2$, and $V_2 = \sup\limits_{\iota,\jmath\in\mathbb{R}_+} \nu_2(\iota,\jmath) = \sup\limits_{\iota,\jmath\in\mathbb{R}_+} \frac{e^{\frac{-\iota\jmath}{2}}}{2\sqrt[5]{1+\iota\jmath}} = \frac{1}{2}$.
In the following, we will introduce other functions in this way:

$$f_1(\iota,\varrho,\wp) = e^{-\iota}\frac{(\arctan|\varrho| + \arctan|\wp|)}{\pi+\iota^4}, \quad |f_1(\iota,\varrho,\wp)| \le m_1(\iota) = e^{-\iota},$$

and

$$\begin{aligned}
|f_1(\iota,\varrho_2,\wp_2) - f_1(\iota,\varrho_1,\wp_1)| &\le \frac{e^{-\iota}}{\pi+\iota^4}\left(|arctg\,|\varrho_2| - arctg\,|\varrho_1|| + |arctg\,|\wp_2| - arctg\,|\wp_1||\right) \\
&\le e^{-\iota}\left(|\varrho_2 - \varrho_1| + |\wp_2 - \wp_1|\right),
\end{aligned}$$

and

$$f_2(\iota,\varrho,\wp)=\frac{(\sin\varrho+\cos\wp)\sqrt[3]{1+\iota^2}}{2+\iota^6},\quad |f_2(\iota,\varrho,\wp)|\le m_2(\iota)=\sqrt[3]{1+\iota^2},$$

and

$$\begin{aligned}|f_2(\iota,\varrho_2,\wp_2)-f_2(\iota,\varrho_1,\wp_1)|&\le\frac{\sqrt[3]{1+\iota^2}}{2+\iota^6}\left(|\sin\varrho_2-\sin\varrho_1|+|\cos\wp_2-\cos\wp_1|\right)\\&\le\sqrt[3]{1+\iota^2}\left(|\varrho_2-\varrho_1|+|\wp_2-\wp_1|\right).\end{aligned}$$

Also

$$\begin{aligned}g_1(\iota,\jmath,\upsilon)&=\ln\left(1+\frac{3\iota^2e^{\upsilon}}{4(\iota^3\ +2)(3+\jmath)}\right),\\g_2(\iota,\jmath,\upsilon)&=\ln\left(1+\frac{|\upsilon|}{\left(3+\iota^2\right)(\jmath+2)\sqrt[3]{1+\iota^2}}\right),\end{aligned}$$

satisfy assumptions (VI) and $(VIII)$. Indeed,

$$\begin{aligned}\int_0^{\iota}|g_1(\iota,\jmath,\upsilon)|\,m_1(\upsilon)\,d\upsilon&=\int_0^{\iota}\ln\left(1+\frac{3\iota^2e^{\upsilon}}{4(\iota^3\ +2)(3+\jmath)}\right)e^{-\upsilon}d\upsilon\\&\le\int_0^{\iota}\frac{3\iota^2e^{\upsilon}}{4(\iota^3\ +2)(3+\jmath)}e^{-\upsilon}d\upsilon\\&\le\frac{3\iota^3}{4(\iota^3\ +2)(3+\jmath)}.\end{aligned}$$

Hence

$$\begin{gathered}\sup_{\iota,\jmath\in\mathbb{R}_+}\left\{\int_0^{\iota}|g_1(\iota,\jmath,\upsilon)|\,m_1(\upsilon)\,d\upsilon\right\}\le G_1=\frac{1}{4},\\\lim_{\iota,\jmath\to\infty}\int_0^{\iota}|g_1(\iota,\jmath,\upsilon)|\,m_1(\upsilon)\,d\upsilon=0.\end{gathered}$$

And

$$\begin{aligned}\int_0^{\iota}|g_2(\iota,\jmath,\upsilon)|\,m_2(\upsilon)\,d\upsilon&=\int_0^{\iota}\ln\left(1+\frac{|\upsilon|}{\left(3+\iota^2\right)(\jmath+2)\sqrt[3]{1+\iota^2}}\right)\sqrt[3]{1+\upsilon^2}\,d\upsilon\\&\le\int_0^{\iota}\frac{\upsilon\sqrt[3]{1+\upsilon^2}}{\left(3+\iota^2\right)(\jmath+2)\sqrt[3]{1+\iota^2}}d\upsilon\\&\le\frac{3\left(\left(\iota^2+1\right)\sqrt[3]{\left(1+\iota^2\right)}-1\right)}{8\left(3+\iota^2\right)(\jmath+2)\sqrt[3]{1+\iota^2}}\\&\le\frac{3\left(\iota^2+1\right)\sqrt[3]{1+\iota^2}}{8\left(3+\iota^2\right)(\jmath+2)\sqrt[3]{1+\iota^2}}\\&\le\frac{3\left(\iota^2+1\right)}{8\left(3+\iota^2\right)(\jmath+2)}.\end{aligned}$$

Hence

$$\sup_{\iota,\jmath\in\mathbb{R}_+}\left\{\int_0^{\iota}\left|g_2\left(\iota,\jmath,\upsilon\right)\right|m_2\left(\upsilon\right)d\upsilon\right\}=G_2 \ =\frac{3}{16}.$$
$$\lim_{\iota,\jmath\to\infty}\int_0^{\iota}\left|g_2\left(\iota,\jmath,\upsilon\right)\right|m_2\left(\upsilon\right)d\upsilon=0.$$

In the following, these functions

$$r_1\left(\iota,\jmath,\varrho,\wp\right)=\frac{\arctan\left|\varrho\right|+\arctan\left|\wp\right|}{\left(1+\jmath\right)\left(1+\iota\right)},$$
$$\left|r_1\left(\iota,\jmath,\varrho,\wp\right)\right|\le\frac{\pi}{\left(1+\jmath\right)\left(1+\iota\right)}=n_1\left(\iota,\jmath\right),$$

and

$$\begin{aligned}&\left|r_1\left(\iota,\jmath,\varrho_2,\wp_2\right)-r_1\left(\iota,\jmath,\varrho_1,\wp_1\right)\right|\\&\le\frac{\left|\arctan\left|\varrho_2\right|-\arctan\left|\varrho_1\right|\right|+\left|\arctan\left|\wp_2\right|-\arctan\left|\wp_1\right|\right|}{\left(1+\jmath\right)\left(1+\iota\right)}\\&\le\frac{\left|\varrho_2-\varrho_1\right|+\left|\wp_2-\wp_1\right|}{\left(1+\jmath\right)\left(1+\iota\right)}\\&\le n_1\ \left(\iota,\jmath\right)\left(\left|\varrho_2-\varrho_1\right|+\left|\wp_2-\wp_1\right|\right),\end{aligned}$$

and

$$r_2\left(\iota,\jmath,\varrho,\wp\right)=\frac{\left|\cos\varrho\right|+\left|\sin\wp\right|}{\left(1+\jmath^2\right)\left(1+\iota^2\right)},$$
$$\left|r_2\left(\iota,\jmath,\varrho,\wp\right)\right|\le\frac{2}{\left(1+\jmath^2\right)\left(1+\iota^2\right)}=n_2\ \left(\iota,\jmath\right),$$

and

$$\begin{aligned}&\left|r_2\left(\iota,\jmath,\varrho_2,\wp_2\right)-r_2\left(\iota,\jmath,\varrho_1,\wp_1\right)\right|\\&\le\frac{\left|\left|\cos\varrho_2\right|-\left|\cos\varrho_1\right|\right|+\left|\left|\sin\wp_2\right|-\left|\sin\wp_1\right|\right|}{\left(1+\jmath^2\right)\left(1+\iota^2\right)}\\&\le\frac{\left|\varrho_2-\varrho_1\right|+\left|\wp_2-\wp_1\right|}{\left(1+\jmath^2\right)\left(1+\iota^2\right)}\\&\le n_2\ \left(\iota,\jmath\right)\left(\left|\varrho_2-\varrho_1\right|+\left|\wp_2-\wp_1\right|\right),\end{aligned}$$

satisfy the assumption (VII). Also the following functions,

$$q_1\left(\iota,\jmath,\upsilon,\nu\right)=\frac{\left(1+\upsilon\right)\left(1+\nu\right)}{\left(2+\iota^2\right)\left(3+\jmath^2\right)},$$
$$q_2\left(\iota,\jmath,\upsilon,\nu\right)=\frac{\upsilon^2\nu^2}{\left(4+\jmath^2\right)\left(5+\iota^2\right)},$$

satisfy the assumption (IX), we have,

$$\begin{aligned}\int_0^{\jmath}\int_0^{\iota}\left|q_1\left(\iota,\jmath,\upsilon,\nu\right)\right|n_1\ \left(\upsilon,\nu\right)d\upsilon d\nu&\le\int_0^{\jmath}\int_0^{\iota}\frac{\left(1+\upsilon\right)\left(1+\nu\right)}{\left(2+\iota^2\right)\left(3+\jmath^2\right)}\frac{\pi}{\left(1+\upsilon\right)\left(1+\nu\right)}d\upsilon d\nu\\&\le\frac{\pi}{\left(2+\iota^2\right)\left(3+\jmath^2\right)}\int_0^{\jmath}\int_0^{\iota}d\upsilon d\nu\\&\le\frac{\pi\jmath\iota}{\left(2+\iota^2\right)\left(3+\jmath^2\right)}.\end{aligned}$$

Hence

$$\sup_{\iota,\jmath\in\mathbb{R}_+}\left\{\int_0^{\jmath}\int_0^{\iota}|q_1(\iota,\jmath,\upsilon,\nu)|\,n_1\ (\upsilon,\nu)\,d\upsilon d\nu\right\}\leq Q_1=\frac{\pi}{4\sqrt{6}},$$
$$\lim_{\iota,\jmath\to\infty}\int_0^{\jmath}\int_0^{\iota}|q_1(\iota,\jmath,\upsilon,\nu)|\,n_1(\upsilon,\nu)\,d\upsilon d\nu=0.$$

And

$$\begin{aligned}\int_0^{\jmath}\int_0^{\iota}|q_2(\iota,\jmath,\upsilon,\nu)|\,n_2\ (\upsilon,\nu)\,d\upsilon d\nu &\leq \int_0^{\jmath}\int_0^{\iota}\frac{\upsilon^2\nu^2}{(4+\jmath^2)(5+\iota^2)}.\frac{2}{(1+\upsilon^2)(1+\nu^2)}d\upsilon d\nu\\ &\leq \frac{2}{(4+\jmath^2)(5+\iota^2)}\int_0^{\jmath}\int_0^{\iota}d\upsilon d\nu\\ &\leq \frac{2\jmath\iota}{(4+\jmath^2)(5+\iota^2)}.\end{aligned}$$

Hence

$$\sup_{\iota,\jmath\in\mathbb{R}_+}\left\{\int_0^{\jmath}\int_0^{\iota}|q_2(\iota,\jmath,\upsilon,\nu)|\,n_2\ (\upsilon,\nu)\,d\upsilon d\nu\right\}\leq Q_2\ =\frac{1}{4\sqrt{5}},$$
$$\lim_{\iota,\jmath\to\infty}\int_0^{\jmath}\int_0^{\iota}|q_2(\iota,\jmath,\upsilon,\nu)|\,n_2(\upsilon,\nu)\,d\upsilon d\nu=0.$$

According to the above calculations, we will have:

$$\begin{aligned}\widetilde{H}_1&=\max\left\{\hat{H}_1,\hat{U}_1\right\}=\max\left\{\frac{2}{7},\ln 3\right\}=\ln 3,\\ \widetilde{H}_2&=\max\left\{\hat{H}_2,\hat{U}_2\right\}=\max\left\{\frac{3}{4},\ln 2\right\}=\frac{3}{4},\\ \widetilde{K}_1&=\max\{K_1,V_1\}=\max\left\{\frac{2}{7},\frac{2}{5}\right\}=\frac{2}{5},\\ \widetilde{K}_2&=\max\{K_2,V_2\}=\max\left\{\frac{3}{4},\frac{1}{2}\right\}=\frac{3}{4}.\end{aligned}$$

And

$$\begin{aligned}K_1G_1+V_1Q_1&=\frac{2}{7}\times\frac{1}{4}+\frac{2}{5}\times\frac{\pi}{4\sqrt{6}}\simeq 0.199\ <\frac{1}{2},\\ K_2G_2+V_2Q_2&=\frac{3}{4}\times\frac{3}{16}+\frac{1}{2}\times\frac{1}{4\sqrt{5}}\simeq 0.196\ <\frac{1}{2}.\end{aligned}$$

And

$$\begin{aligned}\widetilde{K}_1(G_1+Q_1)&=\frac{2}{5}\left(\frac{1}{4}+\frac{\pi}{4\sqrt{6}}\right)\simeq 0.228\ <\frac{1}{2},\\ \widetilde{K}_2(G_2+Q_2)&=\frac{3}{4}\left(\frac{3}{16}+\frac{1}{4\sqrt{5}}\right)\simeq 0.224\ <\frac{1}{2}.\end{aligned}$$

Consequently, by using Theorem 9, we deduce that the system of functional-integral equations 20 has at least one solution $\varrho=\varrho(\iota,\jmath)$, $\wp=\wp(\iota,\jmath)$ in the space $\mathbb{B}=BC\left(\mathbb{R}_+^2\right)$.

5 Conclusion

We present some new versions of Darbo's fixed point theorem in m-dimensional spaces using the Cartesian product measure of noncompactness. Using these results, we show that it is possible to investigate the solvability of classes of the system of implicit integral equations in which the dependent operators in these equations include m-operators with m-variables and m-multiple integrals. Another novelty aspect of this study is that different forms of Darbo's theorem can be obtained with various classes of functions in higher-dimensional spaces. Our study encourages the authors to investigate the possible solvability of systems of nonlinear fractional differential equations of order α in m-dimensional spaces.

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