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TRAVELLING WAVE SOLUTIONS AND CONSERVED VECTORS FOR THE KAWAHARA EQUATION WITH DUAL POWER-LAW NONLINEARITIES

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Abstract. The present study aims to propose and thoroughly investigate the Kawahara equation with dual power-law nonlinearities, which is a fifth-order nonlinear partial differential equations that models complex wave phenomena in various scientific disciplines, including fluid dynamics, plasma physics, and nonlinear acoustics. The study employs Lie group analysis in search for exact solutions of the equation. Through this investigation, the Lie point symmetries admitted by the Kawahara equation are identified, exhibiting two translation symmetries. The combination of these symmetries allows the construction of travelling wave solutions. By using these two symmetries, the original equation is reduced to a nonlinear ordinary differential equation (NODE). This NODE is then solved using special analytical methods, including Kudryashov's method and the extended Jacobi elliptic function expansion method, both of which are well-suited for obtaining exact and periodic solutions. To illustrate the physical behaviour of the obtained solutions, 3D, 2D, and density plots are presented. Finally, conserved vectors of the equation are derived using the multiplier technique and Noether's theorem.

Keywords: Kawahara equation, dual power-law nonlinearities, Lie point symmetries, exact solutions, conserved vectors.

AMS Subject Classification: 35B06, 35L65, 37J15, 37K05.

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1 Introduction

In recent years, considerable attention has been dedicated to nonlinear partial differential equations (NLPDEs) due to their wide-ranging applications in various fields, including acoustic wave propagation, nonlinear dynamics, condensed matter physics, optical fiber communication, chemical reaction dynamics, plasma physics, biological membrane modeling, and electromagnetic wave theory Chatterjeeet al. (2025); Wazwaz (2009). Significant progress has recently been made in the study of NLPDEs across these fields, with the aim of gaining deeper insight into their behaviour and modeling complex real-world phenomena Hafiz et al. (2024). A notable example of such NLPDEs is the Boussinesq–Burgers-type system, which incorporates both nonlinear and dispersive effects, as well as viscous dissipation. It is widely used to model the propagation of long, weakly nonlinear shallow water waves near coastal regions and lakes, particularly where wave breaking, bottom friction, or turbulence plays a significant role Gao et al. (2024).

Due to their essential role in modeling nonlinear phenomena, finding exact and traveling wave solutions to NLPDEs remains a significant challenge and an active area of research. Researchers have introduced a variety of techniques to effectively obtain exact explicit solutions. These techniques include: literature. These include the novel computational algorithm for solving PDEs Mutashar et al. (2025), the (G'/G)-expansion method Wang et al. (2005), the Kudyashov's method (Kudryashov (2012)), the simplest equation method Kudryashov (2005), the extended simplest equation method Kudryashov & Loguinova (2008), the $(m+(1/G'))$-expansion method Koc et al. (2024), Hirota's bilinear approach Li et al. (2019), the tanh-coth technique Wazwaz (2007), the rational expansion approach Zeng & Wang (2009), the bifurcation technique Zhang & Khalique (2018), the extended homoclinic test method Darvishi & Najafi (2018), the F-expansion approach Zhou et al. (2003), sine-Gordon equation expansion method Chen & Yan (2005), Painlevé expansion method Weiss et al. (1985), Darboux transformation approach Zhang et al. (2020), Lie group analysis Ovsiannikov (1982); Olver (1993); Ibragimov (1999), and so on.

The Lie group analysis method is a powerful technique for deriving exact solutions of NLPDEs Olver (1993); Ovsiannikov (1982). It was developed by the Norwegian mathematician Sophus Lie in the 19th century. The method plays a critical role in oversimplifying and solving complex mathematical models of science, engineering, and other fields. The first step is to compute the Lie symmetries of the given differential equation (DE). This is usually done by finding a Lie group that acts on the independent and dependent variables of the DE. Once the symmetries are calculated, they can then be used to reduce the number of variables in the DE, making the problem easier to solve. This leads to the construction of invariant solutions or a reduction to a simpler form of DE.

Moreover, conservation laws are recognized as fundamental natural principles that have been extensively studied across diverse scientific disciplines Serre (1999); Leveque (1992); Bodibe & Khalique (2024). These laws include the well-known laws: conservation of momentum, energy, and mass. Conservation laws play a crucial role in analyzing solution stability, examining the behaviour of solutions, and exploring integrability of NLPDEs. Various approaches have been developed to derive conservation laws. The celebrated Noether's theorem Noether (1918) is used to construct conservation laws of DEs with a variational principle. It provides a formula for obtaining the conservation laws by use of symmetries of the action. For DE that do not have a variational principle several methods have been developed recently by the researchers. These include, the multiplier method Olver (1993), partial Lagrangian method Kara & Mahomed (2006), and theorem due to Ibragimov Ibragimov (2007, 2006).

One of the well studied NLPDE is the Kawahara equation (also referred as fifth-order KdV equation)

$$u_t + \delta u u_x + \beta u_{xxx} - u_{xxxxx} = 0 \tag{1}$$

with δ and β being constants. This equation describes the solitary-wave propagation in the media and was first proposed by Kawahara in 1972 Kawahara (1972). Equation (1) was studied numerically by Kawahara and he discovered that it has both oscillatory and monotone solitary wave solutions. Because of the inclusion of the fifth-order dispersive term u_{xxxxx}, equation (1) exhibits weaker dispersion than it occurs for the Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{2}$$

Kawahara also proposed the modified Kawahara equation

$$u_t + \delta u^2 u_x + \beta u_{xxx} - u_{xxxxx} = 0, \tag{3}$$

which arises in the context of shallow water waves. It is also called the singularly perturbed KdV equation and appears in the theory of magnetoacoustic waves in plasmas. Another form of the modified Kawahara equation that has been studied in the literature is

$$u_t + \gamma u_x + \delta u^2 u_x + \beta u_{xxx} - u_{xxxxx} = 0, \tag{4}$$

where γ, δ and β are nonzero arbitrary constants Kwak (2020, 2021). In Kwak (2020), the author examined the Cauchy problem for (4) on one-dimensional torus $\mathbb{T}$, which models capillary-gravity waves in long-wave regimes and established global well-posedness in $L^2(\mathbb{T})$, marking the first such low-regularity result for this equation. This was achieved by adapting Takaoka-Tsutsumi methods to attenuate resonant cubic effects and by leveraging the conservation of the L^2-norm. Additionally, unconditional uniqueness was proved for solutions in Sobolev spaces $H^s(\mathbb{T})$ for $s > \frac{1}{2}$, while for lower regularity $s \leq \frac{1}{2}$, the equation was shown to be weakly ill-posed, as the associated flow map failed to be uniformly continuous. On the other hand, in Kwak (2021) numerical solution of (4) was obtained by employing two efficient methods together. Firstly, Crank-Nicolson discretization algorithm for time integration was used followed by the fifth-order quintic B-spline based differential quadrature method for space integration. Many researchers have studied Kawahara type equations and found their analytical and numerical solutions. See for example, Zara et al. (2022); Hunter & Scheule (1988); Boyd (1991); Sirendaoreji (2004); Wazwaz (2006); Yusufoglu et al. (2008) and references therein.

Recently, two special Kawahara equations, namely

$$u_t + \alpha(2u_x u_{xx} + u u_{xxx}) + \beta u^m u_x + u_{xxxxx} = 0, \tag{5}$$

and

$$u_t + \alpha \left\{(p+1)u^{p-1}u_x u_{xx} + u^p u_{xxx}\right\} + \beta u^m u_x + u_{xxxxx} = 0, \tag{6}$$

where α and β are nonzero constants, and $m, p \geq 2$ are two positive integers, were studied in Wang (2025). Sine-cosine method was used to derive the solitary wave solutions for (5) when $m = 2$ and for (6) when $p = m \geq 2$. Moreover, (5) was solved using a three-level linearized finite difference scheme with fourth-order accuracy together with the initial boundary conditions, whereas (6) was solved by employing a three-level linearized finite difference scheme with second-order accuracy and the initial boundary conditions.

In this study, we consider the generalization of the Kawahara and modified Kawahara equations, which we may call Kawahara equation with dual power-law nonlinearities (KEdpn) that is given by

$$u_t + \gamma u_x + \delta u^n u_x + \mu u^{2n} u_x + \beta u_{xxx} - u_{xxxxx} = 0, \tag{7}$$

where $u(t, x)$ represents the wave profile, γ the strength of advection, δ and μ measures the strengths of the two nonlinear terms, n signifies the power-law nonlinearity parameter, and β is the coefficient of the dispersive term. Firstly, we construct the travelling wave solutions of (7) using Lie symmetry theory along with various other solution techniques. Secondly, conservation laws are derived by the application of two different approaches.

The paper is arranged as follows. In Section 2, we begin by performing Lie group analysis of the KEdpn equation (7) and compute its Lie point symmetries. Thereafter, using the symmetries, equation (7) is transformed to a nonlinear ordinary differential equation (NLODE), which is then solved by various methods such as, Kudryashov's and the extended Jacobi elliptic function expansion methods, to construct exact solutions. In Section 3, the conservation laws of the KEdpn equation (7) are constructed using the multiplier technique and Noether's method. Finally, Section 4 presents the concluding remarks.

2 Travelling wave solutions of (7)

We begin by determining the Lie point symmetries of the KEdpn (7). These will be used in finding the travelling wave solutions and later when deriving the conserved vectors by invoking

Ibragimov's theorem. The vector field $\mathcal{Z}$ given by

$$\mathcal{Z} = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}$$

is a Lie point symmetry of the KEdpn (7) if and only if the invariance condition

$$\mathcal{Z}^{[5]}\left(u_t + \gamma u_x + \delta u^n u_x + \mu u^{2n} u_x + \beta u_{xxx} - u_{xxxxx}\right)|_{(7)} = 0 \tag{8}$$

holds. Here, $\mathcal{Z}^{[5]}$ denotes the fifth prolongation of $\mathcal{Z}$, which is given by

$$\mathcal{Z}^{[5]} = \mathcal{Z} + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} + \zeta_{xxx}\frac{\partial}{\partial u_{xxx}} + \zeta_{xxxx}\frac{\partial}{\partial u_{xxxx}} + \zeta_{xxxxx}\frac{\partial}{\partial u_{xxxxx}}, \tag{9}$$

where $\zeta' s$ are defined as

$$\zeta_{j_1,j_2,\dots,j_k} = D_{j_k}\left(\zeta_{j_1,j_2,\dots,j_{k-1}}\right) - u_{j_1,j_2,\dots,j_{k-1}n}D_{j_k}\left(\xi^n\right), \text{ (sum on } n) \tag{10}$$

and the total differential operators D_t and D_x are defined as

$$\begin{aligned} D_t =& \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + u_{ttt}\frac{\partial}{\partial u_{tt}} + \cdots, \\ D_x =& \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_t} + u_{xx}\frac{\partial}{\partial u_x} + u_{xxx}\frac{\partial}{\partial u_{xx}} + \cdots. \end{aligned} \tag{11}$$

Expanding (8) and splitting the equation based on the derivatives of u, we obtain seven overdetermined system of linear homogeneous PDEs, expressed as

$$\xi^1_t = 0,\ \xi^1_x = 0,\ \ \xi^1_u = 0,\ \ \xi^2_x = 0,\ \xi^2_u = 0,\ \xi^3_u = 0,\ \eta = 0.$$

Solving the above system of PDEs, we obtain the values of the infinitesimals as

$$\xi^1 = C_1,\ \xi^2 = C_2,\ \eta = 0.$$

Thus, we conclude that the KEdpn (7) admits two translation symmetries, given by

$$\begin{aligned} \mathcal{Z}_1 =& \frac{\partial}{\partial t},\ \text{time translation}, \\ \mathcal{Z}_2 =& \frac{\partial}{\partial x},\ \text{space translation}. \end{aligned} \tag{12}$$

We now use these Lie point symmetries (12) to perform symmetry reduction and determine travelling wave solutions of the KEdpn (7). The symmetry reduction will lead us to a NLODE, which is then solved by using different techniques to obtain travelling wave solutions of the KEdpn (7).

Travelling wave solutions using $\mathcal{Z}_1$ and $\mathcal{Z}_2$

Consider the linear combination of the two translation symmetries $\mathcal{Z}_1$ and $\mathcal{Z}_2$, namely $\mathcal{Z} = \mathcal{Z}_1 + c\mathcal{Z}_2$, where c is a constant. The group-invariant solution under $\mathcal{Z}$ will provide us with the travelling wave solutions to the KEdpn (7). Solving the associated Lagrangian system yields the two invariants

$$\epsilon = x - ct,\ \ \Psi(\epsilon) = u.$$

Using these invariants, the KEdpn (7) transforms into a fifth-order NLODE

$$(\gamma - c)\Psi_\epsilon + \delta\Psi^n\Psi_\epsilon + \mu\Psi^{2n}\Psi_\epsilon + \beta\Psi_{\epsilon\epsilon\epsilon} - \Psi_{\epsilon\epsilon\epsilon\epsilon\epsilon} = 0. \tag{13}$$

Solution of (13) by means of Kudryashov's method

We apply Kudryashov's method Kudryashov (2012) to compute an exact solution of (13) when $n = 1$. We assume that the solution of (13) takes the form

$$\Psi(\epsilon) = \sum_{i=0}^{M} \mathcal{A}_i B^i(\epsilon), \tag{14}$$

where M represents a positive integer to be determined via the balancing procedure and $B(\epsilon)$ solves the Riccati equation

$$B'(\epsilon) = B^2(\epsilon) - B(\epsilon), \tag{15}$$

whose exact solution is given by

$$B(\epsilon) = \frac{1}{1 + \exp(\epsilon)}. \tag{16}$$

Utilizing the balancing procedure method on the NLODE (13), that is by equating the highest nonlinear term with the highest-order derivative term, we are able to determine M. Thus, in this case, we get $M = 2$. Consequently, the solution (14) is expressed as

$$\Psi(\epsilon) = \mathcal{A}_0 + \mathcal{A}_1 B(\epsilon) + \mathcal{A}_2 B(\epsilon)^2. \tag{17}$$

Substituting (17) into (13), invoking (15), then equating the coefficients of like powers of $B(\epsilon)$ to zero, yields the following algebraic system of equations in $\mathcal{A}_0, \mathcal{A}_1$, and $\mathcal{A}_2$:

$$\begin{aligned}
&\mu\,{\mathcal{A}_2}^2 - 360 = 0,\\
&5\,\mu\,\mathcal{A}_1{\mathcal{A}_2}^2 - 2\,\mu\,{\mathcal{A}_2}^3 - 120\,\mathcal{A}_1 + 2400\,\mathcal{A}_2 = 0,\\
&\mathcal{A}_1 - \mu\,{\mathcal{A}_0}^2\mathcal{A}_1 - \delta\,\mathcal{A}_0\mathcal{A}_1 - \beta\,\mathcal{A}_1 + c\mathcal{A}_1 - \gamma\,\mathcal{A}_1 = 0,\\
&4\,\mu\,\mathcal{A}_0{\mathcal{A}_2}^2 + 4\,\mu\,{\mathcal{A}_1}^2\mathcal{A}_2 - 5\,\mu\,\mathcal{A}_1{\mathcal{A}_2}^2 + 2\,\delta\,{\mathcal{A}_2}^2 + 24\,\beta\,\mathcal{A}_2 + 360\,\mathcal{A}_1 - 3000\,\mathcal{A}_2 = 0,\\
&6\,\mu\,\mathcal{A}_0\mathcal{A}_1\mathcal{A}_2 - 4\,\mu\,\mathcal{A}_0{\mathcal{A}_2}^2 + \mu\,{\mathcal{A}_1}^3 - 4\,\mu\,{\mathcal{A}_1}^2\mathcal{A}_2 + 3\,\delta\,\mathcal{A}_1\mathcal{A}_2 - 2\,\delta\,{\mathcal{A}_2}^2 + 6\,\beta\,\mathcal{A}_1 - 54\,\beta\,\mathcal{A}_2\\
&\quad - 390\,\mathcal{A}_1 + 1710\,\mathcal{A}_2 = 0,\\
&2\,\mu\,{\mathcal{A}_0}^2\mathcal{A}_2 + 2\,\mu\,\mathcal{A}_0{\mathcal{A}_1}^2 - 6\,\mu\,\mathcal{A}_0\mathcal{A}_1\mathcal{A}_2 - \mu\,{\mathcal{A}_1}^3 + 2\,\delta\,\mathcal{A}_0\mathcal{A}_2 + \delta\,{\mathcal{A}_1}^2 - 3\,\delta\,\mathcal{A}_1\mathcal{A}_2 - 12\,\beta\,\mathcal{A}_1\\
&\quad + 38\,\beta\,\mathcal{A}_2 - 2\,c\mathcal{A}_2 + 2\,\gamma\,\mathcal{A}_2 + 180\,\mathcal{A}_1 - 422\,\mathcal{A}_2 = 0.\\
&\mu\,{\mathcal{A}_0}^2\mathcal{A}_1 - 2\,\mu\,{\mathcal{A}_0}^2\mathcal{A}_2 - 2\,\mu\,\mathcal{A}_0{\mathcal{A}_1}^2 + \delta\,\mathcal{A}_0\mathcal{A}_1 - 2\,\delta\,\mathcal{A}_0\mathcal{A}_2 - \delta\,{\mathcal{A}_1}^2 + 7\,\beta\,\mathcal{A}_1 - 8\,\beta\,\mathcal{A}_2 - c\mathcal{A}_1\\
&\quad + 2\,c\mathcal{A}_2 + \gamma\,\mathcal{A}_1 - 2\,\gamma\,\mathcal{A}_2 - 31\,\mathcal{A}_1 + 32\,\mathcal{A}_2 = 0.
\end{aligned}$$

Solving the above system of equations using Maple, we obtain

$$\begin{aligned}
\mathcal{A}_0 &= \frac{\sqrt{10}\left(10\,\mu - 2\mu\beta - \delta\sqrt{10\mu}\right)}{20\,\sqrt[3]{\mu}},\ \mathcal{A}_1 = -\frac{6\sqrt{10}}{\sqrt{\mu}},\ \mathcal{A}_2 = \frac{6\sqrt{10}}{\sqrt{\mu}},\\
c &= \frac{2\beta^2\mu - 5\delta^2 + 20\gamma\mu + 30\mu}{20\mu}.
\end{aligned}$$

Thus,

$$\Psi(\epsilon) = \frac{\sqrt{10}\left(10\mu - 2\mu\beta - \delta\sqrt{10\mu}\right)}{20\,\sqrt[3]{\mu}} - \frac{6\sqrt{10}}{\sqrt{\mu}}\left\{\left(\frac{1}{1+\exp(\epsilon)}\right) - \left(\frac{1}{1+\exp(\epsilon)}\right)^2\right\}$$

and consequently, the solution of the KEdpn (7) is given by

$$u(t,x) = \frac{\sqrt{10}\left(10\mu - 2\mu\beta - \delta\sqrt{10\mu}\right)}{20\,\sqrt[3]{\mu}} - \frac{6\sqrt{10}}{\sqrt{\mu}}\left\{\left(\frac{1}{1+\exp(\epsilon)}\right) - \left(\frac{1}{1+\exp(\epsilon)}\right)^2\right\}, \tag{18}$$

where $\epsilon = x - ct$. The profile of the solution (18) is depicted in Figure 1.

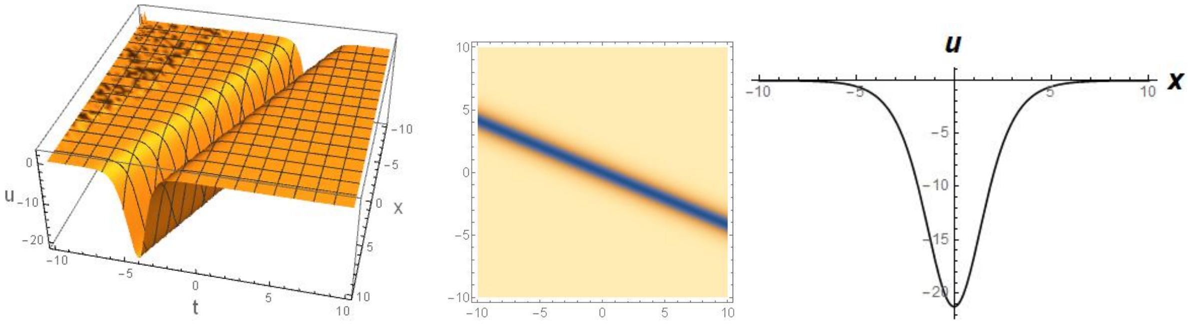

Figure 1: The 3D and 2D profile of solution (18), when $\beta = \delta = \gamma = 1, \mu = 0.05$, $-10 \leq t, x \leq 10$.

Solutions of (13) using extended Jacobi elliptic function expansion method

We now apply the extended Jacobi elliptic function expansion method Farooq et al. (2024) to derive exact solutions of the KEdpn (7). It is assumed that the solutions of the fifth-order NLODE (13) can be represented in the form

$$\Psi(\epsilon) = \sum_{i=-M}^{M} \mathcal{C}_i F(\epsilon)^i, \tag{19}$$

where M represents a positive integer obtained via the balancing procedure, and $F(\epsilon)$ satisfies the following first-order ODEs:

$$F'(\epsilon) = -\sqrt{(1 - F^2(\epsilon))(1 - \omega + \omega F^2(\epsilon))}, \tag{20}$$

$$F'(\epsilon) = \sqrt{(1 - F^2(\epsilon))(1 - \omega F^2(\epsilon))}, \tag{21}$$

$$F'(\epsilon) = \sqrt{(1 - F^2(\epsilon))(\omega - 1 + \omega F^2(\epsilon))}. \tag{22}$$

We recall that the Jacobi cosine-amplitude function

$$F(\epsilon) = \mathrm{cn}(\epsilon|\omega), \tag{23}$$

is a solution to (20), the Jacobi sine-amplitude function

$$F(\epsilon) = \mathrm{sn}(\epsilon|\omega), \tag{24}$$

is a solution to (21), where as the Jacobi delta-amplitude function

$$F(\epsilon) = \mathrm{dn}(\epsilon|\omega), \tag{25}$$

is a solution to (22). Here ω is a parameter such that $0 \leq \omega \leq 1$. We note that as $\omega \to 1$, then $\mathrm{cn}(\epsilon|\omega) \to \mathrm{sech}(\epsilon)$, $\mathrm{sn}(\epsilon|\omega) \to \tanh(\epsilon)$ and $\mathrm{dn}(\epsilon|\omega) \to \mathrm{sech}(\epsilon)$. Similarly, as $\omega \to 0$, then $\mathrm{cn}(\epsilon|\omega) \to \cos(\epsilon)$, $\mathrm{sn}(\epsilon|\omega) \to \sin(\epsilon)$ and $\mathrm{dn}(\epsilon|\omega) \to 1$.

As before, applying the homogeneous balancing principle, we find that $M = 2$. Hence, the solution (19) is expressed in the form

$$\Psi(\epsilon) = \mathcal{C}_{-2}F^{-2} + \mathcal{C}_{-1}F^{-1} + \mathcal{C}_0 + \mathcal{C}_1 F + \mathcal{C}_2 F^2. \tag{26}$$

Cnoidal wave solution

Substituting (26) into the NLODE (13), and using (20), we obtain an algebraic equation, which upon splitting on like powers of F, yields the following set of algebraic equations in $\mathcal{C}_0$, $\mathcal{C}_{-1}$, $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_{-2}$:

$$
\begin{aligned}
&\mu A_2{}^2 - 360\,\omega^2 = 0,\\
&\mu\,\omega\,\mathcal{C}_{-2}{}^3 - \mu\,\mathcal{C}_{-2}{}^3 - 360\,\omega^3\mathcal{C}_{-2} + 1080\,\omega^2\mathcal{C}_{-2} - 1080\,\omega\,\mathcal{C}_{-2} + 360\,\mathcal{C}_{-2} = 0,\\
&\mu\,\omega\,\mathcal{C}_{-2}{}^2\mathcal{C}_{-1} - \mu\,\mathcal{C}_{-2}{}^2\mathcal{C}_{-1} - 24\,\omega^3\mathcal{C}_{-1} + 72\,\omega^2\mathcal{C}_{-1} - 72\,\omega\,\mathcal{C}_{-1} + 72\,\mathcal{C}_{-1} = 0,\\
&12\,\beta\,\omega^2\mathcal{C}_2 - 2\,\mu\,\omega\,\mathcal{C}_0\mathcal{C}_2{}^2 - 2\,\mu\,\omega\,\mathcal{C}_1{}^2\mathcal{C}_2 + 2\,\mu\,\omega\,\mathcal{C}_2{}^3 - \delta\,\omega\,\mathcal{C}_2{}^2 - \mu\,\mathcal{C}_2{}^3 - 1200\,\omega^3\mathcal{C}_2\\
&+ 600\,\omega^2\mathcal{C}_2 = 0,\\
&10\,\mu\,\omega\,\mathcal{C}_1\mathcal{C}_2{}^2 - 3\,\mu\,\omega\,\mathcal{C}_{-1}\mathcal{C}_2{}^2 - 6\,\mu\,\omega\,\mathcal{C}_0\mathcal{C}_1A_2 - \mu\,\omega\,\mathcal{C}_1{}^3 + 6\,\beta\,\omega^2\mathcal{C}_1 - 3\,\delta\,\omega\,\mathcal{C}_1\mathcal{C}_2\\
&- 5\,\mu\,\mathcal{C}_1\mathcal{C}_2{}^2 - 360\,\omega^3\mathcal{C}_1 + 180\,\omega^2\mathcal{C}_1 = 0,\\
&2\,\mu\,\omega\,\mathcal{C}_{-2}{}^2\mathcal{C}_0 - 2\,\mu\,\omega\,\mathcal{C}_{-2}{}^3 + 2\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{-1}{}^2 - 12\,\beta\,\omega^2\mathcal{C}_{-2} + \delta\,\omega\,\mathcal{C}_{-2}{}^2 + \mu\,\mathcal{C}_{-2}{}^3\\
&- 2\,\mu\,\mathcal{C}_{-2}{}^2\mathcal{C}_0 - 2\,\mu\,\mathcal{C}_{-2}\mathcal{C}_{-1}{}^2 + 1200\,\omega^3\mathcal{C}_{-2} + 24\,\beta\,\omega\,\mathcal{C}_{-2} - \delta\,\mathcal{C}_{-2}{}^2 - 3000\,\omega^2\mathcal{C}_{-2}\\
&- 12\,\beta\,\mathcal{C}_{-2} + 2400\,\omega\,\mathcal{C}_{-2} - 600\,\mathcal{C}_{-2} = 0,\\
&3\,\mu\,\omega\,\mathcal{C}_{-2}{}^2\mathcal{C}_1 - 10\,\mu\,\omega\,\mathcal{C}_{-2}{}^2\mathcal{C}_{-1} + 6\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{-1}\mathcal{C}_0 + \mu\,\omega\,\mathcal{C}_{-1}{}^3 - 6\,\beta\,\omega^2\mathcal{C}_{-1} - \mu\,\mathcal{C}_{-1}{}^3\\
&+ 3\,\delta\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{-1} + 5\,\mu\,\mathcal{C}_{-2}{}^2\mathcal{C}_{-1} - 3\,\mu\,\mathcal{C}_{-2}{}^2\mathcal{C}_1 - 6\,\mu\,\mathcal{C}_{-2}\mathcal{C}_{-1}\mathcal{C}_0 + 360\,\omega^3\mathcal{C}_{-1} - 6\,\beta\,\mathcal{C}_{-1}\\
&+ 12\,\beta\,\omega\,\mathcal{C}_{-1} - 3\,\delta\,\mathcal{C}_{-2}\mathcal{C}_{-1} - 900\,\omega^2\mathcal{C}_{-1} + 720\,\omega\,\mathcal{C}_{-1} - 180\,\mathcal{C}_{-1} = 0,\\
&8\,\mu\,\omega\,\mathcal{C}_0\mathcal{C}_2{}^2 - 2\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_2{}^2 - 4\,\mu\,\omega\,\mathcal{C}_{-1}\mathcal{C}_1\mathcal{C}_2 - 2\,\mu\,\omega\,\mathcal{C}_0{}^2\mathcal{C}_2 - 2\,\mu\,\omega\,\mathcal{C}_0\mathcal{C}_1{}^2 - \delta\,\omega\,\mathcal{C}_1{}^2\\
&+ 8\,\mu\,\omega\,\mathcal{C}_1{}^2\mathcal{C}_2 - 2\,\mu\,\omega\,\mathcal{C}_2{}^3 - 64\,\beta\,\omega^2\mathcal{C}_2 - 2\,\delta\,\omega\,\mathcal{C}_0\mathcal{C}_2 + 4\,\delta\,\omega\,\mathcal{C}_2{}^2 2\,\mu\,\mathcal{C}_2{}^3 - 4\,\mu\,\mathcal{C}_0\mathcal{C}_2{}^2\\
&- 4\,\mu\,\mathcal{C}_1{}^2\mathcal{C}_2 + 2912\,\omega^3\mathcal{C}_2 + 32\,\beta\,\omega\,\mathcal{C}_2 + 2\,c\omega\,\mathcal{C}_2 - 2\,\delta\,\mathcal{C}_2{}^2 - 2\,\gamma\,\omega\,\mathcal{C}_2 - 2912\,\omega^2\mathcal{C}_2\\
&+ 512\,\omega\,\mathcal{C}_2 = 0,\\
&6\,\mu\,\omega\,\mathcal{C}_{-1}\mathcal{C}_2{}^2 - 2\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_1\mathcal{C}_2 - 2\,\mu\,\omega\,\mathcal{C}_{-1}\mathcal{C}_0\mathcal{C}_2 - \mu\,\omega\,\mathcal{C}_{-1}\mathcal{C}_1{}^2 - \mu\,\omega\,\mathcal{C}_0{}^2\mathcal{C}_1 + 2\,\mu\,\omega\,\mathcal{C}_1{}^3\\
&- 5\,\mu\,\omega\,\mathcal{C}_1\mathcal{C}_2{}^2 - 14\,\beta\,\omega^2\mathcal{C}_1 - \delta\,\omega\,\mathcal{C}_{-1}\mathcal{C}_2 - \delta\,\omega\,\mathcal{C}_0\mathcal{C}_1 + 6\,\delta\,\omega\,\mathcal{C}_1\mathcal{C}_2 - 3\,\mu\,\mathcal{C}_{-1}\mathcal{C}_2{}^2\\
&- 6\,\mu\,\mathcal{C}_0\mathcal{C}_1\mathcal{C}_2 - \mu\,\mathcal{C}_1{}^3 + 5\,\mu\,\mathcal{C}_1\mathcal{C}_2{}^2 + 376\,\omega^3\mathcal{C}_1 + 7\,\beta\,\omega\,\mathcal{C}_1 + c\omega\,\mathcal{C}_1 - 3\,\delta\,\mathcal{C}_1\mathcal{C}_2 - \gamma\,\omega\,\mathcal{C}_1\\
&- 376\,\omega^2\mathcal{C}_1 + 61\,\omega\,\mathcal{C}_1 + 12\,\mu\,\omega\,\mathcal{C}_0\mathcal{C}_1\mathcal{C}_2 = 0,\\
&4\,\mu\,\omega\,\mathcal{C}_{-2}{}^2\mathcal{C}_0 - 4\,\mu\,\omega\,\mathcal{C}_{-2}{}^2\mathcal{C}_2 + 4\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{-1}{}^2 - 8\,\mu\,\omega\,\mathcal{C}_{-2}A_{-1}\mathcal{C}_1 - 4\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_0{}^2\\
&- 4\,\mu\,\omega\,\mathcal{C}_{-1}{}^2\mathcal{C}_0 - 56\,\beta\,\omega^2\mathcal{C}_{-2} + 2\,\delta\,\omega\,\mathcal{C}_{-2}{}^2 - 4\,\delta\,\omega\,\mathcal{C}_{-2}\mathcal{C}_0 - 2\,\delta\,\omega\,\mathcal{C}_{-1}{}^2 + 2\,\mu\,\mathcal{C}_{-2}{}^2\mathcal{C}_2\\
&+ 4\,\mu\,\mathcal{C}_{-2}\mathcal{C}_{-1}\mathcal{C}_1 + 2\,\mu\,\mathcal{C}_{-2}\mathcal{C}_0{}^2 + 2\,\mu\,\mathcal{C}_{-1}{}^2\mathcal{C}_0 + 1504\,\omega^3\mathcal{C}_{-2} + 56\,\beta\,\omega\,\mathcal{C}_{-2} + 4\,c\omega\,\mathcal{C}_{-2}\\
&+ 2\,\delta\,\mathcal{C}_{-2}\mathcal{C}_0 + \delta\,\mathcal{C}_{-1}{}^2 - 4\,\gamma\,\omega\,\mathcal{C}_{-2} - 2256\,\omega^2\mathcal{C}_{-2} - 8\,\beta\,\mathcal{C}_{-2} - 2\,c\mathcal{C}_{-2} + 2\,\gamma\,\mathcal{C}_{-2}\\
&+ 816\,\omega\,\mathcal{C}_{-2} - 32\,\mathcal{C}_{-2} = 0,\\
&4\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_2{}^2 + 8\,\mu\,\omega\,\mathcal{C}_{-1}\mathcal{C}_1\mathcal{C}_2 + 4\,\mu\,\omega\,\mathcal{C}_0{}^2\mathcal{C}_2 + 4\,\mu\,\omega\,\mathcal{C}_0\mathcal{C}_1{}^2 - 4\,\mu\,\omega\,\mathcal{C}_0\mathcal{C}_2{}^2 - 4\,\mu\,\omega\,\mathcal{C}_1{}^2\mathcal{C}_2\\
&+ 56\,\beta\,\omega^2\mathcal{C}_2 + 4\,\delta\,\omega\,\mathcal{C}_0\mathcal{C}_2 + 2\,\delta\,\omega\,\mathcal{C}_1{}^2 - 2\,\delta\,\omega\,\mathcal{C}_2{}^2 - 2\,\mu\,\mathcal{C}_{-2}\mathcal{C}_2{}^2 - 4\,\mu\,\mathcal{C}_{-1}\mathcal{C}_1\mathcal{C}_2\\
&- 2\,\mu\,\mathcal{C}_0{}^2\mathcal{C}_2 - 2\,\mu\,\mathcal{C}_0\mathcal{C}_1{}^2 + 4\,\mu\,\mathcal{C}_0\mathcal{C}_2{}^2 + 4\,\mu\,\mathcal{C}_1{}^2\mathcal{C}_2 - 1504\,\omega^3\mathcal{C}_2 - 56\,\beta\,\omega\,\mathcal{C}_2 - 4\,c\omega\,\mathcal{C}_2\\
&- 2\,\delta\,\mathcal{C}_0\mathcal{C}_2 - \delta\,\mathcal{C}_1{}^2 + 2\,\delta\,\mathcal{C}_2{}^2 + 4\,\gamma\,\omega\,A_2 + 2256\,\omega^2\mathcal{C}_2 + 8\,\beta\,\mathcal{C}_2 + 2\,c\mathcal{C}_2 - 2\,\gamma\,\mathcal{C}_2\\
&- 816\,\omega\,\mathcal{C}_2 + 32\,\mathcal{C}_2 = 0,
\end{aligned}
$$

$$
\begin{aligned}
&2\,\mu\,\omega\,\mathcal{C}_{-2}{}^{3}-8\,\mu\,\omega\,\mathcal{C}_{-2}{}^{2}\mathcal{C}_{0}+2\,\mu\,\omega\,\mathcal{C}_{-2}{}^{2}\mathcal{C}_{2}-8\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{-1}{}^{2}+4\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{-1}\mathcal{C}_{1}\\
&+2\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{0}{}^{2}+2\,\mu\,\omega\,\mathcal{C}_{-1}{}^{2}\mathcal{C}_{0}+64\,\beta\,\omega^{2}\mathcal{C}_{-2}-4\,\delta\,\omega\,\mathcal{C}_{-2}{}^{2}+2\,\delta\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{0}\\
&+\delta\,\omega\,\mathcal{C}_{-1}{}^{2}+4\,\mu\,\mathcal{C}_{-2}{}^{2}\mathcal{C}_{0}-2\,\mu\,\mathcal{C}_{-2}{}^{2}\mathcal{C}_{2}+4\,\mu\,\mathcal{C}_{-2}\mathcal{C}_{-1}{}^{2}-4\,\mu\,\mathcal{C}_{-2}\mathcal{C}_{-1}\mathcal{C}_{1}-2\,\mu\,\mathcal{C}_{-2}\mathcal{C}_{0}{}^{2}\\
&-2\,\mu\,\mathcal{C}_{-1}{}^{2}\mathcal{C}_{0}-2912\,\omega^{3}\mathcal{C}_{-2}-96\,\beta\,\omega\,\mathcal{C}_{-2}-2\,c\omega\,\mathcal{C}_{-2}+2\,\delta\,\mathcal{C}_{-2}{}^{2}-2\,\delta\,\mathcal{C}_{-2}\mathcal{C}_{0}\\
&-\delta\,\mathcal{C}_{-1}{}^{2}+2\,\gamma\,\omega\,\mathcal{C}_{-2}+5824\,\omega^{2}\mathcal{C}_{-2}+32\,\beta\,\mathcal{C}_{-2}+2\,c\mathcal{C}_{-2}-2\,\gamma\,\mathcal{C}_{-2}-3424\,\omega\,\mathcal{C}_{-2}\\
&+512\,\mathcal{C}_{-2}=0,\\
&5\,\mu\,\omega\,\mathcal{C}_{-2}{}^{2}\mathcal{C}_{-1}-6\,\mu\,\omega\,\mathcal{C}_{-2}{}^{2}A_{1}-12\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{-1}\mathcal{C}_{0}+2\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{-1}\mathcal{C}_{2}+2\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{0}\mathcal{C}_{1}\\
&-2\,\mu\,\omega\,\mathcal{C}_{-1}{}^{3}+\mu\,\omega\,\mathcal{C}_{-1}{}^{2}\mathcal{C}_{1}+\mu\,\omega\,\mathcal{C}_{-1}\mathcal{C}_{0}{}^{2}+14\,\beta\,\omega^{2}\mathcal{C}_{-1}-6\,\delta\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{-1}+\delta\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{1}\\
&+\delta\,\omega\,\mathcal{C}_{-1}\mathcal{C}_{0}+3\,\mu\,\mathcal{C}_{-2}{}^{2}\mathcal{C}_{1}+6\,\mu\,\mathcal{C}_{-2}\mathcal{C}_{-1}\mathcal{C}_{0}-2\,\mu\,\mathcal{C}_{-2}\mathcal{C}_{-1}\mathcal{C}_{2}-2\,\mu\,\mathcal{C}_{-2}\mathcal{C}_{0}\mathcal{C}_{1}+\mu\,\mathcal{C}_{-1}{}^{3}\\
&-\mu\,\mathcal{C}_{-1}{}^{2}\mathcal{C}_{1}-\mu\,\mathcal{C}_{-1}\mathcal{C}_{0}{}^{2}-376\,\omega^{3}\mathcal{C}_{-1}-21\,\beta\,\omega\,\mathcal{C}_{-1}-c\omega\,\mathcal{C}_{-1}+3\,\delta\,\mathcal{C}_{-2}\mathcal{C}_{-1}-\delta\,\mathcal{C}_{-2}\mathcal{C}_{1}\\
&-\delta\,\mathcal{C}_{-1}\mathcal{C}_{0}+\gamma\,\omega\,\mathcal{C}_{-1}+752\,\omega^{2}\mathcal{C}_{-1}+7\,\beta\,\mathcal{C}_{-1}+c\mathcal{C}_{-1}-\gamma\,\mathcal{C}_{-1}-437\,\omega\,\mathcal{C}_{-1}+61\,\mathcal{C}_{-1}=0,\\
&2\,\mu\,\omega\,\mathcal{C}_{-2}{}^{2}\mathcal{C}_{2}+4\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{-1}\mathcal{C}_{1}+2\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{0}{}^{2}-2\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{2}{}^{2}+2\,\mu\,\omega\,\mathcal{C}_{-1}{}^{2}\mathcal{C}_{0}\\
&-4\,\mu\,\omega\,\mathcal{C}_{-1}\mathcal{C}_{1}\mathcal{C}_{2}-2\,\mu\,\omega\,\mathcal{C}_{0}{}^{2}\mathcal{C}_{2}-2\,\mu\,\omega\,\mathcal{C}_{0}\mathcal{C}_{1}{}^{2}+16\,\beta\,\omega^{2}\mathcal{C}_{-2}-16\,\beta\,\omega^{2}\mathcal{C}_{2}+2\,\delta\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{0}\\
&+\delta\,\omega\,\mathcal{C}_{-1}{}^{2}-2\,\delta\,\omega\,\mathcal{C}_{0}\mathcal{C}_{2}-\delta\,\omega\,\mathcal{C}_{1}{}^{2}+2\,\mu\,\mathcal{C}_{-2}\mathcal{C}_{2}{}^{2}+4\,\mu\,\mathcal{C}_{-1}\mathcal{C}_{1}\mathcal{C}_{2}+2\,\mu\,\mathcal{C}_{0}{}^{2}\mathcal{C}_{2}+2\,\mu\,\mathcal{C}_{0}\mathcal{C}_{1}{}^{2}\\
&-272\,\omega^{3}\mathcal{C}_{-2}+272\,\omega^{3}\mathcal{C}_{2}-8\,\beta\,\omega\,\mathcal{C}_{-2}+24\,\beta\,\omega\,\mathcal{C}_{2}-2\,c\omega\,\mathcal{C}_{-2}+2\,c\omega\,A_{2}+2\,\delta\,\mathcal{C}_{0}\mathcal{C}_{2}\\
&+\delta\,\mathcal{C}_{1}{}^{2}+2\,\gamma\,\omega\,\mathcal{C}_{-2}-2\,\gamma\,\omega\,\mathcal{C}_{2}+272\,\omega^{2}\mathcal{C}_{-2}-544\,\omega^{2}\mathcal{C}_{2}-8\,\beta\,\mathcal{C}_{2}-2\,c\mathcal{C}_{2}+2\,\gamma\,\mathcal{C}_{2}\\
&-32\,\omega\,\mathcal{C}_{-2}+304\,\omega\,\mathcal{C}_{2}-32\,\mathcal{C}_{2}=0,\\
&2\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{-1}\mathcal{C}_{2}+2\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{0}\mathcal{C}_{1}+4\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{1}\mathcal{C}_{2}+\mu\,\omega\,\mathcal{C}_{-1}{}^{2}\mathcal{C}_{1}+\mu\,\omega\,\mathcal{C}_{-1}\mathcal{C}_{0}{}^{2}\\
&+4\,\mu\,\omega\,\mathcal{C}_{-1}\mathcal{C}_{0}\mathcal{C}_{2}+2\,\mu\,\omega\,\mathcal{C}_{-1}\mathcal{C}_{1}{}^{2}-3\,\mu\,\omega\,\mathcal{C}_{-1}\mathcal{C}_{2}{}^{2}+2\,\mu\,\omega\,\mathcal{C}_{0}{}^{2}\mathcal{C}_{1}-6\,\mu\,\omega\,\mathcal{C}_{0}\mathcal{C}_{1}\mathcal{C}_{2}\\
&-\mu\,\omega\,\mathcal{C}_{1}{}^{3}+2\,\beta\,\omega^{2}\mathcal{C}_{-1}+10\,\beta\,\omega^{2}\mathcal{C}_{1}+\delta\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{1}+\delta\,\omega\,\mathcal{C}_{-1}\mathcal{C}_{0}+2\,\delta\,\omega\,\mathcal{C}_{-1}\mathcal{C}_{2}\\
&+2\,\delta\,\omega\,\mathcal{C}_{0}\mathcal{C}_{1}-3\,\delta\,\omega\,\mathcal{C}_{1}\mathcal{C}_{2}-2\,\mu\,\mathcal{C}_{-2}\mathcal{C}_{1}\mathcal{C}_{2}-2\,\mu\,\mathcal{C}_{-1}\mathcal{C}_{0}\mathcal{C}_{2}-\mu\,\mathcal{C}_{-1}\mathcal{C}_{1}{}^{2}+3\,\mu\,\mathcal{C}_{-1}\mathcal{C}_{2}{}^{2}\\
&-\mu\,\mathcal{C}_{0}{}^{2}\mathcal{C}_{1}+6\,\mu\,\mathcal{C}_{0}\mathcal{C}_{1}\mathcal{C}_{2}+\mu\,\mathcal{C}_{1}{}^{3}-16\,\omega^{3}\mathcal{C}_{-1}-152\,\omega^{3}\mathcal{C}_{1}-\beta\,\omega\,\mathcal{C}_{-1}-10\,\beta\,\omega\,\mathcal{C}_{1}\\
&-c\omega\,\mathcal{C}_{-1}-2\,c\omega\,\mathcal{C}_{1}-\delta\,\mathcal{C}_{-1}\mathcal{C}_{2}-\delta\,\mathcal{C}_{0}\mathcal{C}_{1}+3\,\delta\,\mathcal{C}_{1}\mathcal{C}_{2}+\gamma\,\omega\,\mathcal{C}_{-1}+2\,\gamma\,\omega\,\mathcal{C}_{1}+16\,\omega^{2}\mathcal{C}_{-1}\\
&+228\,\omega^{2}\mathcal{C}_{1}+\beta\,\mathcal{C}_{1}+c\mathcal{C}_{1}-\gamma\,\mathcal{C}_{1}-\omega\,\mathcal{C}_{-1}-78\,\omega\,\mathcal{C}_{1}+\mathcal{C}_{1}=0,\\
&3\,\mu\,\omega\,\mathcal{C}_{-2}{}^{2}\mathcal{C}_{1}+6\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{-1}\mathcal{C}_{0}-4\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{-1}\mathcal{C}_{2}-4\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{0}\mathcal{C}_{1}-2\,\mu\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{1}\mathcal{C}_{2}\\
&+\mu\,\omega\,\mathcal{C}_{-1}{}^{3}-2\,\mu\,\omega\,\mathcal{C}_{-1}{}^{2}\mathcal{C}_{1}-2\,\mu\,\omega\,\mathcal{C}_{-1}\mathcal{C}_{0}{}^{2}-2\,\mu\,\omega\,\mathcal{C}_{-1}\mathcal{C}_{0}\mathcal{C}_{2}-\mu\,\omega\,\mathcal{C}_{-1}\mathcal{C}_{1}{}^{2}-\mu\,\omega\,\mathcal{C}_{0}{}^{2}\mathcal{C}_{1}\\
&-10\,\beta\,\omega^{2}\mathcal{C}_{-1}-2\,\beta\,\omega^{2}\mathcal{C}_{1}+3\,\delta\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{-1}-2\,\delta\,\omega\,\mathcal{C}_{-2}\mathcal{C}_{1}-2\,\delta\,\omega\,\mathcal{C}_{-1}\mathcal{C}_{0}-\delta\,\omega\,\mathcal{C}_{-1}\mathcal{C}_{2}\\
&-\delta\,\omega\,\mathcal{C}_{0}\mathcal{C}_{1}+2\,\mu\,\mathcal{C}_{-2}\mathcal{C}_{-1}\mathcal{C}_{2}+2\,\mu\,\mathcal{C}_{-2}\mathcal{C}_{0}\mathcal{C}_{1}+2\,\mu\,\mathcal{C}_{-2}\mathcal{C}_{1}\mathcal{C}_{2}+\mu\,\mathcal{C}_{-1}{}^{2}\mathcal{C}_{1}+\mu\,\mathcal{C}_{-1}\mathcal{C}_{0}{}^{2}\\
&+2\,\mu\,\mathcal{C}_{-1}\mathcal{C}_{0}\mathcal{C}_{2}+\mu\,\mathcal{C}_{-1}\mathcal{C}_{1}{}^{2}+\mu\,\mathcal{C}_{0}{}^{2}\mathcal{C}_{1}+152\,\omega^{3}\mathcal{C}_{-1}+16\,\omega^{3}\mathcal{C}_{1}+10\,\beta\,\omega\,\mathcal{C}_{-1}+3\,\beta\,\omega\,\mathcal{C}_{1}\\
&+2\,c\omega\,\mathcal{C}_{-1}+c\omega\,\mathcal{C}_{1}+\delta\,\mathcal{C}_{-2}\mathcal{C}_{1}+\delta\,\mathcal{C}_{-1}\mathcal{C}_{0}+\delta\,\mathcal{C}{-1}\mathcal{C}_{2}+\delta\,\mathcal{C}_{0}\mathcal{C}_{1}-2\,\gamma\,\omega\,\mathcal{C}_{-1}\\
&-\gamma\,\omega\,\mathcal{C}_{1}-228\,\omega^{2}\mathcal{C}_{-1}-32\,\omega^{2}\mathcal{C}_{1}-\beta\,\mathcal{C}_{-1}-\beta\,\mathcal{C}_{1}-c\mathcal{C}_{-1}-c\mathcal{C}_{1}+\gamma\,\mathcal{C}_{-1}+\gamma\,\mathcal{C}_{1}\\
&+78\,\omega\,\mathcal{C}_{-1}+17\,\omega\,\mathcal{C}_{1}-\mathcal{C}_{-1}-\mathcal{C}_{1}=0.
\end{aligned}
$$

Solving the above equations using Maple, we obtain

$$
\begin{aligned}
&\mathcal{C}_{-1}=0,\ \mathcal{C}_{1}=0,\ \mathcal{C}_{0}=-\frac{\delta}{2\mu},\ \mathcal{C}_{2}=\frac{6\sqrt{10}\,\omega}{\sqrt{\mu}},\ \mathcal{C}_{-2}=-\frac{6\sqrt{10}(\omega-1)}{\sqrt{\mu}},\\
&\beta=20(2\omega+1),\ \ c=\frac{704\mu\omega(1-\omega)+4\,\mu\,\gamma-\delta^{2}+256\,\mu}{4\mu}.
\end{aligned}
$$

Hence from (26), we obtain

$$\Psi(\epsilon) = -\frac{6\sqrt{10}(\omega-1)}{\sqrt{\mu}}\mathrm{cn}(\epsilon|\omega)^{-2} - \frac{\delta}{2\mu} + \frac{6\sqrt{10}\,\omega}{\sqrt{\mu}}\mathrm{cn}(\epsilon|\omega)^{2}. \tag{27}$$

Subsequently, the cnoidal wave solution of the KEdpn (7) can be expressed as

$$u(t,x) = \frac{6\sqrt{10}\,\omega}{\sqrt{\mu}}\mathrm{cn}(x-ct|\omega)^{2} - \frac{\delta}{2\mu} - \frac{6\sqrt{10}(\omega-1)}{\sqrt{\mu}}\mathrm{cn}(x-ct|\omega)^{-2}. \tag{28}$$

The profile of the cnoidal wave solution (28) is depicted in Figure 2.

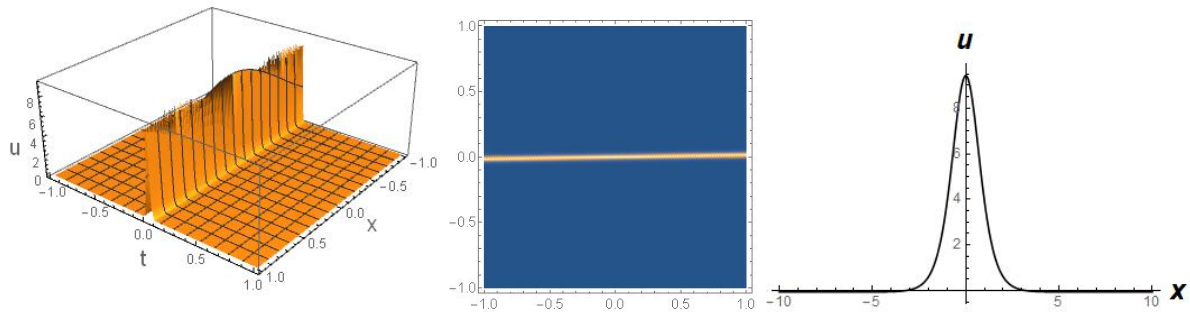

Figure 2: The 3D and 2D profile of the cnoidal wave solution (28), where $\delta = 0.5$, $\gamma = 1$, $\omega = 1$, $\mu = 4$, $-10 \le t, x \le 10$.

Snoidal wave solution

In the same manner as above, but using the first-order NLODE (21), we find that

$$\mathcal{C}_{-1} = 0,\ \mathcal{C}_1 = 0,\ \mathcal{C}_0 = -\frac{\delta}{2\mu},\ \mathcal{C}_2 = \frac{6\sqrt{10}\,\omega}{\sqrt{\mu}},\ \mathcal{C}_{-2} = -\frac{6\sqrt{10}}{\sqrt{\mu}},$$
$$\beta = -20(\omega+1),\ \ c = \frac{256\,\mu(\omega^2+1) + 4\,\mu\,\gamma - \delta^2 - 1216\,\mu\,\omega}{4\mu}.$$

Thus, the snoidal wave solution of the KEdpn (7) is given by

$$u(t,x) = \frac{6\sqrt{10}\,\omega}{\sqrt{\mu}}\mathrm{sn}(x-ct|\omega)^{2} - \frac{\delta}{2\mu} - \frac{6\sqrt{10}}{\sqrt{\mu}}\mathrm{sn}(x-ct|\omega)^{-2}. \tag{29}$$

The profile of the snoidal wave solution (29) is depicted in Figure 3.

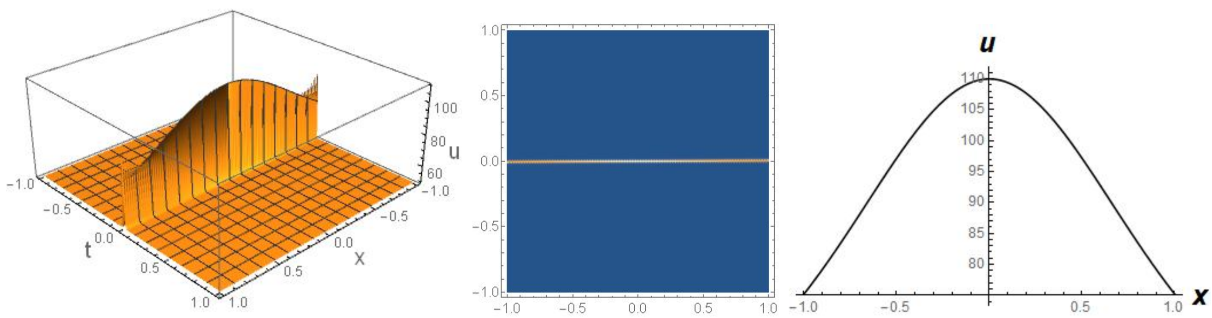

Figure 3: The 3D and 2D profile of the snoidal wave solution (29), where $\delta = 10, \gamma = -10$, $\omega = 1$, $\mu = -0.1$, $-1 \le t, x \le 1$.

Dnoidal wave solution

Proceeding as above, but utilizing the NLODE (22), we obtain

$$\mathcal{C}_{-1}=0,\ \mathcal{C}_1=0,\ \mathcal{C}_0=-\frac{\delta}{2\mu},\ \mathcal{C}_2=\frac{6\sqrt{10}}{\sqrt{\mu}},\ \mathcal{C}_{-2}=\frac{6\sqrt{10}(\omega-1)}{\sqrt{\mu}},$$
$$\beta=-20(\omega-2),\ c=\frac{704\mu(\omega-1)+4\,\mu\gamma-\delta^2+256\,\mu\,\omega^2}{4\mu}.$$

Consequently, the dnoidal wave solution to the KEdpn (7) takes the form

$$u(t,x)=\frac{6\sqrt{10}(\omega-1)}{\sqrt{\mu}}\mathrm{dn}(x-ct|\omega)^{-2}-\frac{\delta}{2\mu}+\frac{6\sqrt{10}}{\sqrt{\mu}}\mathrm{dn}(x-ct|\omega)^2. \tag{30}$$

The profile of the dnoidal wave solution (30) is depicted in Figure 4.

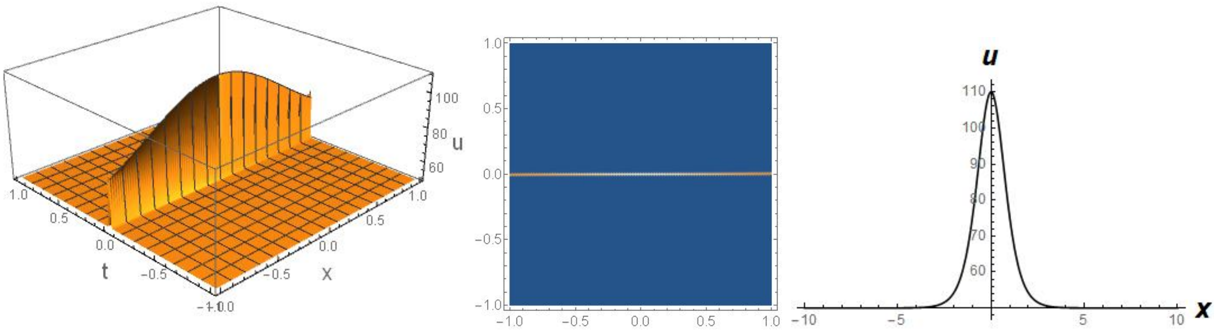

Figure 4: The 3D and 2D profile of the dnoidal wave solution (30),where $\delta=10,\gamma=60,\ \omega=1,\mu=-0.1,\ -1\leq t,x\leq 1$.

3 Conserved vectors of (7)

In this section, we construct the conserved vectors of the KEdpn (7) through the application of two different approaches. Firstly, we utilize the multiplier approach. The importance of this method is that it can be used even when a variational structure is not known for the differential equation Olver (1993). On the other hand, Noether's theorem is applied to those differential equations, that come from a variational principle Noether (1918).

3.1 Conserved vectors of (7) using the multiplier approach

We compute the conserved vectors for the KEdpn (7) using the multiplier approachOlver (1993). We seek the zeroth-order multipliers $\Lambda=\Lambda(t,x,u)$. To compute the multipliers we make use of the determining equation

$$\frac{\delta}{\delta u}\left[\Lambda(t,x,u)\left\{u_t+\gamma u_x+\delta u^n u_x+\mu u^{2n}u_x+\beta u_{xxx}-u_{xxxxx}\right\}\right]=0, \tag{31}$$

where

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}-D_t\frac{\partial}{\partial u_t}-D_x\frac{\partial}{\partial u_x}-D_x^3\frac{\partial}{\partial u_{xxx}}-D_x^5\frac{\partial}{\partial u_{xxxxx}}$$

is the Euler-Lagrange operator and the total differential operators D_t and D_x are as defined in (11). Consequently, expanding (31) and splitting the resultant equation based on the derivatives of u, we obtain a system of linear homogeneous PDEs. Solving these PDEs, we get the multiplier Λ in the form

$$\Lambda=C_1+C_2u$$

with C_1, C_2 being constants, which is a linear combination of two nontrivial multipliers

$$\Lambda_1 = 1, \;\; \Lambda_2 = u.$$

To obtain the conserved vectors we invoke the determining equation

$$\Lambda\left\{u_t + \gamma u_x + \delta u^n u_x + \mu u^{2n} u_x + \beta u_{xxx} - u_{xxxxx}\right\} = D_t C^t + D_x C^x, \tag{32}$$

where C^t denotes the conserved density, and C^x is the spatial flux. Thus, we have the following two cases.

Case 1. The conserved vector $({C_1}^t, {C_1}^x)$ that relates to $\Lambda_1 = 1$ is given by

$$\begin{aligned}
{C_1}^t &= u,\\
{C_1}^x &= \frac{1}{2n^3 + 7n^2 + 7n + 2}\Big\{\mu n^2 u u^{2n} + 3\mu n u u^{2n} + 2\mu u u^{2n} + 2\delta n^2 u u^n + 5\delta n u u^n\\
&\quad + 2\delta u u^n + 2\gamma n^3 u + 2\beta n^3 u_{xx} - 2n^3 u_{xxxx} + 7\gamma n^2 u + 7\beta n^2 u_{xx} - 7n^2 u_{xxxx}\\
&\quad + 7\gamma n u + 7\beta n u_{xx} - 7n u_{xxxx} + 2\gamma u + 2\beta u_{xx} - 2u_{xxxx}\Big\}.
\end{aligned}$$

Case 2. The corresponding conserved vector $({C_2}^t, {C_2}^x)$ associated with $\Lambda_2 = u$ is

$$\begin{aligned}
{C_2}^t &= \frac{1}{2}u^2,\\
{C_2}^x &= \frac{1}{2(2n^3 + 7n^2 + 7n + 2)}\Big\{2\mu u^2 u^{2n} + 2\delta u^2 u^n + 2\gamma n^3 u^2 - 2\beta n^3 u_x^2 - 2u_{xx}^2\\
&\quad - 4n^3 u u_{xxxx} + 4n^3 u_x u_{xxx} + 7\gamma n^2 u^2 - 7\beta n^2 u_x^2 - 14n^2 u u_{xxxx} + 14n^2 u_x u_{xxx}\\
&\quad + 7\gamma n u^2 - 7\beta n u_x^2 - 14n u u_{xxxx} + 14n u_x u_{xxx} + 4\beta u u_{xx} - 2n^3 u_{xx}^2 - 7n^2 u_{xx}^2\\
&\quad - 7n u_{xx}^2 - 4u u_{xxxx} + 2\gamma u^2 - 2\beta u_x^2 + 4u_x u_{xxx} + 2\mu n^2 u^2 u^{2n} + 5\mu n u^2 u^{2n}\\
&\quad + 4\delta n^2 u^2 u^n + 6\delta n u^2 u^n + 4\beta n^3 u u_{xx} + 14\beta n^2 u u_{xx} + 14\beta n u u_{xx}\Big\}.
\end{aligned}$$

Remark. It should be noted that the multiplier $\Lambda_1 = 1$ reveals that the KEdpn (7) is itself in conserved form.

As a special case we now consider the KEdpn (7) when $n = 1$ and seek fourth-order multipliers $\Lambda = \Lambda(t, x, u, u_x, u_{xx}, u_{xxx}, u_{xxxx})$. Following the procedure described above, we obtain the multiplier Λ in the form

$$\Lambda = \; J_1 + J_2 u + J_3\left(u_{xxxx} - \beta u_{xx} - \frac{1}{2}\delta u^2 - \frac{1}{3}\mu u^3\right)$$

with J_1, J_2, J_3 being constants, which provides us with three nontrivial multipliers

$$\Lambda_1 = 1, \;\; \Lambda_2 = u, \;\; \Lambda_3 = u_{xxxx} - \beta u_{xx} - \frac{1}{2}\delta u^2 - \frac{1}{3}\mu u^3.$$

Consequently, we have the following three cases.

Case 1. The conserved vector $({C_1}^t, {C_1}^x)$ corresponding to $\Lambda_1 = 1$ is given by

$$\begin{aligned}
{C_1}^t &= u,\\
{C_1}^x &= \gamma u + \beta u_{xx} + \frac{1}{2}\delta u^2 + \frac{1}{3}\mu u^3 - u_{xxxx}.
\end{aligned}$$

Case 2. The conserved vector $({C_2}^t, {C_2}^x)$ associated with $\Lambda_2 = u$ is

$$\begin{aligned}
{C_2}^t &= \frac{1}{2}u^2,\\
{C_2}^x &= \frac{1}{2}\gamma u^2 + \frac{1}{3}\delta u^3 + \frac{1}{4}\mu u^4 - \frac{1}{2}\beta u_x^2 - \frac{1}{2}u_{xx}^2 + \beta u u_{xx} + u_x u_{xxx} - u u_{xxxx}.
\end{aligned}$$

Case 3. Conserved vector $({C_3}^t, {C_3}^x)$ connected to $\Lambda_3 = u_{xxxx} - \beta u_{xx} - \frac{1}{2}\delta u^2 - \frac{1}{3}\mu u^3$ is

$$
\begin{aligned}
{C_3}^t =& -\frac{1}{12}\mu u^4 - \frac{1}{6}\delta u^3 - \frac{1}{2}\beta u u_{xx} + \frac{1}{2}u u_{xxxx},\\
{C_3}^x =& \frac{1}{2}\beta u u_{tx} - \frac{1}{2}\beta^2 u_{xx}^2 - \frac{1}{3}\mu\beta u^3 u_{xx} + \beta u_{xx}u_{xxxx} - \frac{1}{2}\beta\gamma\, u_x^2 - \frac{1}{2}\beta u_t u_x\\
& - \frac{1}{18}\mu^2 u^6 - \frac{1}{12}\mu\gamma u^4 + \frac{1}{3}\mu u^3 u_{xxxx} + \gamma u_x u_{xxx} - \frac{1}{2}\gamma\, u_{xx}^2 - \frac{1}{2}\, u u_{txxx}\\
& - \frac{1}{2}u_{xxxx}^2 + \frac{1}{2}\, u_t u_{xxx} + \frac{1}{2}u_{txx}u_x - \frac{1}{2}u_{tx}u_{xx} - \frac{1}{2}\beta\delta u^2 u_{xx} + \frac{1}{2}\delta u^2 u_{xxxx}\\
& - \frac{1}{6}\delta\mu u^5 - \frac{1}{6}\gamma\delta u^3 - \frac{1}{8}\delta^2 u^4.
\end{aligned}
$$

Remark. The multiplier $\Lambda_1 = 1$ indicates that the KEdpn (7) is in conserved form.

3.2 Conserved vectors of (7) using Noether's approach

In this subsection, we compute the conserved vectors for the KEdpn (7) using Noether's theorem Noether (1918). Equation (7) is a fifth-order equation and does not have a Lagrangian. To make the equation variational, we let $u = U_x$. Thus, equation (7) becomes

$$U_{tx} + \gamma U_{xx} + \delta U_x^n U_{xx} + \mu U_x^{2n} U_{xx} + \beta U_{xxxx} - U_{xxxxxx} = 0, \tag{33}$$

which has a third-order Lagrangian $\mathcal{L}$ given by

$$\mathcal{L} = -\frac{1}{2}U_t U_x - \frac{\gamma}{2}U_x^2 - \frac{\delta}{(n+1)(n+2)}U_x^{n+2} - \frac{\mu}{(2n+1)(2n+2)}U_x^{2n+2} + \frac{\beta}{2}U_{xx}^2 + \frac{1}{2}U_{xxx}^2$$

as $\delta\mathcal{L}/\delta U = 0$ holds on the equation. Here $\delta/\delta U$ is the Euler-Lagrange operator that is defined as

$$\frac{\delta}{\delta U} = \frac{\partial}{\partial U} - D_t\frac{\partial}{\partial U_t} - D_x\frac{\partial}{\partial U_x} + D_x^2\frac{\partial}{\partial U_{xx}} - D_x^3\frac{\partial}{\partial U_{xxx}} + \cdots,$$

and D_t and D_x are the total differential operators as defined in (11). The equation that determines the Noether point symmetries of (33) is given by

$$\mathcal{Q}^{[3]}(\mathcal{L}) + \mathcal{L}\left[D_t(\xi^1) + D_x(\xi^2)\right] = D_t(W^1) + D_x(W^2), \tag{34}$$

where $W^1 = W^1(t,x,U)$ and $W^2 = W^2(t,x,U)$ are gauge terms and $\mathcal{Q}^{[3]}$ is the third extension of the infinitesimal generator

$$\mathcal{Q} = \xi^1(t,x,U)\frac{\partial}{\partial t} + \xi^2(t,x,U)\frac{\partial}{\partial x} + \eta(t,x,U)\frac{\partial}{\partial U}.$$

Expanding (34), and splitting on various derivatives of U, we obtain the following fourteen overdetermined systems of linear homogeneous PDEs:

$$
\begin{aligned}
&\xi_x^1 = 0,\ \xi_U^1 = 0,\ \xi_U^2 = 0,\ \eta_x = 0,\ \eta_U = 0,\ \xi_{xx}^2 = 0,\ \xi_t^1 - 3\xi_x^2 = 0, \xi_t^1 - 5\xi_x^2 = 0,\ \eta_t + 2\,W_U^2 = 0,\\
&\xi_t^1 + (n+1)\,\xi_x^2 = 0,\ \xi_t^1 + (2n+1)\,\xi_x^2 = 0,\ \xi_t^2 + \gamma\left(\xi_x^2 - \xi_t^1\right) = 0,\ W_u^1 = 0,\ W_t^1 + W_x^2 = 0.
\end{aligned}
$$

Solving these PDEs, we obtain the following Noether point symmetries and gauge functions:

$$
\begin{aligned}
\mathcal{Q}_1 &= \frac{\partial}{\partial t},\ W^1 = 0,\ W^2 = 0,\\
\mathcal{Q}_2 &= \frac{\partial}{\partial x},\ W^1 = 0,\ W^2 = 0,\\
\mathcal{Q}_g &= g(t)\frac{\partial}{\partial U},\ W^1 = 0,\ W^2 = -\frac{1}{2}g'(t)U + h(t).
\end{aligned}
$$

Conserved vectors (P_i^t, P_i^x) corresponding to each Noether point symmetry are derived using the following formulae Sarlet (2010):

$$
\begin{aligned}
&{P_i}^t = \mathcal{L}\xi^1 + \left(\eta - U_t\xi^1 - U_x\xi^2\right)\frac{\partial\mathcal{L}}{\partial U_t} + \left(\zeta_2 - U_{tx}\xi^1 - U_{xx}\xi^2\right)\frac{\partial\mathcal{L}}{U_{tx}} - W^1,\\
&{P_i}^x = \mathcal{L}\xi^2 + \left(\eta - U_t\xi^1 - U_x\xi^2\right)\left\{\frac{\partial\mathcal{L}}{\partial U_x} - D_t\left(\frac{\partial\mathcal{L}}{\partial U_{tx}}\right) - D_x\left(\frac{\partial\mathcal{L}}{\partial U_{xx}}\right)\right\}\\
&+ \left(\zeta_2 - U_{tx}\xi^1 - U_{xx}\xi^2\right)\frac{\partial\mathcal{L}}{U_{xx}} - W^2.
\end{aligned}
$$

Therefore, we have the following three cases:

Case 1. $\mathcal{Q}_1 = \partial/\partial t$ provides the conserved vector $({P_1}^t, {P_1}^x)$ with components given by

$$
\begin{aligned}
{P_1}^t =& \frac{1}{2}\beta U_{xx}^2 - \frac{\delta}{(n+1)(n+2)}g(t)U_tU_x^{n+2} - \frac{\mu}{(2n+1)(2n+2)}g(t)U_tU_x^{2n+2}\\
&- \frac{1}{2}\gamma U_x^2 + \frac{1}{2}U_{xxx}^2,\\
{P_1}^x =& \frac{1}{2}U_t^2 + \frac{\delta}{(n+1)}g(t)U_x^{n+2} + \frac{\mu}{(2n+1)}g(t)U_x^{2n+2} + \gamma U_tU_x - \beta U_{tx}U_{xx}\\
&+ \beta U_tU_{xxx} + U_{tx}U_{xxxx} - U_{txx}U_{xxx} - U_tU_{xxxxx}.
\end{aligned}
$$

Case 2. $\mathcal{Q}_2 = \partial/\partial x$ yields the conserved vector $({P_2}^t, {P_2}^x)$ with components

$$
\begin{aligned}
{P_2}^t =& \frac{1}{2}U_x^2,\\
{P_2}^x =& \frac{1}{2}\gamma U_x^2 + \frac{\delta}{(n+1)}g(t)U_x^{n+2} + \frac{\mu}{(2n+1)}g(t)U_x^{2n+2} - \frac{\delta}{(n+1)(n+2)}g(t)U_x^{n+2}\\
&- \frac{\mu}{(2n+1)(2n+2)}g(t)U_x^{2n+2} - \frac{1}{2}\beta U_{xx}^2 + \beta U_xU_{xxx} - \frac{1}{2}U_{xxx}^2 + U_{xx}U_{xxxx}\\
&- U_xU_{xxxxx}.
\end{aligned}
$$

Case 3. $\mathcal{Q}_g = g(t)\partial/\partial U$ produces the conserved vector $({P_g}^t, {P_g}^x)$ whose components are

$$
\begin{aligned}
{P_g}^t =& -\frac{1}{2}g(t)U_x,\\
{P_g}^x =& \frac{1}{2}g'(t)U - \frac{1}{2}g(t)U_t - \frac{\delta}{n+1}g(t)U_x^{n+1} - \frac{\mu}{2n+1}g(t)U_x^{2n+1}\\
&- \gamma g(t)U_x - h(t) - \beta g(t)U_{xxx} + g(t)U_{xxxxx}.
\end{aligned}
$$

Reverting to the original variable u, we have

$$
\begin{aligned}
{P_1}^t =& \frac{1}{2}\beta u_x^2 - \frac{\delta}{(n+1)(n+2)}g(t)\left(\int u_t dx\right)u^{n+2} - \frac{\mu}{(2n+1)(2n+2)}g(t)\left(\int u_t dx\right)u^{2n+2} \\
&- \frac{1}{2}\gamma u^2 + \frac{1}{2}u_{xx}^2, \\
{P_1}^x =& \frac{1}{2}\left(\int u_t dx\right)^2 + \frac{\delta}{(n+1)}g(t)u^{n+2} + \frac{\mu}{(2n+1)}g(t)u^{2n+2} + \gamma\left(\int u_t dx\right)u - \beta u_t u_x \\
&+ \beta\left(\int u_t dx\right)u_{xx} + u_t u_{xxx} - u_{tx}u_{xx} - \left(\int u_t dx\right)u_{xxxx}.
\end{aligned}
$$

$$
\begin{aligned}
{P_2}^t =& \frac{1}{2}u^2, \\
{P_2}^x =& \frac{1}{2}\gamma u^2 + \frac{\delta}{(n+1)}g(t)u^{n+2} + \frac{\mu}{(2n+1)}g(t)u^{2n+2} - \frac{\delta}{(n+1)(n+2)}g(t)u^{n+2} \\
&- \frac{1}{2}\beta u_x^2 - \frac{\mu}{(2n+1)(2n+2)}g(t)u^{2n+2} + \beta u u_{xx} - \frac{1}{2}u_{xx}^2 + u_x u_{xxx} - u u_{xxxx}.
\end{aligned}
$$

$$
\begin{aligned}
{P_g}^t =& -\frac{1}{2}g(t)u, \\
{P_g}^x =& \frac{1}{2}g'(t)\left(\int u dx\right) - \frac{1}{2}g(t)\left(\int u_t dx\right) - \frac{\delta}{n+1}g(t)u^{n+1} - \frac{\mu}{2n+1}g(t)u^{2n+1} \\
&- \gamma g(t)u - h(t) - \beta g(t)u_{xx} + g(t)u_{xxxx}.
\end{aligned}
$$

4 Concluding remarks

In this paper, we studied, for the first time, the Kawahara equation with dual power-law nonlinearities (7), which represents an important higher-order nonlinear evolution equation describing complex wave phenomena encountered in several branches of applied sciences, including fluid dynamics, plasma physics, and nonlinear acoustics. This equation generalized the classical Kawahara model by subsuming dual nonlinear terms, thereby permitting for a more thorough description of nonlinear dispersive wave propagation in physical media specified by nonstandard nonlinear responses. Firstly, travelling wave solutions were obtained by invoking Lie group analysis on the equation. The calculation of infinitesimal generators revealed that the equation had two translation symmetries, corresponding to invariance under translations in the spatial and temporal variables. Using these symmetries, we performed a symmetry reduction, which systematically reduced the original equation to a nonlinear ordinary differential equation (NLODE). Several analytical solution techniques were utilized on the NLODE and travelling wave solutions were constructed. These methods yielded a rich diversity of exact solutions expressed in terms of exponential, hyperbolic, and elliptic functions, which represent dissimilar physical wave structures. The obtained solutions furnished valuable insight into the qualitative behaviour and propagation characteristics of nonlinear dispersive waves governed by the dual power-law Kawahara equation. Additionally, the physical behaviour of the solutions are illustrated graphically in Figures 1–4 through three-dimensional, two-dimensional, and density plots. Finally, we constructed the conserved vectors associated with equation (7) by invoking two different approaches, namely the multiplier method and Noether's theorem. Each of these approaches provided a systematic framework for constructing conserved quantities. The resulting conserved vectors were explicitly derived and shown to correspond to the conservation of fundamental physical quantities such as energy and momentum.

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