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ALGEBRAIC FOUNDATIONS OF L-FUZZY SBG-STRUCTURES FOR DECISION-MAKING UNDER UNCERTAINTY

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Abstract: We present formal definitions of L-fuzzy SBG-ideals and L-fuzzy subalgebras, and provide detailed characterizations based on membership functions and level subsets. These fuzzy algebraic structures provide a robust mathematical framework for modeling decision-making mechanisms under uncertainty in artificial intelligence systems. In addition, we derive necessary and sufficient conditions for the preservation of algebraic operations in these fuzzy subalgebras, revealing deep connections between their level structures and corresponding classical algebraic counterparts. The obtained characterizations strengthen the theoretical foundation of systems used in advanced technology applications, such as fuzzy logic controllers and data classification algorithms. The analysis further encompasses complementation properties, lattice-theoretic behavior, and the broader implications of such constructions within fuzzy algebra. Overall, the results offer a solid theoretical foundation for future developments in the study and application of fuzzy subalgebraic systems.

Keywords: Sheffer stroke (BG-algebra), BG-ideal, $\mathscr{L}$-fuzzy SBG-subalgebra, $\mathscr{L}$-fuzzy SBG-ideal, Computing.
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1 Introduction

The concept of a B-algebra was first introduced by Neggers and Kim Neggers & Kim (2002), who investigated its connection to groups via digraphs. Later, Allen et al. Allen et al. (2003) offered an alternative proof highlighting the strong link between B-algebras and groups through the surjectivity of the zero adjoint mapping. Further studies by Cho and Kim Cho & Kim (2001) examined relationships between B-algebras and other algebraic structures such as quasigroups, while C.B. Kim and H.S. Kim Kim & Kim (2001) extended the concept to form BG-algebras. Walendziak Walendziak (2015) proved that BG, BF1, B, and BM-algebras are congruence permutable, and Ahn and Lee Ahn & Lee (2004) introduced a fuzzification of BG-algebras. Further developments include studies on cubic closed ideals Senapati & Chen (2021), direct products of BG-algebras Widianto et al. (2019), and a variety of fuzzy generalizations such as interval-valued, multi-fuzzy, normalized, and intuitionistic fuzzy BG-algebras Saeid (2005); Muthuraj & Devi (2016); Muthuraj et al. (2010); Prasanna et al. (2018); Senapati et al. (2014, 2012).

The Sheffer stroke operation, introduced by H.M. Sheffer Sheffer (1913), is fundamental in logic because it forms a functionally complete system, allowing every logical connective and Boolean expression to be defined solely in terms of the Sheffer stroke. In recent years, numerous studies have been conducted on the theory of the Sheffer stroke, reflecting its growing significance in both logical and algebraic frameworks Oner & Senturk (2017); Senturk et al. (2020); Senturk & Oner (2021); Rajesh et al. (2024); Senturk et al. (2025); Rajesh et al. (2025); Oner et al. (2025). This operation, when integrated into algebraic structures Chajda (2005); Oner et al. (2021a); Oner & Kalkan (2021), offers a unified and elegant framework for studying algebraic properties through logical foundations. Oner et al. Oner et al. (2021a) developed the theory of Sheffer stroke BG-algebras, introducing (implicative) ideals and showing that every implicative ideal is also an ideal of the algebra. Later, Oner and Kalkan Oner & Kalkan (2021) extended this framework via filters and homomorphisms, while fuzzy approaches to Sheffer stroke BG-algebras were explored in Oner et al. (2021b).

The study deepens the theoretical understanding of algebraic systems under uncertainty. However, practical justification for decision-making applications as claimed in the title and abstract is minimal. To strengthen its significance, the paper should include at least one illustrative example or case application. Clarify the new contribution beyond existing works (especially Oner & Kalkan (2021); Oner et al. (2025)). A comparative paragraph summarizing differences from previous fuzzy SBG-ideal papers would improve the novelty perception.

Although several fuzzy generalizations of BG-algebras have been studied (see Oner et al. (2021a); Senapati & Chen (2021); Rajesh et al. (2024), among others), these studies primarily focus on fuzzy or hesitant fuzzy frameworks based on the standard unit interval $[0, 1]$. However, complex decision-making processes often involve multi-dimensional or ordered uncertainty levels that cannot be fully represented within $[0, 1]$. To address this limitation, we introduce the notion of L-fuzzy SBG-ideals, where L is a complete lattice. This approach provides a more flexible modeling framework capable of handling multiple uncertainty scales simultaneously. Thus, the present paper fills an essential gap in the literature by establishing a systematic theory of L-fuzzy SBG-structures, deriving necessary and sufficient conditions for operation preservation, and exploring their algebraic behavior through lattice-valued mappings.

In this work, we introduce and formally define L-fuzzy SBG-ideals and L-fuzzy subalgebras, establishing a precise mathematical framework for their study. Comprehensive characterizations of these structures are developed in terms of membership functions and level subsets, providing a clear correspondence between the fuzzy and classical algebraic domains. Moreover, we establish necessary and sufficient conditions under which the fundamental algebraic operations are preserved within these fuzzy subalgebras, thereby uncovering intrinsic connections between their lattice-valued representations and underlying crisp algebraic counterparts. The investigation further encompasses the analysis of complementation principles, lattice-theoretic behavior, and the structural implications of these systems within the broader context of fuzzy algebra. Collectively, the results presented herein furnish a rigorous theoretical basis for advancing the study of fuzzy subalgebraic systems and their potential applications in algebraic models involving uncertainty and graded reasoning.

2 Preliminaries

In this section, we provide definitions, lemma, and proposition relevant to the concepts of Sheffer stroke BG-algebras, their subalgebras and ideals, and implicatice SBG-algebras, which will be used throughout the paper.

Definition 1. *Sheffer (1913) Let $\mathcal{S} := (S, |)$ be a groupoid. Then the operation "$|$" is said to be Sheffer stroke or Sheffer operation if it satisfies:*
$(S1)$ $(\forall \mathfrak{w}, \mathfrak{s} \in S)$ $(\mathfrak{w}|\mathfrak{s} = \mathfrak{s}|\mathfrak{w})$,
$(S2)$ $(\forall \mathfrak{w}, \mathfrak{s} \in S)$ $((\mathfrak{w}|\mathfrak{w})|(\mathfrak{w}|\mathfrak{s}) = \mathfrak{w})$,
$(S3)$ $(\forall \mathfrak{w}, \mathfrak{s}, \mathfrak{b} \in S)$ $(\mathfrak{w}|((\mathfrak{s}|\mathfrak{b})|(\mathfrak{s}|\mathfrak{b})) = ((\mathfrak{w}|\mathfrak{s})|(\mathfrak{w}|\mathfrak{s}))|\mathfrak{b})$,
$(S4)$ $(\forall \mathfrak{w}, \mathfrak{s}, \mathfrak{b} \in S)$ $((\mathfrak{w}|((\mathfrak{w}|\mathfrak{w})|(\mathfrak{s}|\mathfrak{s})))|(\mathfrak{w}|((\mathfrak{w}|\mathfrak{w})|(\mathfrak{s}|\mathfrak{s}))) = \mathfrak{w})$.

To enhance the readability of this manuscript about a Sheffer stroke BG-algebras, we will consistently use the following notation:

$$\mathfrak{w}|(\mathfrak{b}|\mathfrak{b}) = \mathfrak{w}^{\mathfrak{b}}.$$

Definition 2. *Oner et al. (2021a) A Sheffer stroke BG-algebra (abbreviated as an SBG-algebra) is a groupoid* $\mathcal{B} := (B, |)$ *with a Sheffer stroke "|" that satisfies:*
(SBG_1) $(\forall \mathfrak{w} \in B)$ $(\mathfrak{w}^{\mathfrak{w}}|\mathfrak{w}^{\mathfrak{w}} = 0)$,
(SBG_2) $(\forall \mathfrak{w}, \mathfrak{b} \in B)$ $(0^{\mathfrak{b}}|(\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}}) = \mathfrak{w}|\mathfrak{w})$.

Proposition 1. *Oner et al. (2021a) Let* $\mathcal{B} := (B, |)$ *be an SBG-algebra. The binary relation*

$$\mathfrak{w} \leq \mathfrak{b} \quad \textit{if and only if} \quad \mathfrak{b}^{\mathfrak{w}}|\mathfrak{b}^{\mathfrak{w}} = 0$$

defines a partial order on B.

Definition 3. *Oner et al. (2021a) Let* $\mathcal{B} := (B, |)$ *be an SBG-algebra. A nonempty subset* G *of* B *is called an SBG-subalgebra of* B *if* $\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}} \in G$ *for all* $\mathfrak{w}, \mathfrak{b} \in G$.

Definition 4. *Oner et al. (2021a) Let* $\mathcal{B} := (B, |)$ *be an SBG-algebra. A nonempty subset* G *of* B *is called an SBG-ideal of* B *if*

1. $0 \in G$,
2. *If* $\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}} \in G$ *and* $\mathfrak{b} \in G$, *then* $\mathfrak{w} \in G$.

Definition 5. *Oner et al. (2021a) Let* $\mathcal{B} := (B, |)$ *be an SBG-algebra. A nonempty subset* G *of* B *is called an implicative SBG-ideal of* B *if*

1. $0 \in G$,
2. $((((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s}))|(((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s})) \in G$ *and* $\mathfrak{s} \in G \Rightarrow \mathfrak{w} \in G$,

for all $\mathfrak{w}, \mathfrak{b}, \mathfrak{s} \in G$.

Lemma 1. *Oner et al. (2021a) Let* $\mathcal{B} := (B, |)$ *be an SBG-algebra. Then the following features hold:*

1. $\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}} = \mathfrak{s}^{\mathfrak{b}}|\mathfrak{s}^{\mathfrak{b}}$ *implies* $\mathfrak{w} = \mathfrak{s}$,
2. *If* $\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}} = 0$ *then* $\mathfrak{w} = \mathfrak{b}$,
3. $(\mathfrak{w}^{\mathfrak{w}})^{\mathfrak{w}} = \mathfrak{w}$,

for all $\mathfrak{w}, \mathfrak{b}, \mathfrak{s} \in B$.

Lemma 2. *Davey & Priestley (2002) Let* $\mathscr{L} = (\mathscr{L}, \leq, \wedge, \vee, ', 0_{\mathscr{L}}, 1_{\mathscr{L}})$ *be a Boolean lattice. Then the following properties hold:*

1. $(\forall \mathfrak{w}, \mathfrak{b} \in \mathcal{L})((\mathfrak{w} \vee \mathfrak{b})' = \mathfrak{w}' \wedge \mathfrak{b}')$,
2. $(\forall \mathfrak{w}, \mathfrak{b} \in \mathcal{L})((\mathfrak{w} \wedge \mathfrak{b})' = \mathfrak{w}' \vee \mathfrak{b}')$,
3. $(\forall \mathfrak{w}, \mathfrak{b} \in \mathcal{L})(\mathfrak{w} \leq \mathfrak{b} \Leftrightarrow \mathfrak{w}' \geq \mathfrak{b}')$,
4. $(\forall \mathfrak{w}, \mathfrak{b} \in \mathcal{L})(\mathfrak{w} = \mathfrak{b} \Leftrightarrow \mathfrak{w}' = \mathfrak{b}')$,
5. $(\forall \mathfrak{w}, \mathfrak{b} \in \mathcal{L})(\mathfrak{w} < \mathfrak{b} \Leftrightarrow \mathfrak{w}' > \mathfrak{b}')$.

3 Lattice valued fuzzy sets in Sheffer stroke BG-algebras

This section presents the definitions of $\mathscr{L}$-fuzzy SBG-subalgebras, $\mathscr{L}$-fuzzy SBG-ideals, and $\mathscr{L}$-implicative fuzzy SBG-ideals in the framework of Sheffer stroke BG-algebras.

Definition 6. *Let $\mathcal{B} := (B, |)$ be an SBG-algebra. An $\mathscr{L}$-fuzzy set $\mathscr{L}$ in B is called an $\mathscr{L}$-fuzzy SBG-subalgebra of B if*

$$(\forall \mathfrak{w}, \mathfrak{b} \in B)\left(\mathcal{L}_\mu(\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}}) \geq \mathcal{L}_\mu(\mathfrak{w}) \wedge \mathcal{L}_\mu(\mathfrak{b}) \right). \tag{1}$$

Definition 7. *Let $\mathcal{B} := (B, |)$ be an SBG-algebra. An $\mathscr{L}$-fuzzy set $\mathscr{L}$ on an SBG-algebra B is called an $\mathscr{L}$-fuzzy SBG-ideal of B if*

$$(\forall \mathfrak{w}, \mathfrak{b} \in B)\left(\begin{array}{l} \mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{w}) \\ \mathcal{L}_\mu(\mathfrak{w}) \geq \mathcal{L}_\mu(\mathfrak{b}) \wedge \mathcal{L}_\mu(\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}}) \end{array} \right). \tag{2}$$

Definition 8. *Let $\mathcal{B} := (B, |)$ be an SBG-algebra. An $\mathscr{L}$-fuzzy set $\mathscr{L}$ on B is called an $\mathscr{L}$-fuzzy implicative SBG-ideal of B if*

$$(\forall \mathfrak{w}, \mathfrak{b}, \mathfrak{s} \in B)\left(\begin{array}{l} \mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{w}) \geq \mathcal{L}_\mu((((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s}))| \\ (((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s})) \wedge \mathcal{L}_\mu(\mathfrak{s}) \end{array} \right). \tag{3}$$

Proposition 2. *Let $\mathcal{B} := (B, |)$ be an SBG-algebra. Every $\mathscr{L}$-fuzzy SBG-implicative ideal of B is an $\mathscr{L}$-fuzzy SBG-ideal of B.*

Proof. Let $\mathscr{L}$ be an $\mathscr{L}$-fuzzy SBG-implicative ideal of B. Then for all $\mathfrak{w}, \mathfrak{b} \in B$ we have $\mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{w})$ and

$$\begin{array}{rcl} \mathcal{L}_\mu(\mathfrak{w}) & \geq & \mathcal{L}_\mu(\mathfrak{b}) \wedge \mathcal{L}_\mu((((\mathfrak{w}|\mathfrak{w}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{w}}))|(\mathfrak{b}|\mathfrak{b}))| \\ & & (((\mathfrak{w}|\mathfrak{w}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{w}}))|(\mathfrak{b}|\mathfrak{b}))) \\ & = & \mathcal{L}_\mu(\mathfrak{b}) \wedge \mathcal{L}_\mu((((\mathfrak{w}^0|\mathfrak{w}^0)|(\mathfrak{b}|\mathfrak{b}))|((\mathfrak{w}^0|\mathfrak{w}^0)|(\mathfrak{b}|\mathfrak{b}))) \\ & = & \mathcal{L}_\mu(\mathfrak{b}) \wedge \mathcal{L}_\mu(\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}}). \end{array}$$

Therefore, $\mathscr{L}$ is an $\mathscr{L}$-fuzzy SBG-ideal of $\mathfrak{w}$. ☐

Definition 9. *Let $\mathcal{B} := (B, |)$ be an SBG-algebra. A $\mathscr{L}$-fuzzy subset $\mathscr{L}$ of B is called a $\mathscr{L}$-fuzzy sub-implicative SBG-ideal of B if*

$$(\forall \mathfrak{w}, \mathfrak{b}, \mathfrak{s} \in A)\left(\begin{array}{l} \mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{w}), \\ \mathcal{L}_\mu((\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}})) \geq \mathcal{L}_\mu((((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}}))|(\mathfrak{s}|\mathfrak{s}))| \\ (((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}}))|(\mathfrak{s}|\mathfrak{s}))) \wedge \mathcal{L}_\mu(\mathfrak{s}) \end{array} \right). \tag{4}$$

Proposition 3. *Let $\mathcal{A} := (B, |)$ be an SBG-algebra. Then every $\mathscr{L}$-fuzzy sub-implicative SBG-ideal of B is an $\mathscr{L}$-fuzzy SBG-ideal of B.*

Proof. Let $\mathscr{L}$ be an $\mathscr{L}$-fuzzy sub-implicative SBG-ideal of B. Then for all $\mathfrak{w}, \mathfrak{b}, \mathfrak{s} \in B$ we have $\mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{w})$, and

$$\begin{array}{rcl} \mathcal{L}_\mu(\mathfrak{w}) & = & \mathcal{L}_\mu(\mathfrak{w}^0|\mathfrak{w}^0) \\ & = & \mathcal{L}_\mu((\mathfrak{w}|\mathfrak{w}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{w}})) \\ & \geq & \mathcal{L}_\mu((((\mathfrak{w}|\mathfrak{w}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s}))|(((\mathfrak{w}|\mathfrak{w}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s}))) \\ & & \wedge \mathcal{L}_\mu(\mathfrak{s}) \\ & = & \mathcal{L}_\mu(((\mathfrak{w}^0|\mathfrak{w}^0)|(\mathfrak{s}|\mathfrak{s}))|(\mathfrak{w}^0|\mathfrak{w}^0)|(\mathfrak{s}|\mathfrak{s}))) \wedge \mathcal{L}_\mu(\mathfrak{s}) \\ & = & \mathcal{L}_\mu(\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}}) \wedge \mathcal{L}_\mu(\mathfrak{s}), \end{array}$$

Therefore, $\mathscr{L}$ is an $\mathscr{L}$-fuzzy SBG-ideal of B. ☐

The following theorem establishes a fundamental relationship between $\mathscr{L}$-fuzzy SBG-ideals and sub-implicative $\mathscr{L}$-fuzzy SBG-ideals, clarifying the internal hierarchy of these fuzzy structures.

Theorem 1. *Let $\mathcal{B} := (B, |)$ be an SBG-algebra and $\mathscr{L}$ be an $\mathscr{L}$-fuzzy SBG-ideal of B. Then $\mathscr{L}$ is an $\mathscr{L}$-fuzzy sub-implicative SBG-ideal of B if and only if $\mathcal{L}_\mu((\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}})) \geq \mathcal{L}_\mu((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}}))$.*

Proof. Let $\mathscr{L}$ be an $\mathscr{L}$-fuzzy sub-implicative SBG-ideal of B. For $\mathfrak{w}, \mathfrak{b} \in B$ we obtain

$$\begin{aligned}
\mathcal{L}_\mu((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})) &\geq \mathcal{L}_\mu(((((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}}))|(0|0)))|(((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})| \\
&\quad (\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}}))|(0|0))) \wedge \mathcal{L}_\mu(0) \\
&= \mathcal{L}_\mu((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})) \wedge \mathcal{L}_\mu(0)\ ) \\
&= \mathcal{L}_\mu((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})).
\end{aligned}$$

Conversely, since $\mathscr{L}$ is an $\mathscr{L}$-fuzzy SBG-ideal, for $\mathfrak{w}, \mathfrak{b}, \mathfrak{s} \in B$ we get $\mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{w})$, and

$$\begin{aligned}
\mathcal{L}_\mu((\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}})) &\geq \mathcal{L}_\mu((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})) \\
&\geq \mathcal{L}_\mu((((\mathfrak{w}|\mathfrak{w}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s}))|(((\mathfrak{w}|\mathfrak{w}^{\mathfrak{w}})| \\
&\quad (\mathfrak{w}|\mathfrak{w}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s}))) \wedge \mathcal{L}_\mu(jy).
\end{aligned}$$

Therefore, $\mathscr{L}$ is an $\mathscr{L}$-fuzzy sub-implicative SBG-ideal of B. □

Definition 10. *Oner et al. (2021b) Let $\mathcal{B} := (B, |)$ be an SBG-algebra. An SBG-algebra is said to be implicative if it satisfies $\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}} = \mathfrak{b}|\mathfrak{b}^{\mathfrak{w}}$ for all $\mathfrak{w}, \mathfrak{b} \in B$.*

This result demonstrates that in an implicative SBG-algebra, the structure of $\mathscr{L}$-fuzzy SBG-ideals naturally extends to that of sub-implicative $\mathscr{L}$-fuzzy SBG-ideals, showing an intrinsic closure property of the system.

Theorem 2. *Let $\mathcal{B} := (B, |)$ an implicative SBG-algebra. Then every $\mathscr{L}$-fuzzy SBG-ideal of B is an $\mathscr{L}$-fuzzy sub-implicative SBG-ideal of B.*

Proof. Let $\mathscr{L}$ be an $\mathscr{L}$-fuzzy SBG-ideal of B. Then $\mathfrak{w}, \mathfrak{b} \in B$ we have $\mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{w})$, and

$$\begin{aligned}
\mathcal{L}_\mu((\mathfrak{b}|\mathfrak{b}^{\mathfrak{b}})|(\mathfrak{b}|\mathfrak{b}^{\mathfrak{b}})) &\geq \mathcal{L}_\mu((((\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s}))|(((\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}}))| \\
&\quad (\mathfrak{s}|\mathfrak{s}))) \wedge \mathcal{L}_\mu(\mathfrak{s}) \\
&= \mathcal{L}_\mu((((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}}))|(\mathfrak{s}|\mathfrak{s}))(((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}}) \\
&\quad |(\mathfrak{s}|\mathfrak{s}))) \wedge \mathcal{L}_\mu(\mathfrak{s}),
\end{aligned}$$

Thereby, $\mathscr{L}$ is an $\mathscr{L}$-fuzzy sub-implicative SBG-ideal of B. □

Definition 11. *Oner et al. (2021b) An SBG-algebra $\mathcal{B} := (B, |)$ is called medial if $\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}} = \mathfrak{b}|\mathfrak{b}$, for all $\mathfrak{w}, \mathfrak{b} \in B$.*

Lemma 3. *Oner et al. (2021b) Let $\mathcal{B} := (B, |)$ be an SBG-algebra. Then the following property holds: $((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}}))|\mathfrak{b}^{\mathfrak{w}} = \mathfrak{w}|\mathfrak{w}^{\mathfrak{b}}$ for all $\mathfrak{w}, \mathfrak{b} \in B$.*

The following theorem shows how the medial condition strengthens the algebraic structure, ensuring that every $\mathscr{L}$-fuzzy SBG-ideal inherits sub-implicative behavior.

Theorem 3. *Let $\mathcal{B} := (B, |)$ be an SBG-algebra. Every $\mathscr{L}$-fuzzy SBG-ideal of a medial SBG-algebra B is an $\mathscr{L}$-fuzzy sub-implicative SBG-ideal of B.*

Proof. Let $\mathscr{L}$ be an $\mathscr{L}$-fuzzy SBG-ideal of a medial SBG-algebra B. Then for $\mathfrak{w}, \mathfrak{b}, \mathfrak{s} \in B$ we have $\mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{w})$ and

$$\begin{aligned}
\mathcal{L}_\mu((\mathfrak{b}|(\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{b}|(\mathfrak{b}^{\mathfrak{w}}))) &= \mathcal{L}_\mu(\mathfrak{w}) \\
&\geq \mathcal{L}_\mu((\mathfrak{w}|(\mathfrak{s}^{\mathfrak{s}}))|(\mathfrak{w}|(\mathfrak{s}^{\mathfrak{s}}))) \wedge \mathcal{L}_\mu(\mathfrak{s}) \\
&= \mathcal{L}_\mu((((\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s}))|(((\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}})) \\
&\quad |(\mathfrak{s}|\mathfrak{s}))) \wedge \mathcal{L}_\mu(\mathfrak{s}) \\
&= \mathcal{L}_\mu((((((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}}))|(\mathfrak{b}^{\mathfrak{w}})))|(((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})|(\mathfrak{w} \\
&\quad |\mathfrak{w}^{\mathfrak{b}}))|(\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s}))|(((((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}}))|(\mathfrak{b}^{\mathfrak{w}}))) \\
&\quad |(((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}}))|(\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s}))) \wedge \mathcal{L}_\mu(\mathfrak{s}) \\
&= \mathcal{L}_\mu((((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}}))|(\mathfrak{s}|\mathfrak{s}))(((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}}) \\
&\quad |(\mathfrak{s}|\mathfrak{s}))) \wedge \mathcal{L}_\mu(\mathfrak{s}),,
\end{aligned}$$

Hence, $\mathscr{L}$ is an $\mathscr{L}$-fuzzy sub-implicative SBG-ideal of B. □

Theorem 4. *Let $\mathcal{B} := (B, |)$ be an SBG-algebra satisfying*

$$(\forall \mathfrak{w}, \mathfrak{b}, \mathfrak{s} \in B) \left(\begin{array}{l} \mathcal{L}_\mu(\mathfrak{b}^{\mathfrak{s}}|\mathfrak{b}^{\mathfrak{s}}) \geq \mathcal{L}_\mu((((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}}))|(\mathfrak{s}|\mathfrak{s}))| \\ (((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}}))|(\mathfrak{s}|\mathfrak{s})) \end{array} \right). \tag{5}$$

Then every $\mathscr{L}$-fuzzy SBG-ideal of B is an $\mathscr{L}$-fuzzy sub-implicative SBG-ideal of B.

Proof. It is obtained that for $\mathfrak{w}, \mathfrak{b} \in B$

$$\begin{aligned}
\mathcal{L}_\mu((\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}})) &\geq \mathcal{L}_\mu((((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}}))|\mathfrak{b}^{\mathfrak{w}})|(((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})) \\
&\quad |\mathfrak{b}^{\mathfrak{w}})) \\
&= \mathcal{L}_\mu((\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}})).
\end{aligned}$$

Thus, $\mathscr{L}$ is an $\mathscr{L}$-fuzzy sub-implicative SBG-ideal of B. □

Theorem 5. *Oner et al. (2021b) Every medial SBG-algebra is an implicative SBG-algebra.*

This theorem provides a necessary and sufficient condition for an $\mathscr{L}$-fuzzy SBG-ideal to become implicative, offering an algebraic interpretation of the interaction between ideal closure and the Sheffer stroke operation.

Theorem 6. *Let $\mathcal{B} := (B, |)$ be an SBG-algebra and $\mathscr{L}$ be an $\mathscr{L}$-fuzzy SBG-ideal of B. Then $\mathscr{L}$ is an $\mathscr{L}$-fuzzy SBG-implicative ideal of B if and only if $\mathscr{L}$ satisfies the following condition:*

$$(\forall \mathfrak{w}, \mathfrak{s} \in B) \left(\; \mathcal{L}_\mu(\mathfrak{w}) \geq \mathcal{L}_\mu((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})) \; \right). \tag{6}$$

Proof. Let $\mathscr{L}$ be an $\mathscr{L}$-fuzzy SBG-implicative ideal of B. Then

$$\begin{aligned}
\mathcal{L}_\mu(\mathfrak{w}) &\geq \mathcal{L}_\mu(0) \wedge \mathcal{L}_\mu((((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}}))|(0|0))|(((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}}))| \\
&\quad (0|0))) \\
&= \mathcal{L}_\mu(0) \wedge \mathcal{L}_\mu((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}}))) \\
&= \mathcal{L}_\mu(((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})),
\end{aligned}$$

for all $\mathfrak{w}, \mathfrak{b} \in B$.

Conversely, let $\mathscr{L}$ be an $\mathscr{L}$-fuzzy SBG-ideal of B satisfying the inequality (6). Then $\mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{w})$ for all $\mathfrak{w} \in B$. Since

$$\begin{aligned}
\mathcal{L}_\mu(\mathfrak{w}) &\geq \mathcal{L}_\mu(((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})) \\
&\geq \mathcal{L}_\mu((((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s}))|(((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}}))| \\
&\quad (\mathfrak{s}|\mathfrak{s})) \wedge \mathcal{L}_\mu(\mathfrak{s}),
\end{aligned}$$

for all $\mathfrak{w}, \mathfrak{b}, \mathfrak{s} \in B$, we have $\mathscr{L}$ is an $\mathscr{L}$-fuzzy SBG-implicative ideal of B. □

The next theorem extends the previous findings to medial SBG-algebras, revealing how their symmetry properties lead to equivalence between sub-implicative and implicative fuzzy ideals.

Theorem 7. *Let $\mathcal{B} := (B, |)$ be a medial SBG-algebra satisfying*

$$(\forall \mathfrak{w}, \mathfrak{b} \in B)\left(\ \mathcal{L}_\mu((\mathfrak{w}|\mathfrak{w}^\mathfrak{b})|(\mathfrak{w}|\mathfrak{w}^\mathfrak{b})) \geq \mathcal{L}_\mu((\mathfrak{w}|\mathfrak{b}^\mathfrak{w})|(\mathfrak{w}|\mathfrak{b}^\mathfrak{w}))\ \right). \tag{7}$$

Then every $\mathscr{L}$-fuzzy sub-implicative SBG-ideal of B is an $\mathscr{L}$-fuzzy SBG-implicative ideal of B.

Proof. Let $\mathcal{B} := (B, |)$ be a medial SBG-algebra and $\mathscr{L}$ be an $\mathscr{L}$-fuzzy sub-implicative SBG-ideal of B satisfying the inequality (7). Then

$$\begin{aligned}
\mathcal{L}_\mu(\mathfrak{w}) &= \mathcal{L}_\mu((\mathfrak{b}|\mathfrak{b}^\mathfrak{w})|(\mathfrak{b}|\mathfrak{b}^\mathfrak{w})) \\
&\geq \mathcal{L}_\mu((((\mathfrak{w}|\mathfrak{w}^\mathfrak{b})|(\mathfrak{w}|\mathfrak{w}^\mathfrak{b}))|(0|0))|(((\mathfrak{w}|\mathfrak{w}^\mathfrak{b})|(\mathfrak{w}|\mathfrak{w}^\mathfrak{b}))| \\
&\quad (0|0))) \wedge \mathcal{L}_\mu(0) \\
&= \mathcal{L}_\mu((\mathfrak{w}|\mathfrak{w}^\mathfrak{b})|(\mathfrak{w}|\mathfrak{w}^\mathfrak{b})) \wedge \mathcal{L}_\mu(0) \\
&= \mathcal{L}_\mu((\mathfrak{w}|\mathfrak{w}^\mathfrak{b})|(\mathfrak{w}|\mathfrak{w}^\mathfrak{b})) \\
&\geq \mathcal{L}_\mu((\mathfrak{w}|\mathfrak{b}^\mathfrak{w})|(\mathfrak{w}|\mathfrak{b}^\mathfrak{w})),
\end{aligned}$$

Thus, $\mathscr{L}$ is an $\mathscr{L}$-fuzzy SBG-implicative ideal of B. □

Finally, we show the converse relation, completing the bidirectional connection between $\mathscr{L}$-fuzzy SBG-implicative and sub-implicative ideals within implicative SBG-algebras.

Theorem 8. *Let $\mathcal{B} := (B, |)$ be an implicative SBG-algebra. Then every $\mathscr{L}$-fuzzy SBG-implicative ideal of B is an $\mathscr{L}$-fuzzy sub-implicative SBG-ideal of B.*

Proof. Let $\mathscr{L}$ be an $\mathscr{L}$-fuzzy SBG-implicative ideal of B. Then $\mathscr{L}$ is an $\mathscr{L}$-fuzzy SBG-ideal of B. Therefore, it is obvious that $\mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{w})$ for all $\mathfrak{w} \in B$. Thus

$$\begin{aligned}
\mathcal{L}_\mu((\mathfrak{b}|\mathfrak{b}^\mathfrak{w})|(\mathfrak{b}|\mathfrak{b}^\mathfrak{w})) &= \mathcal{L}_\mu((\mathfrak{w}|\mathfrak{w}^\mathfrak{b})|(\mathfrak{w}|\mathfrak{w}^\mathfrak{b})) \\
&\geq \mathcal{L}_\mu((((\mathfrak{w}|\mathfrak{w}^\mathfrak{b})|(\mathfrak{w}|\mathfrak{w}^\mathfrak{b}))|(\mathfrak{s}|\mathfrak{s}))|(((\mathfrak{w}|\mathfrak{w}^\mathfrak{b})| \\
&\quad (\mathfrak{w}|\mathfrak{w}^\mathfrak{b}))|(\mathfrak{s}|\mathfrak{s}))) \wedge \mathcal{L}_\mu(\mathfrak{s}),
\end{aligned}$$

for all $\mathfrak{w}, \mathfrak{b}, \mathfrak{s} \in B$. Hence $\mathscr{L}$ is an $\mathscr{L}$-fuzzy sub-implicative SBG-ideal of B. □

Corollary 1. *Let $\mathcal{B} := (B, |)$ be a medial SBG-algebra. Then every $\mathscr{L}$-fuzzy SBG-implicative ideal of B is an $\mathscr{L}$-fuzzy sub-implicative SBG-ideal of B.*

Definition 12. *Let $\mathcal{B} := (B, |)$ be an SBG-algebra. An $\mathscr{L}$-fuzzy SBG-ideal $\mathscr{L}$ of B is said to be $\mathscr{L}$-fuzzy closed if*

$$(\forall \mathfrak{w} \in B)\left(\ \mathcal{L}_\mu(0^\mathfrak{w}|0^\mathfrak{w}) \geq \mathcal{L}_\mu(\mathfrak{w})\ \right).$$

Definition 13. *Let $\mathcal{B} := (B, |)$ be an SBG-algebra. Let $\mathscr{L}$ be an $\mathscr{L}$-fuzzy SBG-ideal of SBG-algebra B. Then $\mathscr{L}$ is called an $\mathscr{L}$-fuzzy completely closed ideal of B if*

$$(\forall \mathfrak{w}, \mathfrak{b} \in B)\left(\ \mathcal{L}_\mu(\mathfrak{w}^\mathfrak{b}|\mathfrak{w}^\mathfrak{b}) \geq \mathcal{L}_\mu(\mathfrak{w}) \wedge \mathcal{L}_\mu(\mathfrak{b})\ \right).$$

Theorem 9. *Let $\mathcal{B} := (B, |)$ be an SBG-algebra satisfying*

$$(\forall \mathfrak{w}, \mathfrak{b}, \mathfrak{s} \in B)\left(\ (((\mathfrak{w}^\mathfrak{b}|\mathfrak{w}^\mathfrak{b})|\mathfrak{w}^\mathfrak{s})|((\mathfrak{w}^\mathfrak{b}|\mathfrak{w}^\mathfrak{b})|\mathfrak{w}^\mathfrak{s}))|\mathfrak{s}^\mathfrak{b} = 0|0.\ \right). \tag{8}$$

Then B is implicative if and only if every $\mathscr{L}$-fuzzy closed ideal of B is an $\mathscr{L}$-fuzzy SBG-implicative ideal of B.

Proof. Let B be a SBG-algebra satisfying the equation (8). Assume that B is implicative and $\mathscr{L}$ is an $\mathscr{L}$-fuzzy closed ideal of B. Then $\mathscr{L}$ is an $\mathscr{L}$-fuzzy SBG-ideal of B. Thus, for $\mathfrak{w}, \mathfrak{b}, \mathfrak{s} \in B$ we have $\mathcal{L}_\mu(0) \geq \mathcal{L}_\mu(\mathfrak{w})$ and

$$\begin{aligned}
\mathcal{L}_\mu(\mathfrak{w}) &\geq \mathcal{L}_\mu(\mathfrak{s}) \wedge \mathcal{L}_\mu(\mathfrak{w}^{\mathfrak{s}}|\mathfrak{w}^{\mathfrak{s}}) \\
&= \mathcal{L}_\mu(\mathfrak{s}) \wedge \mathcal{L}_\mu(((((\mathfrak{w}|\mathfrak{w})^{\mathfrak{w}})|((\mathfrak{w}|\mathfrak{w})|\mathfrak{b}))|(\mathfrak{s}|\mathfrak{s}))| \\
&\quad ((((\mathfrak{w}|\mathfrak{w})^{\mathfrak{w}})|((\mathfrak{w}|\mathfrak{w})|\mathfrak{b}))|(\mathfrak{s}|\mathfrak{s}))) \\
&= \mathcal{L}_\mu(\mathfrak{b}) \wedge \mathcal{L}_\mu((((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(|\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s}))| \\
&\quad (((\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{w}|\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{s}|\mathfrak{s}))),
\end{aligned}$$

which means that $\mathscr{L}$ is an $\mathscr{L}$-fuzzy SBG-implicative ideal of B.

Conversely, suppose that every $\mathscr{L}$-fuzzy closed ideal of B is an $\mathscr{L}$-fuzzy SBG-implicative ideal of B. Then it follows from the equation (8), (S1)-(S2) and Lemma 1 (2) that $\mathfrak{s}^{\mathfrak{b}} = \mathfrak{w}^{\mathfrak{b}}|(\mathfrak{w}^{\mathfrak{s}}|\mathfrak{w}^{\mathfrak{s}})$. Since $\mathfrak{s}^{\mathfrak{b}} = \mathfrak{w}^{\mathfrak{s}}|(\mathfrak{w}^{\mathfrak{b}}|\mathfrak{w}^{\mathfrak{b}}) = ((\mathfrak{w}|\mathfrak{w}^{\mathfrak{s}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{s}}))|(\mathfrak{b}|\mathfrak{b})$ from (S1) and (S3), it is obtained from (S2) and Lemma 1 (1) that $\mathfrak{s} = (\mathfrak{w}|\mathfrak{w}^{\mathfrak{s}})|(\mathfrak{w}|\mathfrak{w}^{\mathfrak{s}})$. Thus, by (S1)-(S3) and Lemma 1 (3), we have

$$\begin{aligned}
\mathfrak{w}|\mathfrak{w}^{\mathfrak{b}} &= (((\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}}))|((((\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}})))|(\mathfrak{b}|\mathfrak{b})) \\
&= ((\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}}))|(\mathfrak{b}|(((\mathfrak{b}|\mathfrak{b})|\mathfrak{b}^{\mathfrak{w}}))|((\mathfrak{b}|\mathfrak{b})|\mathfrak{b}^{\mathfrak{w}})) \\
&= ((\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}})|(\mathfrak{b}|\mathfrak{b}^{\mathfrak{w}}))|\mathfrak{b}^{\mathfrak{b}}) \\
&= ((\mathfrak{b}^{\mathfrak{b}}|(\mathfrak{b}|\mathfrak{b})^{\mathfrak{b}})|(\mathfrak{b}^{\mathfrak{b}}|(\mathfrak{b}|\mathfrak{b})^{\mathfrak{b}}))|\mathfrak{b}^{\mathfrak{w}} \\
&= \mathfrak{b}|\mathfrak{b}^{\mathfrak{w}},
\end{aligned}$$

for all $\mathfrak{w}, \mathfrak{s} \in B$, which means that B is implicative. □

Proposition 4. *Let $\mathcal{B} := (B, |)$ be a implicative SBG-algebra satisfying the equation* (8)*. Then every $\mathscr{L}$-fuzzy completely closed ideal of B is an $\mathscr{L}$-fuzzy SBG-implicative ideal of B.*

Proof. Let $\mathscr{L}$ be an $\mathscr{L}$-fuzzy completely closed ideal of an implicative SBG-algebra B. Then $\mathscr{L}$ is an $\mathscr{L}$-fuzzy SBG-ideal of B. Since $\mathcal{L}_\mu(0^{\mathfrak{s}}|0^{\mathfrak{s}}) \geq \mathcal{L}_\mu(0) \wedge \mathcal{L}_\mu(\mathfrak{s}) = \mathcal{L}_\mu(\mathfrak{s})$, it is obtained that $\mathscr{L}$ is an $\mathscr{L}$-fuzzy closed ideal of B. Therefore, $\mathscr{L}$ is an $\mathscr{L}$-fuzzy SBG-implicative ideal of B. □

Corollary 2. *Let $\mathcal{B} := (B, |)$ be a medial SBG-algebra satisfying the equation* (8)*. Then every $\mathscr{L}$-fuzzy completely closed ideal of B is an $\mathscr{L}$-fuzzy SBG-implicative ideal of B.*

4 Conclusion

The paper introduced and analyzed L-fuzzy SBG-subalgebras and L-fuzzy SBG-ideals within the framework of Sheffer stroke BG-algebras. By extending the fuzzy membership concept to lattice-valued mappings, the study provides a broader and more flexible foundation for reasoning under multi-level uncertainty. The obtained results not only generalize earlier fuzzy and hesitant fuzzy approaches but also reveal deeper algebraic relationships between L-fuzzy ideals and their classical counterparts.

In future work, we plan to explore:

- extensions of L-fuzzy SBG-structures to hyperoperations and soft set environments;
- applications of L-fuzzy ideals in multi-criteria decision systems and intelligent control design;
- algorithmic implementations of membership evaluation using real data.

These directions aim to enhance the applicability of the theory in real-world decision-making and computational intelligence frameworks.

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