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SYNTHESIS OF FORCED-ROBUST CONTROL FOR UNCERTAIN SYSTEMS

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Abstract. The main result of this article is the replacement of the Lyapunov Energy Function (LEF), which is a solution to the Lyapunov equation, with a quadratic form. This function consists of the most informative indicators of the system — the tracking error and its derivatives which allows for easy determination of the stability of nonlinear systems in real time. Moreover, to suppress uncertainties, a "high-gain coefficient" is used as a control multiplier, giving the system robust properties without compromising stability. A qualitative analysis using Matlab/Simulink, based on a "peak gyroscope" control example, demonstrated the effectiveness of the proposed approach. Currently, robust control methods for uncertain systems pose complex challenges both in terms of mathematical synthesis and practical implementation. This article shows that the use of a high open-loop gain (forcing) makes it possible to compensate for total uncertainty to an arbitrarily small value without violating stability. The High-Gain Robust Control (HGRC) method was first proposed by the Russian scientist M. Meerov. The obtained results in terms of tracking accuracy, response speed, and implementation simplicity demonstrate a high scientific and technical level.

Keywords: robustness, energy function, stability, sign indicator, phase portrait, attractor, peak gyroscope, Matlab.

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1 Introduction

The main distinction between robust systems (from *robust* — strong, resilient) and adaptive systems is that, despite changes in object characteristics, the parameters and structure of the controller remain unchanged (Gerasimov & Nikiforov, 2015).

Despite being synthesized once, the "magic controller" ensures specified performance metrics even when system characteristics vary widely (Zhou et al., 1996). However, some authors (e.g., F. Chaki) classify robust systems as "passive adaptive systems" (Chaki, 1975).

At present, robust system synthesis methods are well developed (Rustamov & Rustamov, 2022; Lazarev, 2015; Baklanov & Vokhryshev, 2014; Rustamov, 2015, 2014). This paper proposes the synthesis of a robust control system with a high-gain coefficient. The effectiveness of such systems has been demonstrated in studies (Rustamov, 2015, 2014; Meerov, 1967; Alkhateeb et al., 2009; Vostrikov, 2008).

A key problem of the proposed approach is that increasing the gain may lead to instability.

2 Problem Statement

We consider an uncertain nonlinear object described by an n-th order differential equation:

$$y^{(n)} = f(\mathbf{y}, t) + b(\mathbf{y}, t)u, \ t \in [0, \infty], \tag{1}$$

Here: $\mathbf{y} = (y, \dot{y}, \ldots, y^{(n-1)})^T = (x_1, x_2, ..., x_n)^T \in R^n$ - measurable or estimated state vector; $y \in R$ – controlled output; $u \in R$ – control signal; $f(\mathbf{y}, t)$, $b(\mathbf{y}, t)$ – generally nonlinear, unknown, and bounded functions.

The main components of the model $f(\cdot)$ and $b(\cdot)$, which serve as the source of uncertainty, include both unknown and known elements.

However, these can be treated as parasitic dynamics and suppressed using a sufficiently high gain.

As a result, the closed-loop control system acquires the desired dynamics.

The goal is to determine a control signal u, such that, after the transient process, the output $y(t)$ of the uncertain object (1) follows a given reference $y_d(t)$ with required precision: $|y_d(t) - y(t)| \leq \delta_s, \quad t \geq t_s$, where $\delta_s = (1-5)\%$ is the allowable steady-state error, and t_s is the settling time.

3 Solution Approach

Control signal u is determined based on the requirement that the Lyapunov function derivative is negative:

$$dV(\boldsymbol{e})/dt < 0. \tag{2}$$

We define the Lyapunov function as:

$$V = 1/2s^2 \tag{3}$$

$$s = c_1 e + c_2 \dot{e} + ... + e^{(n-1)} \tag{4}$$

Derivative of the compound function (3):

$$dV/dt = \dot{V} = s\dot{s}. \tag{5}$$

$$\dot{s} = c_1 \dot{e} + c_2 \ddot{e} + ... + e^{(n)}. \tag{6}$$

To determine the control that satisfies the sufficient condition for stability (2) $\dot{V} < 0$, it is necessary to align u with the corresponding expression $\dot{V}$. For this purpose, in expression (6), taking into account the system equation (1) we will substitute the highest-order derivative $e^{(n)}$:

$$e^{(n)} = y_d^{(n)} - y^{(n)} = y_d^{(n)} - f(\boldsymbol{y}, t) - b(\boldsymbol{y}, t)u.$$

And rewrite:

$$\dot{s} = x(t) - b(\boldsymbol{y}, t)u, \qquad b > 0. \tag{7}$$

Here

$$x(t) = \sum_{i=1}^{n-1} c_i e^{(i)} + y_d^{(n)} - f(\boldsymbol{y}, t).$$

From expression (7), it follows that, under a condition $x(t) = 0$, satisfying the given relation $dV(\boldsymbol{e})/dt = s\dot{s} < 0$, the control can be chosen in the form of a proportional P-control law:

$$u = u_s = k \cdot s(t). \tag{8}$$

The influence of $x(t)$ on the system dynamics can be reduced by increasing the gain coefficient K.
Taking (8) into account in expression (7), we can write:

$$s(t) = b^{-1}[x(t) - \dot{s}(t)]/k.$$

As k tends to infinity, in the limit

$$\lim_{k \to \infty} s(t) = 0.$$

This means that, in the limit, the system equation can be described by the hyperplane equation (4):

$$s = c_1 e + c_2 \dot{e} + ... + e^{(n-1)} = 0, \qquad s(0) = s_0.$$

For the system to be stable, the corresponding equation written in operator form

$$H(p) = p^{n-1} + c_{n-1} p^{n-2} + ... + c_1 \tag{9}$$

must be a Hurwitz polynomial (according to the Routh criterion and others).
This means that the coefficients c_i must be selected in such a way that the roots of equation (9) satisfy the condition $Re(p_i)<0$.

In Fig. 1, the error trajectories are shown for various values of K:
$|y_d(t) - y(t)| \leq \delta_s, t \geq t_s$— a) and the phase portrait of the system for K =120 — b).

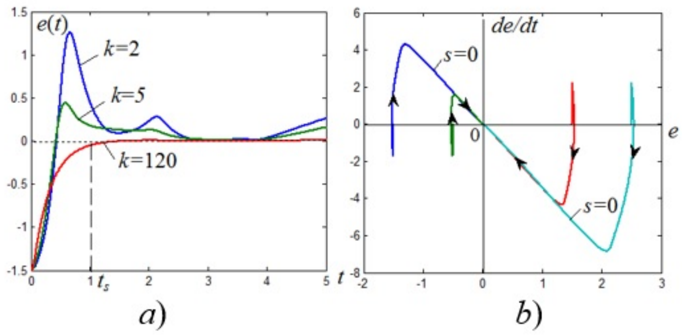

Figure 1: a) trajectories of change in control error, b) the phase portrait of the system for $K = 120$

At $K = 120$, overshoot and oscillations are eliminated, and the required quality indicators are achieved.

Tuning Parameter Determination. For sufficiently large values of K, from any initial state $s(0) = s_0$ the system rapidly (fast motion) reaches the plane $s = 0$. Then, along this hyperplane, it moves toward the origin (slow motion) (Fig. 1, b). For this reason, the system's motion from any initial position can be accurately described by the hyperplane equation s=0 .

Controller Tuning. The system's performance indicators (settling time t_s, overshoot σ%, etc.) depend on the angular coefficients c_i, i=1,. . . , c-1, which define the plane s=0, as well as on the gain coefficient K in the control equation (8).
To determine the tuning parameters c_i, it is advisable to use the modal control method.
In this case, one can apply characteristic polynomials such as those of Chebyshev or Butterworth, or use polynomials that minimize a given integral quality criterion.

Monotonic Transient Response - $\sigma = 0\%$**.** We require that the transient response with respect to the error $e(t)$ has no overshoot $-$ in other words, that it is monotonic. To achieve this, the roots p_i of the closed-loop system (9) are chosen according to a binomial distribution. In this case, all roots are equal:

$$p_1 = p_2 = ... = p_{n-1}.$$

When the roots are repeated, it becomes possible to assign their values based on the specified settling time t_s.

In the case of repeated roots, expression (9) can be written as:

$$W_{SYS}(p) = H(p) = (p + p_1)^{n-1}.$$

Let us define the root p_1. For repeated roots, the solution to equation (4) has the form:

$$e(t) = (C_1 + C_2 t + C_3 t^2 + ... + C_n t^{n-2})e^{-p_1 t}.$$

By determining the integration constants based on the known initial condition $\mathbf{e_s}(0)$, the solution can be written in the following vector form:

$$e(t) = C^T \theta e^{-p_1 t}, \quad C = Z^{-1}\mathbf{e}_s(0), \tag{10}$$

Here

$$\theta = \begin{pmatrix} 1 \\ t \\ t^2 \\ \cdots \\ t^{(n-2)} \end{pmatrix}, \quad \mathbf{e_s}(0) = \begin{pmatrix} e(0) \\ \dot{e}(0) \\ \ddot{e}(0) \\ .... \\ e^{(n-2)}(0) \end{pmatrix} = \mathbf{y}_{ds}(0) - \mathbf{y}_s(0);$$

$Z(t)|_{t=t_0=0}$ − quadratic matrix of $(n-1) \times (n-1)$r dimension;

$$Z = \begin{bmatrix} z_1(t) & z_2(t) \text{ .... } z_{n-1}(t) \\ \dot{z}_1(t) & \dot{z}_2(t) \text{ .... } \dot{z}_{n-1}(t) \\ . \quad . \quad . \quad . & . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ z_1^{(n-2)}(t) & z_2^{(n-2)}(t) \text{ ... } z_{n-1}^{(n-2)}(t) \end{bmatrix}_{t=t_0=0}.$$

Fundamental system of solutions: $z_1 = e^{-p_1 t}$, $z_2 = te^{-p_1 t}$, $z_3 = t^2 e^{-p_1 t}$, ... , $z_{n-1} = t^{n-2}e^{-p_1 t}$.

In the similar problems for the n-th initial condition of the system, it is common to assume $y^{(n-1)}(0)$=0.

Suppose the settling time t=t_s is specified for an error δ_s=±(2–5)%. Then, based on expression (10), an expression for determining the root p_1 can be obtained

$$e(t_s) = \pm\delta_s/100 = C^T \cdot \theta|_{t=t_s} \cdot e^{-p_1 t_s}.$$

If e_0>0, then δ_s >0 (convergence from above); otherwise, δ_s<0.

The corresponding transcendental equation is:

$$f(p_1) = C^T \cdot \theta|_{t=t_s} e^{-p_1 t_s} \mp \delta_s/100 = 0. \tag{11}$$

For n=2 $\theta = 1$,$\mathbf{e}_s(0) = e(0), Z^{-1} = 1 \Rightarrow C^T = e(0)$.

Equation (11) becomes

$$f(p_1) = e(0)\exp(-p_1 t_s) \mp \delta_s/100 = 0. \tag{12}$$

For given values of δ_s, t_s and $\boldsymbol{e}_s(0)$, equation (11) can be solved using the Matlab function *solve*('f(p1)=0').

After determining the root p_1, the tuning parameters c_i, $i = 1, ..., k-1$ are defined as the coefficients of the polynomial:

$$W_{SYS}(p) = (p + p_1)^{n-1} = p^{n-1} + a_{n-1}p^{n-2} + ... + a_2 p + a_1. \tag{13}$$

In this case

$$c_1 = a_1, c_2 = a_2, ..., c_{n-1} = a_{n-1}. \tag{14}$$

When implementing the controller (8) in the form of a transfer function

$$W_C(p) \approx K\frac{(p+p_1)^{n-1}}{(T_d p+1)^{n-1}} \quad , \tag{15}$$

the coefficients $c_i,\ i=1,...,n-1$ do not need to be explicitly calculated.

Determining the Gain Coefficient K This parameter is closely related to the real object and is therefore tuned for maximum impact. For this reason, K is determined either directly on the physical system by creating an extreme operating mode, or during design on a computer (simulation task).

By using higher values of K, it is possible to achieve the required quality indicators – $\sigma\%$ and t_s (see fig. 1, b). Nevertheless, this may lead to energy consumption exceeding acceptable limits.

In non-nominal (emergency) operating modes, an "extreme regulator" can be applied.

Example. The robust system considered as an example belongs to the class of *interval systems*. This is due to the fact that the model parameters, which are sources of uncertainty, are defined by their boundary values.

Oscillating Pendulum. Let us consider the control problem of a pendulum with periodically varying coefficients and load.

The system equation (Vostrikov, 2008):

$$\ddot{y} = -a\frac{\dot{R}}{R}\dot{y} - g\frac{1}{R}\sin(y) + \frac{1}{mR^2}u + n(t),$$

Here $m = 1kg,\ \ g = 9.81m/s^2,\ \ a = 2;\ 0 < R_m \le R \le R_M$. Initial condition of the object is $\boldsymbol{y}(0) = (0,\ 0)^T$.

The resulting equation of motion for the load:

$$R = 0.8 + 0.1\sin(8t) + 0.3\cos(4t).$$

Let the reference trajectory be given as:

$$y_d = 0.5\sin(0.5t) + 0.5\cos(t).$$

Initial condition: $y_d(0) = (0.5; 0.25)^T$.

In this case, the main components of the model are:

$$f(\boldsymbol{y},t) = -2\frac{\dot{R}}{R}\dot{y} - g\frac{1}{R}\sin(y) + n(t), \quad b(\boldsymbol{y},t) = \frac{1}{mR^2}.$$

In Fig. 2, the kinematic diagram of the pendulum (a) and the variation of the coefficients (b) are shown.

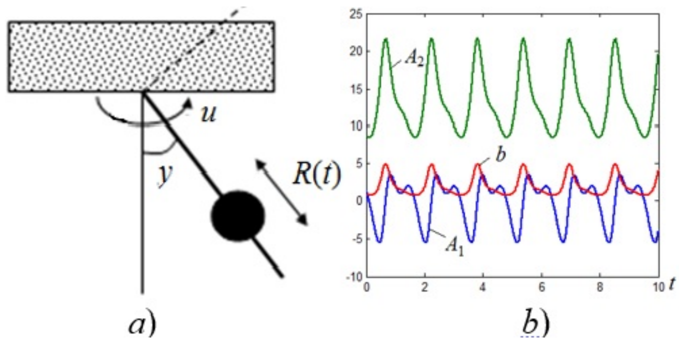

Figure 2: Kinematic diagram of the pendulum and the behavior of its coefficients

In the figure: $A_1 = 2\dot{R}/R, A_2 = g/R, \quad b = 1/mR^2$.

Required quality indicators: overshoot σ=0%; for an allowable error $\delta = \pm 2$ the settling time is specified $t_s = 1\ s$.

Since the system order is$n = 2$, the controller has two tuning parameters: c_1 and K.

Determining c_1**.** Given $n = 2$ the system's initial condition: $\boldsymbol{e}(0) = \mathrm{y}_d(0) - \mathrm{y}(0) = (0.5;\ 0.25)^T$.

Then equation (12) becomes:

$$f(\cdot) = e(0)e^{-p_1 t_s} - 0.02 = 0.5e^{-p_1} - 0.02 = 0.$$

Using MATLAB and the *solve(.)* function we determine the root p_1=3.23. Then equation (13) becomes:

$$W_{SYS} = p + 3.23 \Rightarrow c_1 = 3.23.$$

Corresponding MATLAB code:

```
>> p1=solve('0.5*exp(-p1)-0.02=0')
```

Result: p_1 =3.2188.

As a result, the controller equation (8) is $u = K(3.23e + \dot{e})$.

In Fig. 3, a and b, for the values a =2, m=1 the dependence of the error $e(t)$ on the gain coefficient K is shown, as well as the phase portrait for K=120.

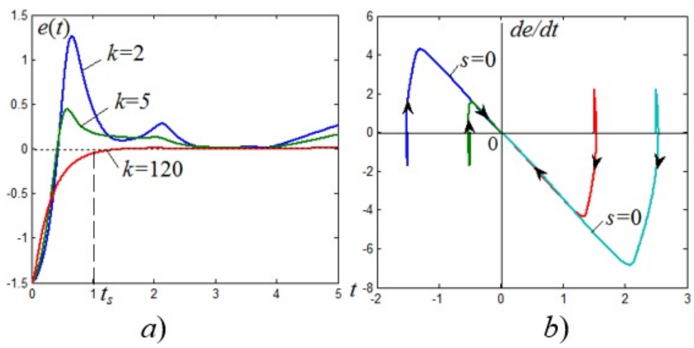

Figure 3: Transient characteristics and phase portrait of the system

At K=120, the required quality indicators are already achieved $\sigma = 0\%$, $t_s = 1s$.

With further increases in K, despite a slight improvement in performance indicators, more energy is required.

In Fig.3 a and c, the transient characteristics and phase portrait are shown for K=120 with various initial conditions $y(0) = [2,\ 1, -1, -2]$,$\dot{y}\,(0) = [2,\ 2, -2, -2]$. As can be seen, at K=120 all phase trajectories (phase flow) are attracted to the line $s = \dot{e} + 3.23e = 0$.

Let us now consider the case of parametric uncertainty. Suppose that the coefficient a for $\dot{y}(t)$ and the mass of the load m vary approximately by $\pm$50%: $a = [1; 1.5; 2; 2.5]$;$m = [0.5; 1; 1.5; 2]$;

In Fig. 4, for K=120 and a given initial condition$\boldsymbol{y}(0) = (2; 0)^T$, the following are shown: a bundle of transient responses {y(t)}(Fig. 4a), the corresponding errors$\{e(t)\}$ (Fig. 4 b) and the control signals $\{u(t)\}$ (Fig. 4 c).

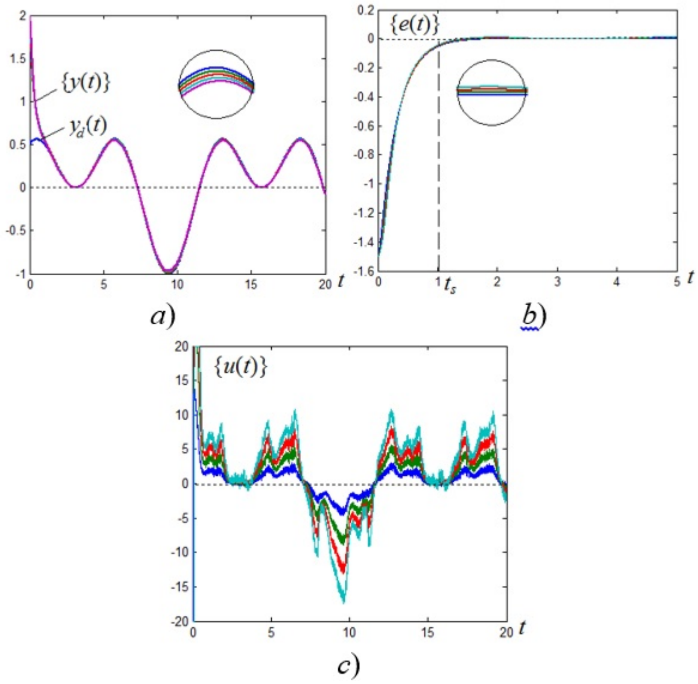

Figure 4: Dynamic characteristics of the system under varying parameters a and m

As can be seen, the concentration (density) of the bundles $\{y(t)\}$ and $\{e(t)\}$ is quite high. After $t_s \approx 1$ c the controlled variable $\{y(t)\}$ begins to closely follow the reference trajectory (setpoint) with high accuracy $y_d(t)$.

4 Conclusion

In this work, one of the tracking robust control systems with a high gain coefficient has been examined. The methodology is based on the *Lyapunov function method*, specifically the *method of quadratic forms*. The proposed approach allows one to avoid the complex problem of control synthesis for nonlinear and nonstationary systems under uncertainty by using a PD controller.

However, the proposed synthesis method has the following drawbacks:

- lack of an analytical formula for determining the gain coefficient K;
- the robust controller is implemented in the form of a PD-controller;
- it is known that the D-component of the controller, by amplifying high-frequency noise, increases the variance of the output $y(t)$.
- not every system allows for the implementation of a large K value.

By choosing an appropriate Lyapunov function, the need for the D-component can be eliminated.

Solving the applied problem in Matlab/Simulink demonstrated high practical effectiveness.

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